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ENTROPY DRIVEN MECHANISM FOR ORDERING, PHASE SEPARATION AND PATTERN FORMATION PROCESSES IN STOCHASTIC SYSTEMS

D. O. Kharchenko[1], A. V. Dvornichenko[2], V. O. Kharchenko[1]
[1]*Institute of Applied Physics, 58, Petropavlivska St., 40030, Sumy, Ukraine*
[2]*Sumy State University, 2, Rimskii-Korsakov St., 40007, Sumy, Ukraine*
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We present study concerns a generalization of the model for extended stochastic systems with a field-dependent kinetic coefficient and a noise source satisfying fluctuation-dissipation relation. Phase transitions with entropy driven mechanism are investigated in systems with conserved and nonconserved dynamics. It is found that in stochastic systems with a relaxational flow and a symmetric local potential reentrant phase transitions can be observed. We have studied the entropy-driven mechanism leading to stationary patterns formation in stochastic systems of reaction diffusion kind. It is shown that a multiplicative noise fulfilling a fluctuation-dissipation relation is able to induce and sustain stationary structures. Our mean-field results are verified by computer simulations.
Key words: noise, phase transition, spatial pattern.

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I. INTRODUCTION

It is well known that nonlinear systems that exhibit disordered behaviour in the absence of fluctuations can be organized to sustain ordered states when an additional amount of noise is added [1, 2]. In recent decades, many studies have focused on investigations of noise-induced phenomena which demonstrate a counterintuitive role for fluctuations leading to self-organization effects, such as noise-induced transitions in zero-dimensional systems (see Ref. [1] and citations therein), stochastic resonance [3], noise-induced spatial patterns and phase transitions [2, 4, 5], and phase transitions induced by cross-correlations of noises [6, 7]. As will be discussed in this paper, one of the most interesting effects is an ordering phase transition in extended systems, where the ordered phase (in a thermodynamic sense) results only if a randomly fluctuating source is introduced into the dynamical system, which must possess spatial degrees of freedom. Such problems are actual not only in statistical physics. They arise naturally in the physics of lasers and electronics [8, 9], in physics of irradiation induced effects and microstructure transformations [10], in solid-state physics to describe the reconstruction of a defect structure [11], chemistry and biology [12], etc.

Most of the works concerning the above phenomena have focused on the problems concerning the influence of external noise. Analytically, numerically, and experimentally it was found that an external noise source only plays an organizing role if its amplitude depends on the field variable (see Refs. [1, 2, 13, 14]). This result was explained as follows: in systems with fluctuations having a bounded frequency spectrum, the ordered phase exists for a particular range of the system parameters such as the control parameter, the noise intensity, and the intensity of spatial coupling (see Refs. [2,7,13]). Such reentrant phase transitions correspond to cases wherein an increase in one of the above parameters leads to an ordering dynamics once a first critical threshold is crossed, but after a second threshold is passed, the system becomes disordered. The above reentrance appears as a result of the combined effect of the nonlinearity of the system, the spectrally variant nature of the noise, and the spatial coupling. From a fundamental point of view, such effects have a dynamic origin: in the short-time limit, external fluctuations destabilize the disordered homogeneous state.

Recently, a new class of phase transitions was found [15] where fluctuations do not lead to instability in the disordered phase (homogeneous mixture). Here, the ordered (separated) phase appears due to the balance between relaxing forces moving the system to the homogeneous state, and field-variable dependent fluctuations pulling the system away from the disordered state. This mechanism belongs to a set of entropy driven phase transitions, which are the extension of noise-induced unimodal-bimodal transitions in zero-dimensional systems [1]. The origin of such phase transitions is in changing the form of the nonequilibrium potential [15–17]. The novelty of this phase transition lies in the fact that it arises entirely from an energy functional-like relaxation dynamics. Its occurrence indicates the presence of two elements in the stochastic dynamics: a field-dependent kinetic coefficient and a fluctuation dissipation relation. It allows on to interpretate the corresponding fluctuations as internal noise with intensity proportional to the bath temperature. For such a class of stochastic systems, the corresponding distribution function, free energy and an associated effective potential are known exactly. Therefore, noise-induced phase transitions can be analyzed without any dynamic reference.

The main goal of this work is to study the internal noise influence onto the system evolution and formation of stationary states in cases of both nonconserved and conserved dynamics. We shall show that phase transitions and phase separation processes are driven by special types of mechanism with no short-time instability of a disordered state. Considering systems of reaction-diffusion kind we study the ability of the noise to sustain stationary spatially patterns.

II. STOCHASTIC SYSTEMS WITH INTERNAL FLUCTUATIONS

Let us consider the following generic deterministic model for a real field $x(\mathbf{r},t)$

$$\partial_t x(\mathbf{r},t) = -\mathcal{M}[\nabla, x(\mathbf{r},t)]\frac{\delta\mathcal{F}}{\delta x(\mathbf{r},t)}. \quad (1)$$

This equation corresponds to a relaxational flow in a potential $\mathcal{F}[x]$ with a field dependent kinetic coefficient $M[\nabla, x]$ [18]. For the systems with nonconserved dynamics (models of $\mathcal{A}$ class with $\int \mathrm{d}\mathbf{r}x(\mathbf{r},t) \neq \mathrm{const}$) the kinetic coefficient can depend on the field x only, if dynamics is conserved (models of $\mathcal{B}$ class with $\int \mathrm{d}\mathbf{r}x(\mathbf{r},t) = \mathrm{const}$), then $\mathcal{M}$ depends on the operator $\nabla = \partial_\mathbf{r}$ and x.

As was shown by Ibanes, et al [15] the introduction of a stochastic source into Eq. (1) for the system with nonconserved dynamics according to the fluctuation-dissipation relation leads to the Langevin equation of the form

$$\partial_t x(\mathbf{r},t) = -M[x(\mathbf{r},t)]\frac{\delta\mathcal{F}}{\delta x(\mathbf{r},t)} + \sqrt{M[x(\mathbf{r},t)]}\zeta(\mathbf{r},t). \quad (2)$$

Here the Gaussian noise ζ has the following properties $\langle\zeta(\mathbf{r},t)\rangle = 0$, $\langle\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')\rangle = 2\sigma^2\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, σ^2 is the noise intensity. Considering the system with conserved dynamics, the corresponding Langevin equation can be rewritten as follows [2]:

$$\partial_t x = \nabla\cdot\left(M[x(\mathbf{r},t)]\nabla\frac{\delta\mathcal{F}}{\delta x}\right) + \nabla\sqrt{M[x(\mathbf{r},t)]}\zeta(\mathbf{r},t). \quad (3)$$

With this construction a stationary distribution $P_s[x]$ obtained as a solution of the corresponding Fokker-Planck equation has the Boltzmann form

$$P_s[x] = N\exp\left(-\mathcal{U}_{\mathrm{eff}}[x]/\sigma^2\right),$$
$$\mathcal{U}_{\mathrm{eff}}[x] = \mathcal{F}[x] + \frac{\sigma_0^2}{2}\int \mathrm{d}\mathbf{r}\ln M[x], \quad (4)$$

here σ_0^2 is the renormalized noise intensity proportional to σ^2 including the ultraviolet cut off (see [15]). For the system with conserved dynamics the form of the effective potential is the same [19]. It is principally important that this potential can be bistable only in the presence of noise. From the formal viewpoint if we associate the Lyapunov functional $\mathcal{F}$ with the free energy, then the second term in $\mathcal{U}_{\mathrm{eff}}$ is reduced to the nonequilibrium entropy, whereas the noise intensity σ^2 has the meaning of the nonequilibrium temperature. Therefore, using thermodynamic relations one associates the functional $\mathcal{U}_{\mathrm{eff}}$ with the effective internal energy for the model.

We shall show in what manner this class of systems can exhibit noise induced phase transitions. Suppose that $\mathcal{F}$ is described by a deterministic free energy potential of the usual Ginzburg-Landau form $\mathcal{F} = \int \mathrm{d}\mathbf{r}\left[f(x) + (\beta/2)(\nabla x)^2\right]$, where $f(x)$ is the free energy density, β is the gradient energy term related to the interaction radius $r_0 \equiv \sqrt{\partial^2\mathcal{F}/\partial(\nabla x)^2|_{\nabla x=0}}$ as $\beta = r_0^2$. In our study we use the following construction $f(x) = -\varepsilon/2x^2 + x^4/4$, where ε is the control parameter, related to the temperature counted off a critical one. The kinetic coefficient we assume in a bell-shaped form. This kind of model describes large fluctuations in disordered/diluted state and small fluctuations in ordered/dense state. In our study we use the following approximation $M[x] = (1+\alpha x^2)^{-1}$, $\alpha > 0$.

The last model presented here is the model for reaction-diffusion systems where the noise can sustain stationary patterns. Usually to investigate the patterns formation the Swift-Hohenberg operator $(1-\nabla^2)^2$ is introduced into the evolution equation instead of the Laplacian ∇^2. In our model we start with the deterministic dynamics given by Eq. (1) and add the term related to the local dynamics $R(x)$. It describes possible chemical reactions in the system. Therefore, the deterministic model for reaction-diffusion system takes the form

$$\partial_t x = R(x) + \nabla\left(M[x]\nabla\frac{\delta\mathcal{F}}{\delta x}\right). \quad (5)$$

Formally, Eq. (5) can be written in a variational form as $\partial_t x = -(M[x])^{-1}\delta\mathcal{U}[x]/\delta x$, where the functional $\mathcal{U}[x]$ is determined through $R[x]$, $M[x]$ and $\mathcal{F}[x]$. It plays a role of a Lyapunov functional for the deterministic dynamics. One can obtain the first variation of $\mathcal{U}[x]$ exactly,

$$\delta\mathcal{U}[x] = -\int \mathrm{d}\mathbf{r}\delta x\left\{R[x]M[x] + M[x]\nabla\left(M[x]\nabla\delta\mathcal{F}/\delta x\right)\right\}. \quad (6)$$

Substituting variational derivative into the reduced equation for x, one arrives at Eq. (5) immediately. In stochastic analysis we introduce a related multiplicative noise in an *ad hoc* form [20],

$$\partial_t x = -\frac{1}{M[x]}\frac{\delta\mathcal{U}[x]}{\delta x} + \frac{1}{\sqrt{M[x]}}\zeta(\mathbf{r},t), \quad (7)$$

where the fluctuation dissipation relation holds. The stationary distribution can be obtained exactly in the Boltzmann form (4) with

$$\mathcal{U}_{\mathrm{eff}}[x] = \mathcal{U}[x] + \frac{\sigma_0^2}{2}\int \mathrm{d}\mathbf{r}\ln\left(M[x]\right)^{-1}. \quad (8)$$

For a general model one can find only first variation of the nonequilibrium potential $\mathcal{U}_{\mathrm{eff}}[x]$. It allows one to study stationary patterns $x_s(\mathbf{r})$. For the simplest case for the Fickian diffusion the mobility $M[x]$ can be considered as effective diffusion coefficient in the form $\mathcal{D}[x(1-x)]^{\alpha}$, $\alpha > 0$[1].

III. ENTROPY DRIVEN PHASE TRANSITIONS

To study noise induced phase transitions is systems of the class $\mathcal{A}$ analytically one can use the mean-filed approach. To that end we consider the system in a d-dimensional square lattice, and instead of Eq. (2) we get a set of ordinary differential equations for every i-cell variable from the set $\{x_i\}_{i=1}^{N^d}$. In the Weiss mean field approach one can rewrite the discrete gradient operator as follows $(\nabla x)^2 \to (\eta - x)^2$, with the mean field value $\eta \equiv \langle x \rangle$.

As a result, the effective potential acquires a dependence on an unknown mean-field value η:

$$U_{\rm eff}(x;\eta) = f(x) + \frac{\beta}{2}(\eta - x)^2 + \frac{\sigma_0^2}{2}\ln M(x). \qquad (9)$$

The value η can be calculated as the solution of the self-consistency equation

$$\eta = \int x P_s(x;\eta)\mathrm{d}x \equiv \Phi(\eta),$$
$$P_s(x;\eta) = Z^{-1}(\eta)\exp(-U_{\rm eff}(x;\eta)/\sigma_0^2). \qquad (10)$$

Here, Z satisfies the normalization condition, and η plays the role of the order parameter.

From a physical viewpoint, the solution $\eta = 0$ defines the disordered homogeneous phase, and the corresponding distribution function is symmetrical with respect to the origin $x = 0$. If the distribution function is asymmetrical, then the order parameter takes a nontrivial value, $\eta \neq 0$, and the system is ordered. To solve Eq. (10) the standard Newton-Raphson procedure is used. One should note that the right hand side of the function $\Phi(\eta)$, formally, can intersect the left hand side of the function η more than once. A number of intersections gives an equal number of roots of the equation $\eta = \Phi(\eta)$ at the related values of the order parameter η. Generally, the number of roots depends on the form of the function $U_{\rm eff}(x;\eta)$ and the related construction of the normalization constant $Z(\eta)$. In the case under consideration, if the local potential $V(x)$ is of symmetrical form, i. e. $V(-x) = V(x)$, then two equivalent oppositely signed nontrivial solutions $+\eta$ and $-\eta$ appear for a particular range of control parameters. If the potential $V(x)$ is asymmetrical or shifted with respect to the origin $x = 0$, then one can expect more than a unique non vanishing solution ($\eta \neq 0$).

The critical values for the system parameters giving the phase transition line can be obtained from the solution of the problem $\mathrm{d}\Phi(\eta)/\mathrm{d}\eta|_{\eta=0} = 1$. It is easy to see that such derivative defines the generalized susceptibility $\chi = \mu_2 \equiv \langle (x-\eta)^2 \rangle$ (cumulant of the second order), that measures fluctuations around the critical values of the system parameters. The third order derivative of $\Phi(\eta)$ at $\eta = 0$ gives the fourth order cumulant, $\mu_4 \equiv \langle (x-\eta)^4 \rangle$. If $\mu_4 < 0$ and $\mu_2 = 1$, then the corresponding phase transition is of the second kind. After simple algebra one finds the conditions for the critical phase transition:

$$\left.\frac{\mathrm{d}\Phi(\eta)}{\mathrm{d}\eta}\right|_{\eta=0} = \frac{2\beta}{\sigma_0^2}\mu_2, \qquad \mu_2 = \frac{\sigma_0^2}{2\beta};$$
$$\left.\frac{\mathrm{d}^3\Phi(\eta)}{\mathrm{d}\eta^3}\right|_{\eta=0} = \left(\frac{2\beta}{\sigma_0^2}\right)^3 \mu_4, \qquad \mu_4 < 0. \qquad (11)$$

The principle feature of the model with the multiplicative noise satisfying the fluctuation dissipation relation lies in the fact that phase transitions are not related to the short-time instability. Indeed, considering linearized equation for the first moment $\partial_t \langle x \rangle = (\varepsilon - \alpha\sigma^2)\langle x \rangle + \beta\Delta\langle x \rangle$, one can see that at early stages the noise leads to stabilization of the disordered state $\langle x \rangle = 0$. At late stages and in the stationary case the main mechanism of noise induced phase transitions is related to the effective entropy $S_{\rm eff}$ variations. Therefore, the corresponding phase transitions are known as entropy driven phase transitions.

Let us consider solutions of the self-consistency equation for the model with $M(x) = (1 + \alpha x^2)^{-1}$. The order parameter and the generalized susceptibility dependencies versus the noise intensity are shown in Fig. 1a. It is seen that in the case of the bistable potential $f(x)$ with $\varepsilon > 0$ the system undergoes disordering phase transition with an increase in σ^2, for the generalized susceptibility we obtain the peak with the height 1 at the critical σ^2 with $\mu_4 < 0$. If the potential $f(x)$ is monostable ($\varepsilon < 0$), then the reentrant phase transition can be observed: the ordered state exists inside the domain of the noise intensity values $[\sigma_{c1}^2, \sigma_{c2}^2]$, the quantity χ has two peaks placed at critical noise intensity values. Let us consider the phase diagram shown in Fig. 1b. Here the solid lines define critical values for the system parameters, whereas dotted and dashed lines correspond to modality change of the distribution function calculated at the corresponding η values in the denoted domains of the phase diagram. It is seen that during phase transition the probability density function (see insets) changes the modality. At $\varepsilon < 0$ with small β and σ^2 (see point a) the system is disordered, the probability density is of symmetrical form and has one peak centered at $x = 0$. In the domain with point d the system is disordered too, here $P_s(x;\eta)$ is of symmetrical form and has two equivalent peaks. In the domain with point c despite $P_s(x;\eta)$ has one peak, it has broken symmetry due to $\eta \neq 0$. When the dashed line is crossed the modality of $P_s(x;\eta)$ is changed and due to $\eta \neq 0$ the symmetry is still broken. Therefore, the obtained phase transition related to the symmetry breaking of the distribution function is accompanied with its modality change.

[1] A construction for the effective kinetic coefficient can be derived from the nonlinear Fokker–Planck equation containing only the diffusion term and q-deformed version of the logarithm proposed by Tsallis [21].

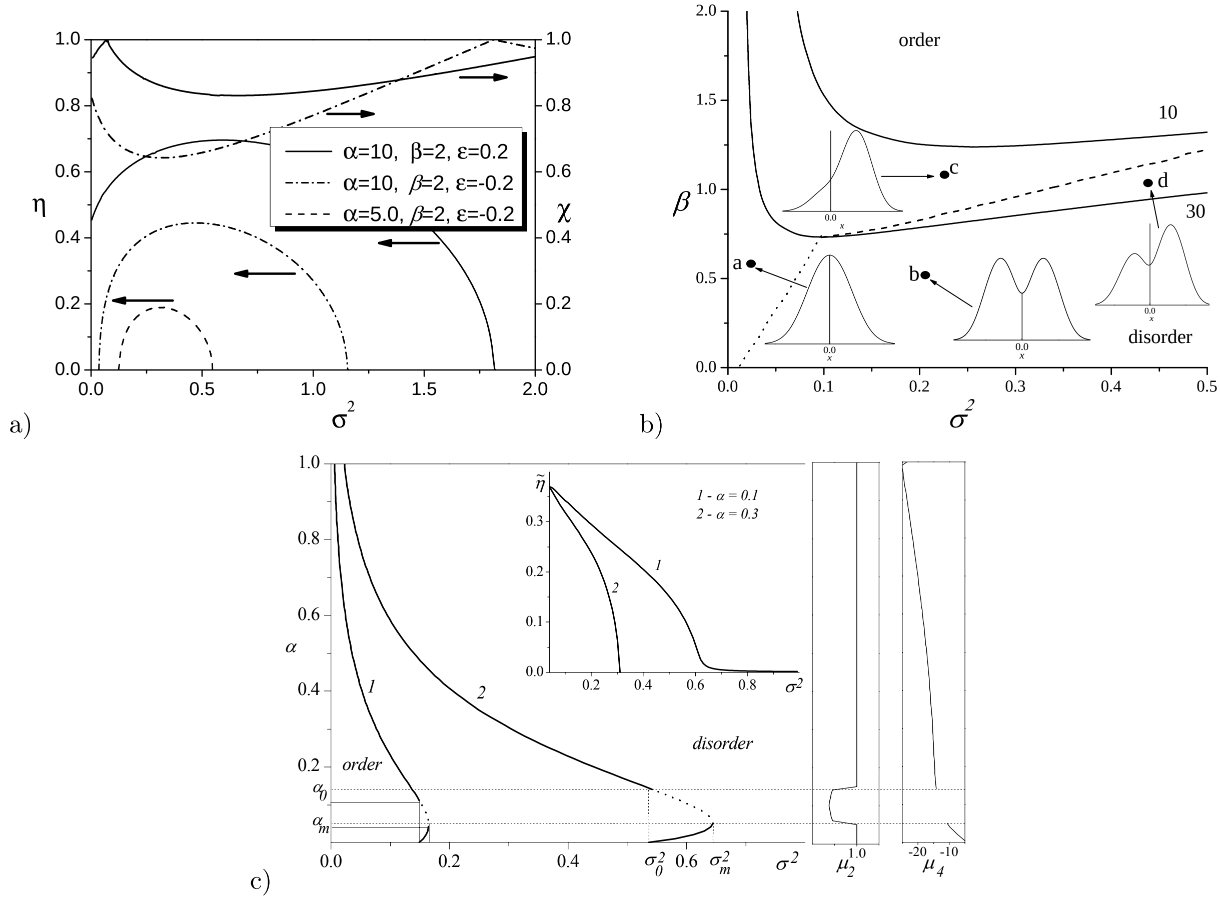

Fig. 1. Mean field diagrams for the entropy-driven phase transitions: order parameter η and susceptibility χ dependencies a) and phase diagram b) at different α with $\varepsilon = -0.2$ for the model with $M = (1 + \alpha x^2)^{-1}$; c) phase diagram for the reaction-diffusion model with $M = \mathcal{D}[x(1-x)]^\alpha$: curve 1 and 2 correspond to $\mathcal{D} = 10,\ 20$ (order parameter is shown in insertion).

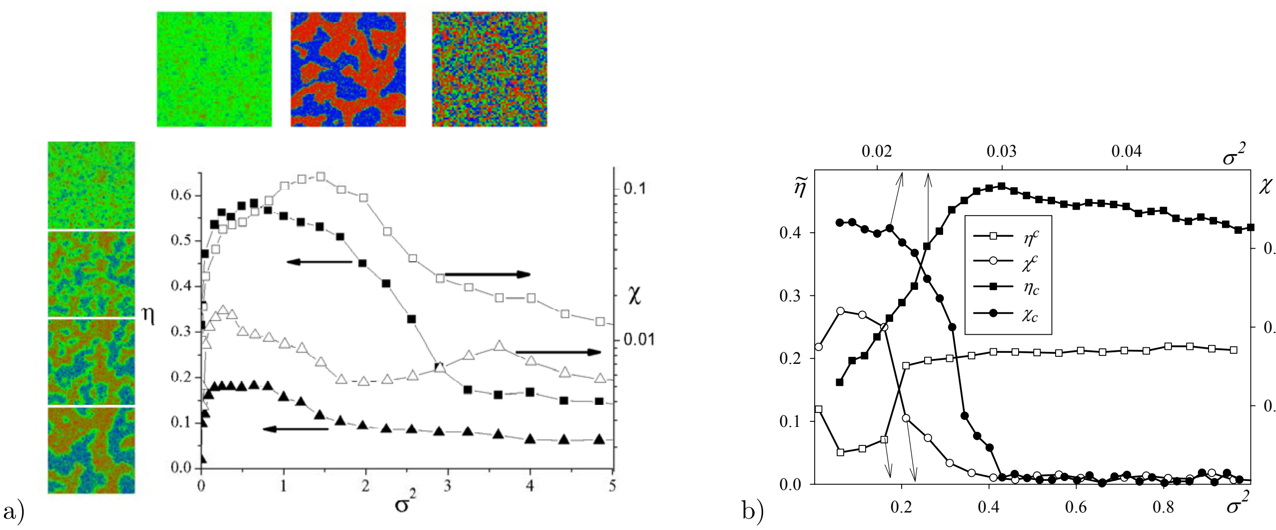

Fig. 2. Results of computer simulations on a 2-dimensional lattice $N^2 = 100 \times 100$: order parameter and susceptibility dependence versus σ^2: a) for the model with $M = (1 + \alpha x^2)^{-1}$; left — evolution of the system; top — typical patterns realized at $\sigma^2 < \sigma^2_{c1}$, $\sigma^2 \in [\sigma^2_{c1}, \sigma^2_{c2}]$ and $\sigma^2 > \sigma^2_{c2}$ from right to left; b) for the model with $M = \mathcal{D}[x(1-x)]^\alpha$.

Our results are in good correspondence with numerical simulations on 2-dimensional lattice (see Fig. 2). Here the order parameter and the generalized susceptibility are evaluated according to the formulas: $\eta = \overline{\langle \sum_i x_i \rangle}$, $\chi = \overline{\langle \sum_i x_i^2 \rangle} - \eta^2$, where $\overline{\cdots}$ and $\langle \cdots \rangle$ means average over time and ensemble, respectively.

IV. PHASE SEPARATION WITH INTERNAL MULTIPLICATIVE NOISE

Let us consider a model related to the class $\mathcal{B}$. At first, we investigate the internal multiplicative noise influence on an instability of a homogeneous phase. In the system with conserved dynamics (see Eq. (5)) the linear stability analysis should be done for the structure function $S_k(t) = \langle x_{\mathbf{k}}(t) x_{-\mathbf{k}}(t) \rangle$. Following the standard approach a linear evolution equation for the spherically averaged structure function can be derived in the form [19, 22]

$$\frac{\mathrm{d}S_k(t)}{\mathrm{d}t} = -k^2\left(\beta k^2 - \varepsilon + \alpha\sigma^2\right) S_k(t) + 2\sigma^2 k^2 - 2\alpha\sigma^2 k^2 \frac{1}{(2\pi)^d} \int \mathrm{d}\mathbf{q} S_q(t). \qquad (12)$$

It is principally important that the noise contribution denoted as $\alpha\sigma^2$ stabilizes the homogeneous state. From exponential solutions of Eq. (12) one can see that only modes with $k < k_c = \sqrt{(\varepsilon - \alpha\sigma^2)/\beta}$ are unstable and grow at early stages of evolution. With an increase in α or σ^2 the size of the unstable modes domain $k < k_c$ decreases. The modes with $k > k_c$ remain stable during the linear regime. One needs to stress that unstable modes cannot be realized at condition $\varepsilon < \alpha\sigma^2$. As it follows, the domain growth should be different for additive and multiplicative noise.

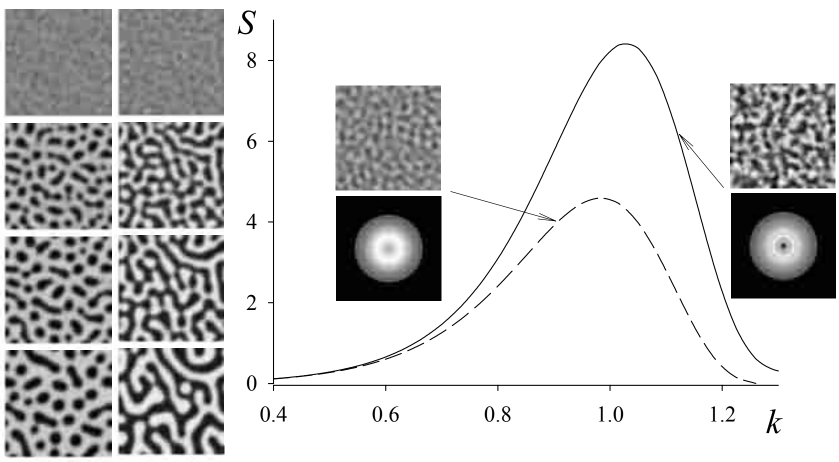

Fig. 3. Evolution of the system at initial concentration difference $\langle x \rangle = 0.2$ (left) and $\langle x \rangle = 0.0$ (right) (a). The structure function behaviour (b) at early stages at different α: solid and dashed lines correspond to $\alpha = 0$ and $\alpha = 0.8$, respectively.

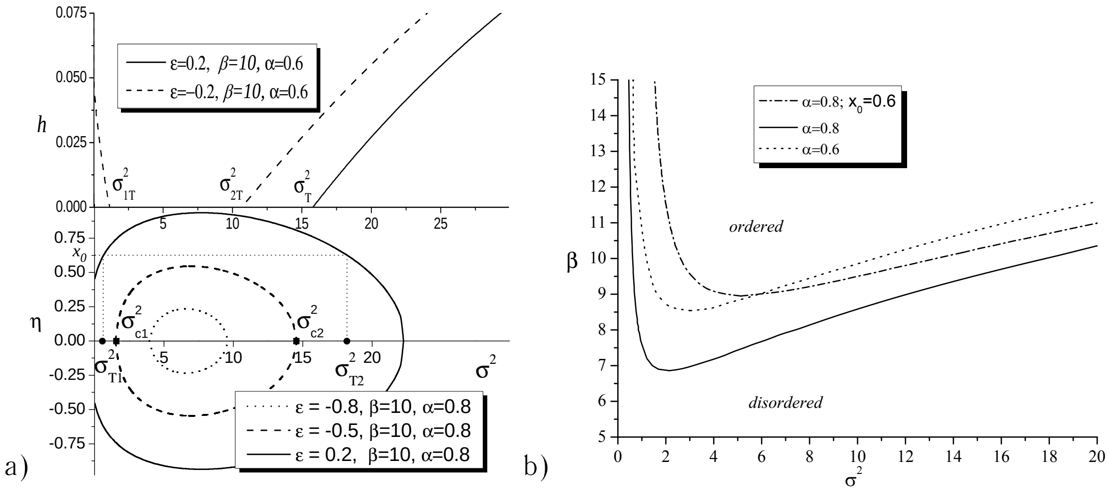

Fig. 4. Constant effective field at $x_0 = 0.2$ and the mean field value at $h = 0$ versus noise intensity (a). Mean filed phase diagram (b) illustrating critical (at $h = 0$) and transition (at $x_0 \neq 0$) points.

In Fig. 3 we present solutions of the Langevin equation (5) at different initial values for the concentration difference $\langle x\rangle$ and solutions of the evolution equation (12) at different values of the parameter α. From Fig. 3a it is seen that at $\langle x\rangle \neq 0$ the nucleation process is observed, at $\langle x\rangle = 0$ the spinodal decomposition is realized. Considering the structure function dynamics one can see that an increase in α leads to a shift of the peak position toward smaller values of k. The peak of $S(k)$ is less pronounced in the multiplicative noise case than in the case of the additive noise. It follows that, if the multiplicative noise is considered, then the dynamic is slowed. A decrease in the peak height means that an interface is more diffuse in the case of multiplicative noise (see insertions in Fig. 3b).

To investigate the steady states we can use an extension of the mean field theory developed for the systems with conserved dynamics [23]. In the framework of this theory one can use thermodynamic suppositions for the deterministic dynamics and afterwards apply it to the stochastic one.

To begin with, let us define transition and critical points [24]. Considering the deterministic case, we use the model $\partial_t x = \nabla M \nabla \delta\mathcal{F}/\delta x$, where the restriction $x_0 = \int_V \mathrm{d}\mathbf{r}\, x(\mathbf{r}, t)$ is taken into account, x_0 is fixed by the initial conditions. For such a system the transition point is $\varepsilon_T(x_0)$: at $\varepsilon < \varepsilon_T(x_0)$ the homogeneous state x_0 is stable; at $\varepsilon > \varepsilon_T(x_0)$ the system separates in bulk phases, x_1 and x_2, with $\langle x\rangle = x_0$. The transition point coincides with critical one for $x_0 = 0$ only, i. e. $\varepsilon_T(0) = \varepsilon_c$.

The corresponding steady state solutions are given as solutions of the equation $\nabla M \nabla \delta\mathcal{F}/\delta x = 0$. If no flux condition is applied, then stationary solutions can be obtained from the equation $\nabla^2 \delta\mathcal{F}/\delta x = 0$, due to the fact that mobility M does not affect the number and extreme positions of the functional $\mathcal{F}$; the mobility leads to the change in the dynamics of the the phase transition only. Hence, the bounded solution is $\delta\mathcal{F}/\delta x = h$, where h is a constant effective field of the system, in equilibrium systems h is a chemical potential. In the homogeneous case the value h does depend on the initial conditions x_0. Above the transition point the steady state is not globally homogeneous, here the system separates into two bulk phases with the values x_1 and x_2. The fraction u of the system can be defined by the lever rule $ux_1 + (1-u)x_2 = x_0$. In the case of the symmetric form of the free energy functional where two phases with $x_1 = -x_2$ are realized we get $h = 0$ [23]. Hence, if the field h becomes trivial, then the transition point can be defined.

As in the previous case we pass to the discrete representation of the system and use the mean field approximation $(\nabla x)^2 \to (\langle x\rangle - x)^2$, the mean field value $\langle x\rangle$ should be defined self-consistently. A stationary probability density function in the mean field approach takes the form [19]

$$P_s(x, \langle x\rangle, h) = N \exp\left(-\frac{1}{\sigma_0^2}\left[f(x) + \frac{\beta}{2}(\langle x\rangle - x)^2 + \frac{\sigma_0^2}{2}\ln M(x) - h\int \frac{\mathrm{d}x'}{M(x')}\right]\right). \qquad (13)$$

In order to determine the unknown quantities h and $\langle x\rangle$ we recall that the considered mean field approach is local and expresses P_s of a field at a given site of the lattice as a function of the field h and of the mean field $\langle x\rangle$ in a neighborhood of the given cell. In the homogeneous case (below the threshold) the mean field is the same everywhere and equals the initial value, i. e. $\langle x\rangle = x_0$. Hence, at the fixed mean field value, solving the self-consistency equation $\langle x\rangle = \int x P_s(x, \langle x\rangle, h)\mathrm{d}x$ we obtain the constant effective field h. Above the threshold the system is separated into two phases with equal $\langle x_1\rangle = -\langle x_2\rangle$ and h must be the same for these two phases and must be zero. Hence, above the threshold only $\langle x\rangle$ should be defined by solving the self-consistency equation with $P_s(x, \langle x\rangle, 0)$.

Let us discuss the mean field $\langle x\rangle$ behaviour. Here we solve the self-consistency equation, setting $h = 0$. The dependence of the constant effective field h and the mean field value η versus noise intensity is shown in Fig. 4a. Here one can see reentrant phase transitions at negative values of the control parameter at large spatial coupling intensity. With an increase in ε the first threshold σ_{1c}^2 is shifted toward small values whereas the second one σ_{2c}^2 becomes larger. Transition points σ_{1T}^2 and σ_{2T}^2 are related to the condition $\langle x\rangle = x_0$. With an increase in the noise intensity at $\varepsilon > 0$ the disordering phase transition is observed. The corresponding phase diagram illustrating reentrant behaviour of the mean field is shown in Fig. 4b. Here the solid and dotted lines correspond to critical noise intensity values, whereas dash-dot line relates to transition points. Here as in the previous case phase separation processes are accompanied with the modality change of the stationary distribution function at $h = 0$ [25].

Our results are in good correspondence with computer simulations on a 2-dimensional lattice. Here due to the fact that the dynamics is conserved the effective order parameter for the phase separation is the second moment of the concentration field $J = \overline{\langle \sum_i x_i^2\rangle}$, whereas the effective susceptibility is $\chi = \overline{\langle (J - \langle J\rangle)^2}$. The corresponding stationary behaviour of such the order parameter and the susceptibility is shown in Fig. 5a. We have studied the system dynamics at late stages. Here we analyzed the domain size growth law $R(t) \propto t^z$. Our calculations show that the variation of the parameter α that governs the kinetic coefficient x-dependence can lead to deviation from the well-known Lifshitz-Slyozov law with the dynamical exponent $z = 1/3$. Indeed, at $\alpha = 0$ that corresponds to additive noise with $M = \mathrm{const}$ the magnitude $z = 1/3$ is revealed. However, with an increase in α the exponent z becomes smaller and at $\alpha = 1$ one has $z = 1/4$. It means that in the case of the multiplicative noise influence with $M = M(x)$ the dynamics is slowed. The corresponding results are shown in Fig. 5b.

V. NOISE INDUCED PATTERNING

Considering the system described by the evolution equation (7) let us assume that the reaction term $R(x)$

can be defined according to a chemical kinetics. Generally, it can be represented through a potential function $V(x)$ in the standard way: $R(x) = -\partial V/\partial x$. As an important special case, we consider in this work the nonlinear force of the form $R(x) = -\prod_i (x - x^0_{(i)})$, where the set $\{x^0_{(i)}\}$ corresponds to zero values of the force and relates to stationary points of the deterministic system. In our model this force is associated with the potential $V(x) = x^4/4 + \mu x^3/3 - \nu x^2/2$, here μ and ν are constants to control the chemical kinetics. This potential has three extrema located at $x_0^{\pm} = -\mu/2 \pm \sqrt{\mu^2 + 4\nu}/2$, and at $x_0 = 0$. A spinodal is given by the equation $\nu = -\mu^2/4$. A prototype model of chemical reactions is $A \rightleftarrows B$, with transient reactions $A + 2X \rightleftarrows 3X$, $X \rightleftarrows B$ [26]. The quantity $x \in [-1, 1]$ measures concentration deviations of spices X from the constant value controlled by parameters ν and μ, related to chemical reaction rates. Another model considered here takes into account the local dynamics described by $R(x) = \varepsilon(x-1/2)-(x-1/2)^3$, the diffusion dynamics is described by the term $\nabla \cdot [\mathcal{D}[x(1-x)]^\alpha]$. As follows from naive consideration such a model can describe phase transitions with pattern formation.

Let us study the short-time instability of the disordered state. The linear stability analysis of the Langevin equation (7) allows to find an evolution equation for the structure function $S(\mathbf{k}, t) = \langle x_{\mathbf{k}}(t) x_{-\mathbf{k}}(t)\rangle$. For the first model the dynamics of the structure function at early stages is given by a linear equation

$$\frac{1}{2}\frac{\mathrm{d}S(k,t)}{\mathrm{d}t} = -(k^2(\beta k^2 - \varepsilon) - \nu - \alpha\sigma^2)S(k,t) + \sigma^2. \quad (14)$$

It follows that such internal multiplicative noise leads to an instability of the null state due to quantity $M[x]$ arising in the denominator of Eq. (7). The same situation is observed for the second model.

To make an appropriate analysis of pattern formation scenario, let us consider a zero-dimensional system, i. e. $x(\mathbf{r}, t) = x(t)$. Due to the entropy driven mechanism related to the modality change of the stationary probability let us discuss at first this problem. For zero-dimensional system the stationary probability density function has the Boltzmann form, $P_{st} \propto \exp(-U_{\text{eff}}(x)/\sigma^2)$. To find the corresponding bifurcation diagrams illustrating the modality change we solve the problem $\mathrm{d}U_{\text{eff}}/\mathrm{d}x = 0$ and find most probable values for x. After simple algebra one can find that the trivial root, $x_0 = 0$, of such an equation exists always. Another two roots $x_{\pm} = -(\mu/2) \pm \frac{1}{2}\sqrt{\mu^2 + 4\varepsilon - 4\sigma^2\alpha}$ are realized if $\sigma^2 < \sigma_c^2$, where $\sigma_c^2 = \alpha^{-1}\left(\nu + \frac{\mu^2}{4}\right)$. At $\sigma^2 = \sigma_c^2$ solutions x_- and x_+ degenerate, and at $\sigma^2 > \sigma_c^2$ only the trivial one, $x_0 = 0$, remains. As follows from naive considerations, the bimodal stationary distribution $P_{st}(x)$ becomes unimodal with an increase in the noise intensity σ^2. At $\sigma^2 = 0$ a form of the effective potential, U_{eff}, is identical topologically to the form of the initial potential, $V(x)$. With an increase in the noise intensity a minimum $U_{\text{eff}}(x_-)$ tends to zero, at $\sigma^2 = \sigma_s^2 = \varepsilon/\alpha$ the effective potential has a double degenerated point, $x_0 = x_- = 0$. Therefore, σ_s^2 values define a spinodal curve. At $\sigma_s^2 < \sigma^2 < \sigma_0^2$ the point x_0 relates to a minimum, whereas x_- defines a maximum position of the function U_{eff}. At $\sigma^2 = \sigma_0^2$ one has $U_{\text{eff}}(0) = U_{\text{eff}}(x_+)$, therefore, σ_0^2 defines a coexistence line (binodal). With a further increase in σ^2 we get $U_{\text{eff}}(0) < U_{\text{eff}}(x_+)$. The equality $U_{\text{eff}}(x_-) = U_{\text{eff}}(x_+)$ is satisfied at $\sigma^2 = \sigma_c^2$, hence the bifurcation point, σ_c^2, defines another spinodal. At $\sigma^2 > \sigma_c^2$ the effective potential has one minimum only. Therefore, at this noise induced transition we have a shift of the potential extreme, transformation of the global minimum into a local one, loss of its stability and, finally, change of number of the extreme of the function U_{eff}.

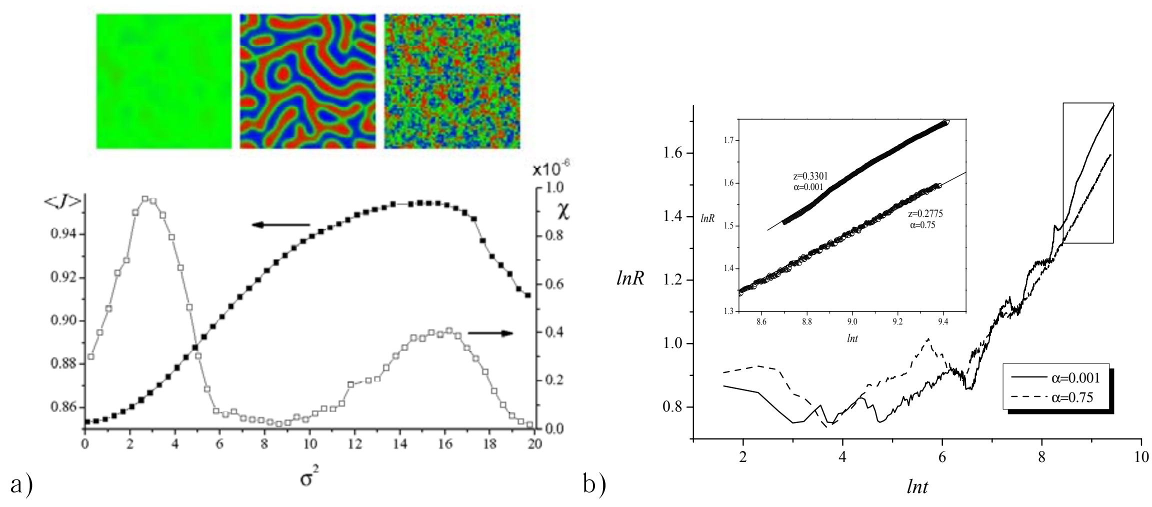

Fig. 5. Results of computer simulations of the stochastic phase separation processes: (a) effective order parameter and the generalized susceptibility versus noise intensity illustrating reentrant phase transition (typical patterns are shown at $\sigma^2 < \sigma_{c1}^2$, $\sigma^2 \in [\sigma_{c1}^2, \sigma_{c2}^2]$ and $\sigma^2 > \sigma_{c1}^2$); (b) domain size growth law at late stages, $R(t) \propto t^z$

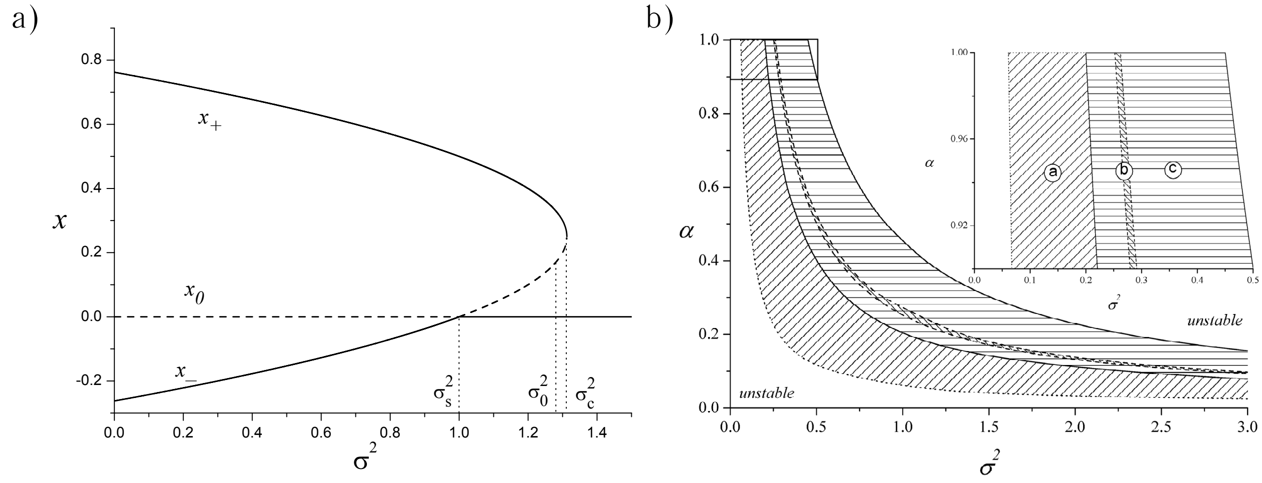

Fig. 6. Bifurcation diagram (a) illustrating modality change of the stationary distribution $P_s(x)$. Phase diagram (b) of reentrant behaviour of self-organization process at $\nu = 0.2$, $\mu = -0.5$, $\beta = 1$, $\varepsilon = 1.0$. Domains of stable solutions are dashed: domain (a) corresponds to a stable stationary structures $x_s(\mathbf{r})$ around the point $x = x_-$); domain (c) is addressed to stable structures around $x = 0$; double filed domain (b) relates to a stable structures around $x = x_-$ and $x = x_+$.

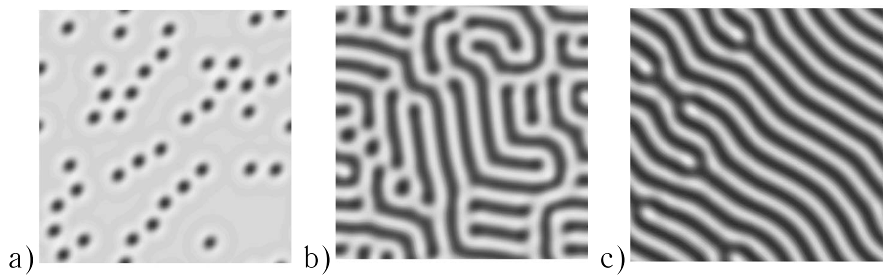

Fig. 7. Two-dimensional solutions of the problem (15): a) $\sigma^2 = 0.2$, $x(0) \approx x_-$; b) $\sigma^2 = \sigma_0^2$, $x(0) = x_+$; c) $\sigma^2 = 2$, $x(0) = 10^{-5}$. Other parameters are: $\alpha = 0.2$, $\nu = 0.2$, $\mu = -0.5$, $\beta = 1.0$, $\varepsilon = 1.0$; initial conditions for spatial derivatives are zero.

Considering a spatially extended system to study patterns formation one needs to solve the variational problem $\delta\mathcal{U}_{\rm eff}/\delta x = 0$. The corresponding equation takes the form

$$M\left[\beta\nabla^4 x - \frac{\partial^2 f}{\partial x^2}\nabla^2 x - \frac{\partial^3 f}{\partial x^3}(\nabla x)^2\right] - \frac{\partial M}{\partial x}\frac{\partial^2 f}{\partial x^2}(\nabla x)^2 = R + \frac{\sigma^2}{2M}\frac{\partial M}{\partial x}. \qquad (15)$$

To proceed, we have performed a stability analysis of solutions of Eq. (15) in the **r**-space. It follows that with the noise intensity increase unstable homogeneous solutions become stable at fixed noise intensity interval $\sigma^2 \in [\sigma_s^2, \sigma_{T0}^2]$, where $\sigma_{T0}^2 = \sigma_s^2 + \varepsilon^2/4\alpha$. Therefore, we get a reentrant picture of self-organization. The corresponding phase diagram illustrating pattern formation is shown in Fig. 6.

The solutions of the variational problem at different σ^2 are shown in Fig. 7. It is seen that at small noise intensity (Fig. 7a) the nuclei are formed, at intermediate values ($\sigma = \sigma_0^2$) a picture type of spinodal decomposition is realized (see Fig. 7b), at large σ^2 (see Fig. 7c) linear defects (dislocations) are observed.

Considering the second model with the Fickian diffusion one can see that here in the ordered state described by the mean field order parameter $\eta \neq 0$ and bimodal stationary distribution the corresponding stationary patterns are described by the harmonic equation following from Eq. (15). Moreover, the corresponding phase transitions can be of critical ($\mu_2 = 1$ and $\mu_4 < 0$) and non-critical ($\mu_2 \neq 1$) character (see Fig. 1c). The character of the order parameter evolution versus noise intensity is shown with the help of insertion in Fig. 1c for the different values of α. Analytical results are verified by computer simulations shown in Fig. 2b.

VI. CONCLUSIONS

We have considered two possible generalizations of entropy-driven phase transitions in physical systems with a relaxation flow and a field-dependent kinetic coefficient. It is shown that the internal multiplicative noise induces the reentrant behaviour of the order parameter in the case of a monostable, symmetrical local potential for the systems with nonconserved dynamics. The phase separation scenario with an entropy driven mechanism of the system with internal multiplicative noise is examined. It was shown that the field-dependent mobility leads to delays in dynamics at early stages and, therefore, leads to delays in domain growth law at late stages. We have found that the system can undergo a reentrant phase transitions when the the mean field becomes nontrivial inside the fixed domain of the noise intensity. A simple model of stochastic reaction-diffusion systems which can qualitatively describe stationary noise patterns is studied. It was found that the stationary distribution can be obtained exactly. Comparing noise induced transitions picture and pattern formation scenario it was shown that the system follows the entropy driven mechanism by analogy with entropy driven phase transitions theory.

Our results can be applied to the investigations of polymer mixtures where relaxational flows are driven by field-dependent coefficients, phase separation in binary alloys, and microstructure phase transitions in systems subjected to irradiation influence.

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[1] W. Horsthemke, R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).

[2] J. Garcia-Ojalvo, J. M. Sancho, *Noise in Spatially Extended System* (Springer-Verlag, New York, 1999).

[3] K. Wiesenfeld, F. Moss, Nature **373**, 33 (1995).

[4] C. Van der Broeck, Phys. Rev. Lett **73**, 3395 (1994).

[5] C. Van der Broeck *et al.* Phys. Rev. E **55**, 4084 (1997).

[6] D. O. Kharchenko, I. A. Knyaz', Eur. Phys. J. B **32**, 375 (2003).

[7] A. I. Olemskoi, D. O. Kharchenko, I. A. Knyaz', Phys. Rev. E **71**, 041101 (2005).

[8] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).

[9] C. W. Gradiner, P. Zoller, *Quantum noise* (Springer-Verlag, Berlin, Heidelberg, New York, 2000).

[10] V. J. Sugakov, *Osnovy synergetyky* (Oberegy, Kyiv, 2001).

[11] A. S. Mikhailov, A. Yu. Loskutov, *Foundations in synergretics II* (Springer Verlag, Berlin, Heidelberg, New York, 1996).

[12] B. Linder, J. Garcia-Ojalvo, A. Neiman, L. Shimansky-Geier, Phys. Rep. **392**, 321 (2004).

[13] D. O. Kharchenko, S. V. Kohan, Eur. Phys. J. B **29**, 97 (2002).

[14] R. Kawai, X. Saller, L. Shimasky-Geier, C. Van der Broeck, Phys. Rev. E **69**, 051104 (2004).

[15] M. Ibanes, J. Garcia-Ojalvo, R. Toral, J. M. Sancho, Phys. Rev. Lett. **87**, 020601 (2001).

[16] O. Carrillo, M. Ibanes, J. Garcia-Ojalvo, J. Casademunt, J. M. Sancho, Phys. Rev. E **67**, 046110 (2003).

[17] F. J. Cao, K. Wood, K. Lindenberg, arXiv:cond-mat/0702524.

[18] P. C. Hohenberg, B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1997).

[19] D. O. Kharchenko, A. V. Dvornichenko, Physica A **387**, 5342 (2008).

[20] S. E. Mangioni, H. S. Wio, Phys. Rev. E **71**, 056203 (2005).

[21] V. O. Kharchenko, Physica A **388**, 268 (2009).

[22] J. Garcia-Ojalvo, A. M. Lacasta, J. M. Sancho, R. Toral, Europhys. Lett. **42**, 125 (1998).

[23] M. Ibanes *et al.* Phys. Rev. E **60**, 3597 (1999).

[24] J. D. Gunton, M. San Miguel, P. S. Sahni, *Phase Transtions and Critical Phenomena, Vol. 8*, edited by C. Domb, J. L. Lebowitz (Academic Press, New York, 1983).

[25] D. O. Kharchenko, A. V. Dvornichenko, I. O. Lysenko, Ukr. Phys. J. **53**, 917 (2008).

[26] F. Schlögl, C. Escher, R. S. Berry, Phys. Rev. A **27**, 2698 (1983).

ЕНТРОПІЙНИЙ МЕХАНІЗМ УПОРЯДКУВАННЯ, ПРОЦЕСИ ФАЗОВОГО РОЗШАРУВАННЯ ТА ФОРМУВАННЯ ПРОСТОРОВИХ СТРУКТУР У СТОХАСТИЧНИХ СИСТЕМАХ

Д. О. Харченко[1], А. В. Дворниченко[2], В. О. Харченко[1]
[1]*Інститут прикладної фізики НАН України, вул. Петропавлівська, 58, 40030, Суми, Україна*
[2]*Сумський державний університет, вул. Римського-Корсакова, 2, 40007, Суми, Україна*

Проведені дослідження стосуються узагальнення моделі розподілених стохастичних систем із залежним від поля кінетичним коефіцієнтом і флюктуаційним внеском, що задовольняє флюктуаційно-дисипаційну теорему. Фазові переходи з ентропійно керованим механізмом досліджено в системах із динамікою, що зберігається та не зберігається. Знайдено, що в стохастичних системах із релаксаційним потоком і симетричним локальним потенціалом відбуваються реверсивні фазові переходи. Вивчено ентропійно керований механізм, що приводить до формування стаціонарних просторових структур у стохастичних системах реакційно-дифузійного типу. Показано, що мультиплікативний шум, що задовольняє флуктуаційно-дисипаційну теорему, може індукувати й підтримувати стаціонарні структури. Результати теорії середнього поля підтверджені комп'ютерними симуляціями.