KRAMERS–WANNIER SYMMETRY AND STRONG-WEAK-COUPLING DUALITY IN 2D Φ^4 FIELD MODELS

B. N. Shalaev
A. F. Ioffe Physical & Technical Institute, Russian Academy of Sciences,
26 Polytechnicheskaya Str., St. Petersburg, 194021, Russia
and
Istituto Nazionale di Fisica Nucleare, Universita di Pavia
Via Bassi, Pavia, 6–27100, Italy
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It is found that the exact beta-function $\beta(g)$ of the continuous 2D $g\Phi^4$ model possesses two types of dual symmetries, these being the Kramers–Wannier (KW) duality symmetry and the weak-strong-coupling symmetry $f(g)$ (or S-duality). All these transformations are explicitly constructed. The S-duality transformation $f(g)$ is shown to connect domains of weak and strong couplings, i.e., above and below g_c. Basically, it means that there is a tempting possibility to compute multi-loop Feynman diagrams for the β-function using high-temperature lattice expansions. The regular scheme developed is found to be strongly unstable. Approximate values of the renormalized coupling constant g_+^* found from duality symmetry equations are in a good agreement with the available numerical results.
Key words: Kramers–Wannier duality, S-duality, renormalization group approuch, beta-function, high-temperature expansion.

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I. INTRODUCTION

The 2D Ising model and some other lattice spin models are known to possess the remarkable Kramers–Wannier (KW) duality symmetry, playing an important role in statistical mechanics, quantum field theory [1–3] as well as in the superstring theory [4]. The self-duality of the isotropic 2D Ising model means that there exists an exact mapping between the high-T and low-T expansions of the partition function [3]. In the transfer-matrix language this implies that the transfer-matrix of the model under discussion is covariant under the duality transformation. If we assume that the critical point is unique, the KW self-duality would yield the exact Curie temperature of the model. This holds for a large set of lattice spin models including systems with quenched disorder (for a review see [3,5]). Recently, the Kramers–Wannier duality symmetry was extended to the continuous 2D $g\Phi^4$ model [6] in the strong-coupling regime, i.e., for $g > g_c$.

This beta-function $\beta(g)$ is to date known only in the five-loop approximation within the framework of conventional perturbation theory at the fixed dimension $d = 2$ [7,8].

The strong coupling expansion for the calculation of the beta-function of the 2D scalar $g\Phi^4$ theory as an alternative approach to the standard perturbation theory was recently developed in [6].

It is well known from quantum field theory and statistical mechanics that any strong coupling expansions are closely connected with the high-temperature (HT) series expansions for lattice models. From the field-theoretical point of view the HT series are nothing but strong coupling expansions for field models, lattices to be considered as a technical device to define cut-off field theories (see [6,8] and references therein).

Calculations of beta-functions are of great interest in statistical mechanics and quantum field theory. The beta-function contains the essential information on the renormalized coupling constant g_+^*, this being important for constructing the equation of state of the 2D Ising model. Duality is known to impose some important constraints on the exact beta-function [10].

In this paper we study other duality symmetries of the beta-function $\beta(g)$ for the 2D $g\Phi^4$ theory regarded as a non-integrable continuum limit of the exactly solvable 2D Ising model. The main purpose is to construct exlicitly the weak-strong (WS) coupling duality transformation $f(g)$ connecting domains of weak and strong couplings, i.e., above and below g_c. The last transformation allows one to compute unknown yet multiloop orders $(6,7,\dots)$ of the β-functions on the basis of lattice expansions [6].

The paper is organized as follows. In Sect. II we set up basic notations and define both the correlation length and beta-function $\beta(g)$. In Sect. III the duality symmetry transformation $\tilde{g} = d(g)$ is derived. Then it is proved that $\beta(d(g)) = d'(g)\beta(g)$. An approximate expression for $d(g)$ is also found. Sect. IV contains an explicit derivation of the weak-strong coupling transformation whilst in Sect. V in order to illustrate our approach the sixth-order term of $\beta(g)$ is approximately computed. Sec. VI. contains disussion and some concluding remarks.

II. CORRELATION LENGTH AND COUPLING CONSTANT

We begin by considering the classical Hamiltonian of the 2D Ising model (in the absence of an external magnetic field), defined on a square lattice with periodic boundary conditions; as usual:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \ , \tag{2.1}$$

where $\langle i,j \rangle$ indicates that the summation is over all the nearest-neighboring sites; $\sigma_i = \pm 1$ are spin variables and J is a spin coupling. The standard definition of the spin-pair correlation function reads:

$$G(R) = \langle \sigma_{\mathbf{R}} \sigma_{\mathbf{0}} \rangle \ , \tag{2.2}$$

where $\langle \ldots \rangle$ stands for a thermal average.

The statistical mechanics definition of the correlation length is given by [11]

$$\xi^2 = \left. \frac{d \ln G(p)}{dp^2} \right|_{p=0} \ . \tag{2.3}$$

The quantity ξ^2 is known to be conveniently expressed in terms of the spherical moments of the spin correlation function itself, namely

$$\mu_l = \sum_{\mathbf{R}} (R/a)^l G(\mathbf{R}) \tag{2.4}$$

with a being some lattice spacing. It is easy to see that

$$\xi^2 = \frac{\mu_2}{2d\mu_0} \ , \tag{2.5}$$

where d is the spatial dimension (in our case $d = 2$).

In order to extend the KW duality symmetry to the continuous field theory we have a need for a "lattice" model definition of the coupling constant g, equivalent to the conventional one exploited in the RG approach. The renormalization coupling constant g of the $g\Phi^4$ theory is closely related to the fourth derivative of the "Helmholtz free energy", namely $\partial^4 F(T,m)/\partial m^4$, with respect to the order parameter $m = \langle \Phi \rangle$. It can be defined as follows (see [11])

$$g(T,h) = -\frac{(\partial^2 \chi/\partial h^2)}{\chi^2 \xi^d} + 3\frac{(\partial \chi/\partial h)^2}{\chi^3 \xi^d} \ , \tag{2.6}$$

where χ is the homogeneous magnetic susceptibility

$$\chi = \int d^2x \, G(x) \ . \tag{2.7}$$

It is in fact easy to show that $g(T,h)$ in Eq. (2.6) is merely the standard four-spin correlation function taken at zero external momenta. The renormalized coupling constant of the critical theory is defined by the double limit

$$g^* = \lim_{h \to 0} \lim_{T \to T_c} g(T,h) \tag{2.8}$$

and it is well known that these limits do not commute with each other. As a result, g^* is a path-dependent quantity in the thermodynamic (T,h) plane [11].

Here we are mainly concerned with the coupling constant on the isochor line $g(T > T_c, h = 0)$ in the disordered phase and with its critical value

$$g_+^* = \lim_{T \to T_c^+} g(T, h = 0) = -\left. \frac{\partial^2 \chi/\partial h^2}{\chi^2 \xi^d} \right|_{h=0} \ . \tag{2.9}$$

The "lattice" coupling constant g_+^* defined in Eq. (2.9) is a given function of the temperature T_c.

III. KRAMERS–WANNIER SYMMETRY

The standard KW duality tranformation is known to be as follows [1–3]

$$\sinh(2\tilde{K}) = \frac{1}{\sinh(2K)} \ . \tag{3.1}$$

We shall see that it will be more convenient to deal with a new variable $s = \exp(2K)\tanh(K)$, where $K = J/T$.

It follows from the definition that s transforms as $\tilde{s} = 1/s$; this implies that the correlation length of the 2D Ising model given by $\xi^2 = s/(1-s)^2$ is a self-dual quantity [6]. Now, on the one hand, we have the formal relation

$$\xi \frac{ds(g)}{d\xi} = \frac{ds(g)}{dg} \beta(g) \ , \tag{3.2}$$

where $s(g)$ is defined as the inverse function of $g(s)$, i.e., $g(s(g)) = g$ and the beta-function is given, as usual, by

$$\xi \frac{dg}{d\xi} = \beta(g) \ . \tag{3.3}$$

On the other hand, it is known from [6]

$$\xi \frac{ds}{d\xi} = \frac{2s(1-s)}{(1+s)} \ . \tag{3.4}$$

Therefore, from Eqs. (3.2)–(3.4), a useful representation of the beta-function in terms of the $s(g)$ function follows

$$\beta(g) = \frac{2s(g)(1-s(g))}{(1+s(g))\,(ds(g)/dg)} \ . \tag{3.5}$$

If one assumes that the fixed point is not singular, then from this equation it would follow that $\omega = \beta'(g)|_{g=g_c} =$

1 in an agreement with the classical paper [14]. (If not, another approach, see in [15] and a discussion in [6]).

Let us define the dual coupling constant $\tilde{g}$ and the duality transformation function $d(g)$ as

$$s(\tilde{g}) = \frac{1}{s(g)}; \qquad \tilde{g} \equiv d(g) = s^{-1}(\frac{1}{s(g)}) , \qquad (3.6)$$

where $s^{-1}(x)$ stands for the inverse function of $x = s(g)$. It is easy to check that a further application of the duality map $d(g)$ gives back the original coupling constant, i.e., $d(d(g)) = g$, as it should be. Notice also that the definition of the duality transformation given by Eq. (3.6) has a form similiar to the standard KW duality equation, Eq. (3.1).

Consider now the symmetry properties of $\beta(g)$. We shall see that the KW duality symmetry property, Eq. (3.1), results in the beta-function being covariant under the operation $g \to d(g)$:

$$\beta(d(g)) = d'(g)\beta(g) . \qquad (3.7)$$

To prove it let us evaluate $\beta(d(g))$. Then Eq. (3.5) yields

$$\beta(d(g)) = \frac{2s(\tilde{g})(1 - s(\tilde{g}))}{(1 + s(\tilde{g}))\,(ds(\tilde{g})/d\tilde{g})} . \qquad (3.8)$$

Bearing in mind Eq. (3.6) one is led to

$$\beta(d(g)) = \frac{2s(g) - 2}{s(g)(1 + s(g))\,(ds(\tilde{g})/d\tilde{g})} . \qquad (3.9)$$

The derivative in the r.h.s. of Eq. (3.9) should be rewritten in terms of $s(g)$ and $d(g)$. It can be easily done by applying Eq. (3.6):

$$\frac{ds(\tilde{g})}{d\tilde{g}} = \frac{d}{d\tilde{g}}\frac{1}{s(g)} = -\frac{s'(g)}{s^2(g)}\frac{1}{d'(g)} . \qquad (3.10)$$

Substituting the r.h.s. of Eq. (3.10) into Eq. (3.9) one obtains the desired symmetry relation, Eq. (3.7).

Therefore, the self-duality of the model allows us to determine the fixed point value in another way, namely from the duality equation $d(g^*) = g^*$.

Making use of a rough approximation for $s(g)$, one gets [6]

$$s(g) \simeq \frac{2}{g} + \frac{24}{g^2} \simeq \frac{2}{g}\frac{1}{1 - 12/g} = \frac{2}{g - 12} . \qquad (3.11)$$

Combining this Padé-approximant with the definition of $d(g)$, Eq. (3.6), one is led to

$$d(g) = 4\frac{3g - 35}{g - 12} . \qquad (3.12)$$

The fixed point of this function, $d(g^*) = g^*$, is easily seen to be $g^*_+ = 14$. The recent numerical and analytical estimates yield $g^*_+ = 14.69$ (see [6,12,13] and references therein).

It is worth mentioning that the above-described approach may be regarded as another method for evaluating g^*_+, fully equivalent to the standard beta-function method.

IV. STRONG-WEAK COUPLING DUALITY

The beta-function of the model under discussion possesses a specific algebraic property (3.6) (KW duality) which allows to develop the weak-strong-duality transformation $f(g)$ connecting both the weak-coupling and strong coupling regimes.

Nowadays both the five-loop approximation results [9] and the strong coupling expansion for the beta-function [6] are known rather well. These are given by

$$\beta_1(g) = 2g - 2g^2 + 1.432347241g^3 - 1.861532885g^4 + 3.164776688g^5 - 6.520837458g^6 + O(g^7) , \quad (4.1)$$

$$\beta_2(g) = -2g + \frac{12}{\pi} - \frac{9}{\pi^2 g} + \frac{27}{\pi^3 g^2} + \frac{81}{8\pi^4 g^3} - \frac{3645}{16\pi^5 g^4} - \frac{15309}{32\pi^6 g^5} + \frac{2187}{64\pi^7 g^6} + O(g^{-7}) . \quad (4.2)$$

Here indices $1, 2$ stand for the weak and strong coupling regimes respectively. The main goal of this Section is to determine a dual transformation $f(g)$ such as $f[f(g)] = g$ relating beta-functions $\beta_1(g)$ and $\beta_2(g)$.

From Eq. (3.5) one can easily find the functions $S_1(g), S_2(g)$ and their inverse functions $G_1(s) = S_1^{-1}(g)$, $G_2(s) = S_2^{-1}(g)$ corresponding to the two regimes. Simple but cumbersome calculations lead to

$$G_1(s) = s + s^2 + 0.3580868104s^3 - 0.1166327797s^4 - 0.1968226859s^5 - 0.1299831557s^6 + O(s^7),$$

$$S_1(g) = g - g^2 + 1.6419131896g^3 - 3.09293317g^4 + 6.361881481g^5 - 13.78545095g^6 + O(g^7) ,$$

$$s \in [0, 1] , \qquad g \in [0, g^*] , \qquad (4.3)$$

$$G_2(s) = \frac{3}{4\pi s} + \frac{9}{2\pi} - \frac{9s}{4\pi} + \frac{18s^2}{\pi} - \frac{108s^3}{\pi} + \frac{618s^4}{\pi} - \frac{3474s^5}{\pi} + \frac{19494s^6}{\pi} + O(s^7),$$

$$S_2(g) = \frac{2\cdot 3}{8\pi g} + \frac{24\cdot 3^2}{(8\pi g)^2} + \frac{264\cdot 3^3}{(8\pi g)^3} + \frac{2976\cdot 3^4}{(8\pi g)^4} + \frac{35136\cdot 3^5}{(8\pi g)^5} + \frac{423680\cdot 3^6}{(8\pi g)^6} + \frac{5149824\cdot 3^7}{(8\pi g)^7} + \frac{63275520\cdot 3^8}{(8\pi g)^8} + O(g^{-9}) ,$$
$$s \in [0,1] , \qquad g \in [g^*, \infty) . \tag{4.4}$$

Having been equipped with these formulas one may easily construct two branches of the same duality transformation function $f_{12}(g)$ and $f_{21}(g)$ defined in different domains of g. The functions are

$$\frac{1}{f_{21}(g)} \equiv \frac{1}{G_2(S_1(g))} = \frac{4\pi g}{3} - \frac{28\pi g^2}{3} + 220.5059303 g^3 - 1766.8145 g^4 + 14816.94007 g^5 - 127842.5955 g^6,$$
$$g \in [0, g^*] , \qquad f_{21}(g) \in [g^*, \infty], \tag{4.5}$$

$$f_{12}(g) \equiv G_1(S_2(g)) = \frac{3}{4\pi g} + \frac{63}{16\pi^2 g^2} + \frac{0.61714739472}{g^3} + \frac{0.9560453953}{g^4} + \frac{1.502156783}{g^5} + \frac{2.368311503}{g^6} + O(g^7) ,$$
$$g \in [g^*, \infty) , \qquad f_{12}(g) \in [0, g^*] . \tag{4.6}$$

The functions found above look like inversion, but they are not so simple. An interesting nontrivial example of the 2D model disordered Dirac fermions was discovered in [16]. It was shown that the beta-function of the (non-integrable) model under consideration also exhibits the strong-weak coupling duality such as $g^* \to \frac{1}{g}$ [16].

It is worth noting that the transformation found is dual indeed

$$f_{12}(f_{21}(g)) = f_{21}(f_{12}(g)) \equiv g . \tag{4.7}$$

Moreover, by definition weak-strong coupling expansions of $\beta(g)$ are related to each other in the following way:

$$\beta_2(g) = \frac{\beta_1(f_{12}(g))}{f'_{12}(g)} , \tag{4.8}$$

$$\beta_1(g) = \frac{\beta_2(f_{21}(g))}{f'_{21}(g)} . \tag{4.9}$$

It is rather amusing that Eq. (4.6) looks like a geometric series. Making use of the Padé method we arrive at

$$f_{12}(g) \approx \frac{0.2387324146 g^2 - 0.0745907136 g + 0.0850867165}{g^3 - 1.983571753 g^2 + 1.086109562 g - 0.6919672492} ,$$
$$g \in [g^*, \infty) , \qquad f_{12}(g) \in [0, g^*] . \tag{4.10}$$

The weak-strong duality equation and strong-coupling expansion yield the following numerical values

$$f_{12}(g) - g = 0, \qquad g^* = 14.38 \qquad \beta_2(g^*) = 0, \qquad g^* = 14.63, \tag{4.11}$$

being in good agreement with modern estimates [17–19].

V. HIGHER-ORDER TERMS FOR BETA-FUNCTION

Finally, let us consider how one can compute the $\beta(g)$ in the multiloop approximation via the strong-coupling expansion and the S-duality function. In order to find that one should exploit Eq. (4.8),Eq. (4.2) and the approximate expression for $f_{12}(g)$ given by Eq. (4.10).

After some tedious but routine calculations we arrive to some polynomial of 7th degree for $\beta_1(g)$:

$$\beta_1(g) = 2g - 2g^2 + 1.432347241 g^3 - 1.861532885 g^4 + 3.164776688 g^5 - 6.520837458 g^6 - 331.454743 g^7. \tag{5.1}$$

It is easily seen that the first 6 terms except for the 7th one are the exact perturbation expansion for $\beta_1(g)$ [9]. It would be tempting but wrong to regard Eq. (5.1) as a $\beta(g)$-function in the 7th loop approximation. In fact, the function in Eq. (4.10) is approximate, so that we have to estimate an accuracy of our calculations.

Suppose, that a difference between the "exact" duality function $f_{12}^{\rm exact}(g)$ and the approximate one given by Eq. (4.10) reads

$$f_{12}^{\rm exact}(g) = \frac{0.2387324146g^2 - 0.0745907136g + 0.0850867165}{g^3 - 1.983571753g^2 + 1.086109562g - 0.6919672492} + b/g^7 \,, \qquad (5.2)$$

with b being an arbitrary parameter. The straightforward calculation shows that a "new" 7th loop contribution computed by making use of the Eq. (5.2) depends on the fitting parameter b and differs vastly from the previous one, namely:

$$\beta_1(g) = 2g - 2g^2 + 1.432347241g^3 - 1.861532885g^4 + 3.164776688g^5 - 6.520837458g^6 + (-331.454743 + 271519.803807b)g^7 \,. \qquad (5.3)$$

Thus, we see that the approach suggested above provides a regular scheme for computing higher-order corrections to the $\beta(g)$-function on the basis of lattice high-order expansions. In other words, one obtains a tempting possibility to compute (approximately) multiloop Feynman diagrams on the basis of Eq. (4.9) and of high-temperature expansions [6]. A serious drawback of that scheme is that it is unstable from a mathematical point of view.

VI. CONCLUDING REMARKS

We have shown that the β-function of the $2D\ g\Phi^4$ theory does have the two types of dual symmetries: (i) the Kramers–Wannier symmetry, and (ii) the weak-strong-coupling symmetry (S-duality).

Our proof of the KW symmetry is based on the properties of $g(s), s(g)$ defined only for $1 \le s < \infty; g_+^* \le g < \infty$ and therefore does not cover the weak-coupling region, $0 \le g \le g^*$. So, the statement is that the beta-function $\beta(g)$ possesses the KW symmetry only in the strong-coupling region.

In contrast to widely held views, the KW symmetry imposes only mild restrictions on $\beta(g)$. It means that this symmetry property fixes only even derivatives of the beta-function $\beta^{(2k)}(g_+^*)(k = 0, 1, \ldots)$ at the fixed point, leaving the odd derivatives free, in particular, the critical exponent ω responsible for corrections to scaling.

We established the existence of the nontrivial weak-strong-coupling dual function $f(g)$ (S-duality) connecting two domains of both weak coupling and strong coupling given both perturbative RG calculations and lattice high-temperature expansions that S-function $f(g)$ can be approximately computed. We also explicitly computed high-order terms for $\beta(g)$. A close analysis of the scheme developed shows that this is strongly unstable.

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СИМЕТРІЯ КРАМЕРСА–ВАНЬЄ Й ДУАЛЬНІСТЬ СИЛЬНО-СЛАБКОГО ЗВ'ЯЗКУ У ДВОВИМІРНИХ ПОЛЬОВИХ МОДЕЛЯХ Φ^4

Б. Н. Шалаєв
Фізико-технічний інститут ім. А. Ф. Йоффе, Російська академія наук,
вул. Політехнічна, 26, Санкт-Петербург, 194021, Росія
Національний інститут ядерної фізики, Університет у Павії,
Віа Басі, Павія, 6–27100, Італія

Знайдено, що точна бета-функція $\beta(g)$ неперервної двовимірної моделі з $g\Phi^4$ має два типи дуальної симетрії, а саме, дуальну симетрію Крамерса–Ваньє (КВ) та симетрію сильно-слабкого зв'язку $f(g)$ (або S-дуальність). Усі ці перетворення отримано явно. Показано, що перетворення S-дуальности з'єднує домени зі слабкими та сильними зв'язками, тобто зі значеннями вище і нижче від g_c. Це означає, що існує приваблива можливість порахувати багатопетлеві діяграми Фейнмана для β-функції, використовуючи високотемпературні ґраткові розклади. Отримана послідовна схема виявляється дуже нестійкою. Знайдені з рівнянь дуальної симетрії наближені значення константи зв'язку g_+^* добре узгоджуються з відомими чисельними результатами.