ISSN 1024-588X. Вісник Львівського університету. Серія фізична. 2023. Випуск 60. С. 18-25
Visnyk of the Lviv University. Series Physics. 2023. Issue 60. P. 18-25

УДК 530.145.61 https://doi.org/10.30970/vph.60.2023.18
PACS 03.65.-w, 03.67.-a

Probability of quantum state determination among N possible ones

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We find a probability of determining an unknown quantum state if some information about it is known. Namely, we know that the state is one of N quantum states. These states are represented as eigenstates of operator of projection of a spin on N directions, corresponding to positive eigenvalues. The probability of determining the unknown state with the smallest number of measurements and for some number of measurements n is obtained. Dependence of these probabilities on the angles between the vectors defining the N directions is studied.

Key words: quantum state determining, quantum measurement, quantum computing

1 Introduction

The idea of quantum computations and quantum computers was born in the 80's [1,2]. For the last decades quantum programming and physical implementations of quantum computing have achieved significant progress and it actively develop today (see, for example [3–7], and references therein). This can help to solve complex physical problems, economical tasks, problems in logistics and many other fields (see, for instance, [8–13]). Quantum computers have a potential to exceed the abilities of classical ones. Quantum supremacy was first demonstrated by a Google experiment conducted with the 53-qubit quantum processor in 2019. It was shown that the quantum computer needs only 200 seconds for solving the same task that a classical one can solve approximately in 10000 years [3].

Determining of an unknown quantum state with a minimal number of measurements can be used to optimize quantum calculations. The problem of determining of a quantum state in the case when it is one of two states was examined in [14]. In this paper we study a problem of determining a quantum state among N possible ones. We find the probability to solve this problem with n measurements and study it dependence on the parameters of N possible states. Also the probability to determine the state with the smallest number of measurements is calculated.

The paper is organized as follows. In the Section 2 the probability of determining of a quantum state among two possible ones is considered. The problem of finding of the state of the spin among N possible ones with n measurements is solved in Section 3. Conclusions are presented in Section 4.

2 Probability of determining of a quantum state among two possible states

Let us consider two quantum states $|\psi^+_{\mathbf{a_1}}\rangle$, $|\psi^+_{\mathbf{a_2}}\rangle$. These states can be represented as eigenstates of the spin-1/2 projection operators on directions given by vectors $\mathbf{a_1}$, $\mathbf{a_2}$ which correspond to positive eigenvalues $\hbar/2$. The operators of projection of spin on $\mathbf{a_1}$, $\mathbf{a_2}$ directions read $(\mathbf{s}\cdot\mathbf{a_1})$, $(\mathbf{s}\cdot\mathbf{a_2})$, here $s_i = \hbar\sigma_i/2$, σ_i are the Pauli matrixes.

We choose the coordinate system with z-axis directed along vector $\mathbf{a_1}$. Therefore we can write

$$|\psi^+_{\mathbf{a_2}}\rangle = \cos\frac{\theta}{2}|\psi^+_{\mathbf{a_1}}\rangle + \sin\frac{\theta}{2}e^{i\phi}|\psi^-_{\mathbf{a_1}}\rangle, \tag{1}$$

where θ, ϕ are azimuthal and polar angles representing vector $\mathbf{a_2}$, and $|\psi^-_{\mathbf{a_1}}\rangle$ is the eigenstate of $(\mathbf{s}\cdot\mathbf{a_1})$ corresponding to opposite eigenvalue $-\hbar/2$.

The task is to determine the state. It is interesting to note that it can be solved with only one measurement with probability given by [14]

$$p_1 = \frac{1}{2}\sin^2\frac{\theta}{2}. \tag{2}$$

Such result is achieved if the spin is in state $|\psi^+_{\mathbf{a_1}}\rangle$, we measure the projection of spin on direction $\mathbf{a_2}$ and obtain negative result. The quantum state can be also determined with one measurement if the spin is in state $|\psi^+_{\mathbf{a_2}}\rangle$, we measure the projection of spin on direction $\mathbf{a_1}$ and obtain negative result. The probability to solve the problem with n measurements reads [14]

$$p_n = 1 - \frac{1}{2^n}\left(1 + \cos^2\frac{\theta}{2}\right)^n. \tag{3}$$

In the next section on the basis of this result we examine the probability to determine quantum state in the case when it is known that it is one of N possible states.

3 Quantum state determination among N possible ones

Let us consider N quantum states $|\psi^+_{\mathbf{a}_i}\rangle$ that are eigenstates of the operators of projection of a spin on $\mathbf{a}_i$ directions, $i = (1..N)$ corresponding to positive eigenvalues $\hbar/2$.

Let the spin is in state $|\psi^+_{\mathbf{a}_k}\rangle$ (here k takes a certain fixed value, $k = (1..N)$). We do not know this information, we just know that the state of the spin is one of N states $|\psi^+_{\mathbf{a}_i}\rangle$, $i = (1...N)$. Let us find the probability to determine the state of the spin performing n measurements.

Suppose in the first measurement we guess the direction which corresponds to $\mathbf{a}_k$. The probability of this is equal to $1/N$. As a result of measurement of the projection of the spin in state $|\psi^+_{\mathbf{a}_k}\rangle$ on $\mathbf{a}_k$ direction we obtain positive result $\hbar/2$. In this case, we cannot determine the state of the spin. This is because the same result one can obtain choosing other directions $\mathbf{a}_i$, $i = (1...N)$ different from $\mathbf{a}_k$.

Let us consider another case when we do not guess the direction which corresponds to $\mathbf{a}_k$. For instance, we choose $\mathbf{a}_l$, $l \neq k$. State $|\psi^+_{\mathbf{a}_k}\rangle$ can be rewritten in the form

$$|\psi^+_{\mathbf{a}_k}\rangle = \cos\frac{\theta_l}{2}|\psi^+_{\mathbf{a}_l}\rangle + \sin\frac{\theta_l}{2}e^{i\phi_l}|\psi^-_{\mathbf{a}_l}\rangle, \tag{4}$$

where θ_l, ϕ_l are azimuthal and polar angles that define $\mathbf{a}_k$ vector in a coordinate system with z axis coinciding with the direction of vector $\mathbf{a}_l$. As a result of measurement of the projection of a spin in state $|\psi^+_{\mathbf{a}_k}\rangle$ on $\mathbf{a}_l$ ($l \neq k$) direction we obtain positive result $\hbar/2$ with probability

$$P_+ = |\langle\psi^+_{\mathbf{a}_k}|\psi^+_{\mathbf{a}_l}\rangle|^2 = \cos^2\frac{\theta_l}{2}. \tag{5}$$

In this case, we cannot say anything about the state of the spin.

Probability to obtain $-\hbar/2$ in the result of measurement reads

$$P_- = |\langle\psi^+_{\mathbf{a}_k}|\psi^-_{\mathbf{a}_l}\rangle|^2 = \sin^2\frac{\theta_l}{2}. \tag{6}$$

In this case we can say that the spin is in one of the states $|\psi^+_{\mathbf{a}_i}\rangle$, $i \neq k$. So, the problem of determining of the state of the spin among N possible ones is reduced to the problem of determining of the state of the spin among $N-1$ possible ones. The probability to obtain such a reduction in one measurement reads

$$\frac{1}{N}\sin^2\frac{\theta_l}{2}, \tag{7}$$

where θ_l is angle between $\mathbf{a}_l$ direction and $\mathbf{a}_k$ direction, which corresponds to the state of the spin $|\psi^+_{\mathbf{a}_k}\rangle$. Multiplier $1/N$ is the probability of choosing $\mathbf{a}_l$ direction among N possible ones. In the second step we choose another direction $\mathbf{a}_m$ among $N-1$ possible ones ($m = (1..N)$, $m \neq l$). The probability to obtain negative result of the measurement $-\hbar/2$ of the projection of the spin on $\mathbf{a}_m$ direction reads

$$P_- = |\langle\psi^+_{\mathbf{a}_k}|\psi^-_{\mathbf{a}_m}\rangle|^2 = \sin^2\frac{\theta_m}{2}, \tag{8}$$

where θ_m is the angle between vectors $\mathbf{a}_m$ and $\mathbf{a}_k$. In the case of such a result of measurement we can say that the state of the spin does not correspond to $|\psi^+_{\mathbf{a}_l}\rangle$ and $|\psi^+_{\mathbf{a}_m}\rangle$. So we obtain additional reduction of the problem to the problem of determining of the state of the spin among $N-2$ possible ones $|\psi^+_{\mathbf{a}_i}\rangle$, $i=(1..N)$, $i \neq= l$, $i \neq m$. The probability of such reduction reads

$$\frac{1}{N-1}\sin^2\frac{\theta_m}{2}, \tag{9}$$

where $1/(N-1)$ is the probability of choosing $\mathbf{a}_m$ direction among $N-1$ possible ones.

On the basis of these results we have that the probability to reduce the problem of determining of the state of the spin among N possible ones to the problem of determining of the state of the spin among $N-2$ possible ones ($|\psi^+_{\mathbf{a}_i}\rangle$, $i=(1..N)$, $i \neq l$, $i \neq m$) on the basis of results of two measurements reads

$$\frac{1}{N(N-1)}\sin^2\frac{\theta_m}{2}\sin^2\frac{\theta_l}{2}. \tag{10}$$

So, we can write the probability of reduction of the problem of determining the unknown state among N possible states $|\psi^+_{\mathbf{a}_i}\rangle$ $i=(1..N)$, to the problem of determining an unknown state among 2 states $|\psi^+_{\mathbf{a}_s}\rangle$, $|\psi^+_{\mathbf{a}_k}\rangle$ in $N-2$ measurements

$$\frac{2}{N!}\prod_{\substack{i=1\\ i\neq s, i\neq k}}^{N}\sin^2\frac{\theta_i}{2}, \tag{11}$$

where θ_i is the angle between $\mathbf{a}_i$ ($i \neq s$, $i \neq k$) direction and $\mathbf{a}_k$ direction, which corresponds to the state of the spin $|\psi^+_{\mathbf{a}_k}\rangle$.

Now we can use the result for the probability to determine the state of a spin in n measurements if it is known that this state is one of two possible ones $|\psi^+_{\mathbf{a}_s}\rangle$, $|\psi^+_{\mathbf{a}_k}\rangle$

$$p_n^{(2)} = 1 - \frac{1}{2^n}\left(1+\cos^2\frac{\theta_s}{2}\right)^n, \tag{12}$$

where θ_s is the angle between $\mathbf{a}_s$ and $\mathbf{a}_k$ directions [14].

On the basis of (11), (12) we find that the probability of determining in which of the N states is the spin for n ($n \geq N-1$) measurements is defined as

$$p_n^{(N)} = \frac{2}{N!}\prod_{i=1}^{N-2}\sin^2\frac{\theta_i}{2}\left(1-\frac{1}{2^{n-N+2}}\left(1+\cos^2\frac{\theta_{N-1}}{2}\right)^{n-N+2}\right). \tag{13}$$

Here for convenience we consider the following notations, θ_i is an angle between vectors $\mathbf{a}_i$ and $\mathbf{a}_N$. Direction $\mathbf{a}_N$ corresponds to the state of the spin $|\psi^+_{\mathbf{a}_N}\rangle$.

The smallest number of measurements which is needed to determine the state of the spin is $N-1$. The probability of this fast determining of the quantum state reads

$$p_n^{(N)} = \frac{2}{N!}\prod_{i=1}^{N-2}\sin^2\frac{\theta_i}{2}\left(1-\frac{1}{2}\left(1+\cos^2\frac{\theta_{N-1}}{2}\right)\right) = \\ = \frac{1}{N!}\prod_{i=1}^{N-1}\sin^2\frac{\theta_i}{2}. \tag{14}$$

This case is realized when we get a negative result in each of the $N-1$ measurements.

Conclusions

We have studied the probability to determine a state of spin 1/2 in the case when we know that the state is one of N quantum states. We have represented the states as eigenstates of the spin projection operators on $\mathbf{a}_i$ directions, $i = (1..N)$ corresponding to positive eigenvalues. We have found the probability of determining the unknown state with the smallest number of measurements and with some number of measurements n. The dependence of these probabilities on the angles between the vectors defining the N directions has been obtained (13), (14). The smallest number of measurements to determine in which of N given states is spin is $N-1$. It is realized when in each of $N-1$ measurements we obtain a negative result for the projection of the spin on a direction. The obtained results and conclusions can be used in further studies in quantum information and can help to optimize calculations on quantum computers.

Acknowledgment

The authors thank Prof. V. M. Tkachuk for useful comments. This work was supported by Project No. 0122U001558 from the Ministry of Education and Science of Ukraine.

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Імовірність визначення квантового стану серед N можливих

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Розглядається задача про визначення квантового стану спіна $1/2$, якщо відома інформація про цей стан. А саме, якщо ми знаємо, що стан є одним із N заданих квантових станів. Відомо, що довільний квантовий стан спіна $1/2$ ми можемо представити як власний стан оператора проекції спіна на певний напрям, який відповідає додатному чи від'ємному власному значенню цього оператора. Ми розглядаємо N заданих квантових станів, як N власних станів оператора проекції спіна на N напрямів з додатніми власними значеннями. Досліджується можливість визначити яким є стан спіна з N можливих квнатових станів за допомогою вимірювання його проекції на певні напрями. Для ефективності розв'язання поставленої задачі під час кожного виміру знаходиться проекція спіна на напрям, який відповідає N заданим квантовим станам. Обчислюється ймовірність визначення квантового стану за n вимірів. Найменша кількість вимірів, за допомогою яких можна визначити у якому з N станів знаходиться спін, є $N-1$. Вона реалізується у тому випадку, коли у кожному з $N-1$ вимірів ми отримали від'ємні результати для проєкції спіна. Лише у цьому випадку після кожного квантового виміру, отримавши від'ємний результат для проекції спіна на певний напрям, ми можемо відкинути цей напрям, як той, що не відповідає стану спіна. Знайдено ймовірність найшвидшого визначення квантового стану та її залежність від початкової інформації про нього, а саме від кутів між векторами, які відповідають N квантовим станам. У випадку, коли результат квантового виміру проекції спіна на певний напрям є додатнім ми не отримаємо жодної інформації про його стан, тому для розв'язання задачі потрібно проводити більше ніж $N-1$ вимірювань. Знайдено ймовірність визначити стан за довільну кількість вимірів n $(n \geq N-1)$ та її залежність від кутів між векторами, які відповідають N квантовим станам. Отримані результати важливі для подальших досліджень в області квантової інформації та можуть бути використані для оптимізації квантових обчислень.

Ключові слова: визначення квантового стану, квантовий вимір, квантові обчислення

Статтю отримано: 11.01.2023
Прийнято до друку: 01.02.2023