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"Faked" Gentile distribution mimicking weakly non-ideal fermions

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This study presents an analysis of an approach to emulate weakly non-ideal fermionic systems by modifying the Gentile statistics. In the original Gentile formulation, the maximum level occupation is restricted to integers, which seems a natural requirement. In this work, we depart from this restriction and introduce a small parameter u thus formally setting the maximum occupation to $1 + u$. By employing this modification, we find connections between the deviation of the system's spectrum from the ideal case and the parameter u. This relationship is derived in the linear approximation over small quantities. We show that an ideal system governed by such a modified Gentile statistics can serve as an approximate model for interacting fermions.

Key words: interacting fermions, Gentile statistics, fractional statistics, effective model

1. Introduction

Back in 1940, Giovanni Gentile, Jr. was first to derive the expression for the occupation numbers generalizing conventional Bose–Einstein and Fermi–Dirac distributions. Notably, Gentile suggested that the maximum level occupation could have a finite value other than unity [1]. This statistics is an example of the so called intermediate or fractional (and even sometimes exotic) types of statistics.

There are certain issues with the Gentile statistics preventing it from being a proper quantum statistics from the field-theoretical standpoint [2, 3]. Still, this statistics can be used as a model in a number of problems. In particular, there is an equivalence between the q-deformed commutator algebra with a complex q and the Gentile statistics [4]. The problem of restricted partitions in number theory can be treated with this statistics [5]. Finite bosonic systems can be approximately described using the Gentile statistics [6].

In recent years, the Gentile statistics has garnered increased attention and has been extensively analyzed in various problem domains. Selvi and Uncu [7] provided a concise overview of its applications, shedding light on specific physical systems where the Gentile statistics finds relevance under particular conditions. Noteworthy examples include the behavior of dilute two-dimensional electron gases,and the Bose–Einstein condensation-like phenomena of quadrupolar systems in certain liquid crystals.

Connections between the Gentile statistics and deformed boson and fermion algebras were analyzed in [8]. Another recent study [9] investigated the transformational links between anyons (particles with fractional statistics) and the Gentile statistics, unraveling new perspectives on their interplay. Shen *et al.* [10] demonstrated that Gentile quasi-particles can effectively describe elementary excitations in spin lattices, expanding the scope of the Gentile statistics to yet another realm of physics.

The present paper aims at finding approximate equivalence between a system of weakly interacting fermions and a modification of the Gentile statistics. The paper has a simple structure. Section 2 introduces the expression for the occupation numbers in the Gentile statistics and discusses its generalization to describe weakly interacting fermions. A model system allowing for analytical results of the majority of expressions involved is considered in Section 3. Brief conclusion is given in Section 4.

2. Gentile distribution

In the Gentile statistics, the occupation numbers are given by

$$n_{\rm G}(\varepsilon,\mu,T) = \frac{1}{e^{(\varepsilon-\mu)/T}-1} - \frac{M+1}{e^{(M+1)(\varepsilon-\mu)/T}-1}, \tag{1}$$

where $1 \le M < \infty$ stands the maximum possible level occupation. It is straightforward to show that $M = 1$ corresponds to the Fermi distribution with

$$n_{\rm F}(\varepsilon,\mu,T) = \frac{1}{e^{(\varepsilon-\mu)/T}+1}. \tag{2}$$

Obviously, in its original formulation, the Gentile statistics accepts only positive integers for the M parameter.

In what follows we depart from such a standard formulation and let the M parameter deviate from unity by a small correction only,

$$M = 1 + u, \tag{3}$$

where $u \ll 1$. Due to essential departure from the original Gentile's proposal assuming integer M, it seems that referring to this distribution as simply “modified” or “deformed” would not be sufficiently justified. Hence, this distribution has been dubbed “faked”. Simultaneously, one should take into account that the chemical potential will slightly differ from that of the ideal Fermi gas,

$$\mu = \mu_{\rm id} + \Delta\mu_{\rm G}. \tag{4}$$

We further write the number of particles using the expressions for the occupation numbers in both the Fermi and the Gentile statistics as follows:

$$N = \int\limits_0^\infty d\varepsilon\, g(\varepsilon) n_{\rm F,G}(\varepsilon,\mu,T), \tag{5}$$

where for the sake of generality the density of states is given by the power-law function

$$g(\varepsilon) = NA\varepsilon^{s-1}. \tag{6}$$

At the same time, the chemical potential of the ideal Fermi gas $\mu_{\rm id}$

$$N = \int_0^\infty d\varepsilon\, g(\varepsilon) n_{\rm F}(\varepsilon, \mu_{\rm id}, T). \tag{7}$$

To be more precise, we consider a weak non-ideality of the Fermi gas, so that the actual spectrum is $\varepsilon + \Delta\varepsilon$, where $\Delta\varepsilon$ is a small correction. Our attempt is to mimic this non-ideality by considering a system with spectrum ε obeying the Gentile statistics with $M = 1 + u$.

So, for the Fermi system we can write:

$$N = \int_0^\infty d\varepsilon\, g(\varepsilon) n_{\rm F}(\varepsilon + \Delta\varepsilon, \mu_{\rm id} + \Delta\mu_{\rm F}, T) = \int_0^\infty d\varepsilon\, g(\varepsilon) n_{\rm F}(\varepsilon, \mu_{\rm id}, T) + \Delta N_{\rm F}, \tag{8}$$

where $\Delta N_{\rm F}$ is a small correction. In the same fashion, for the Gentile system:

$$N = \int_0^\infty d\varepsilon\, g(\varepsilon) n_{\rm G}(\varepsilon, \mu_{\rm id} + \Delta\mu_{\rm G}, T) = \int_0^\infty d\varepsilon\, g(\varepsilon) n_{\rm F}(\varepsilon, \mu_{\rm id}, T) + \Delta N_{\rm G}, \tag{9}$$

with $\Delta N_{\rm G}$ being another small correction.

Taking into account Eq. (7), we immediately see that both small corrections should be zero: $\Delta N_{\rm F} = \Delta N_{\rm G} = 0$. Performing a simple change of variables $x = \varepsilon/T$ in integrals in Eqs. (8)–(9), we obtain in the linear approximation over the small quantities:

$$\Delta\mu_{\rm G} \int_0^\infty dx\, g(x) \frac{e^{x-\nu}}{\left(e^{x-\nu} - 1\right)^2} = uT \int_0^\infty dx\, g(x) \frac{1}{e^{2(x-\nu)} - 1}, \tag{10}$$

where the notation $\nu = \mu_{\rm id}/T$ is introduced for convenience. For fermions, we have

$$\Delta\mu_{\rm F} \int_0^\infty dx\, g(x) \frac{e^{x-\nu}}{\left(e^{x-\nu} + 1\right)^2} = \int_0^\infty dx\, g(x) \eta(x, T) \frac{e^{x-\nu}}{\left(e^{x-\nu} + 1\right)^2}, \tag{11}$$

where $\eta(x, T)$ corresponds to the spectrum correction $\Delta\varepsilon$ after the change of variables. The density of states becomes

$$g(x) = T^{s-1} A x^{s-1}, \tag{12}$$

so that the respective powers of temperature cancel out in both sides of the above equations.

With η given from the problem formulation, three unknown parameters remain in Eqs. (10), (11), namely, $\Delta\mu_{\rm G}$, $\Delta\mu_{\rm F}$, and u. A missing third equation might be obtained from the condition that thermodynamic functions of both the weakly non-ideal Fermi system and the model Gentile system coincide in the linear approximation over the corrections. Having written the expressions for energy,

$$E_{\rm F} = \int\limits_0^\infty d\varepsilon\, \varepsilon g(\varepsilon) n_{\rm F}(\varepsilon + \Delta\varepsilon, \mu + \Delta\mu_{\rm F}, T), \tag{13}$$

$$E_{\rm G} = \int\limits_0^\infty d\varepsilon\, \varepsilon g(\varepsilon) n_{\rm G}(\varepsilon, \mu + \Delta\mu_{\rm G}, T), \tag{14}$$

from $E_{\rm F} = E_{\rm G}$ we get after simple transformations:

$$\Delta\mu_{\rm G} \int\limits_0^\infty dx\, x g(x) \frac{e^{x-\nu}}{(e^{x-\nu}-1)^2} - uT \int\limits_0^\infty dx\, x g(x) \frac{1}{e^{2(x-\nu)}-1}$$

$$= \Delta\mu_{\rm F} \int\limits_0^\infty dx\, x g(x) \frac{e^{x-\nu}}{(e^{x-\nu}+1)^2} - \int\limits_0^\infty dx\, x g(x) \eta(x,T) \frac{e^{x-\nu}}{(e^{x-\nu}+1)^2}, \tag{15}$$

We have thus obtained in a general form three equations (10), (11), and (15) for three parameters $\Delta\mu_{\rm G}$, $\Delta\mu_{\rm F}$, and u.

3. Model example

In order to obtain some analytical results, we consider a simplified model with a constant density of states $g(\varepsilon) = NA$ corresponding to $s = 1$. The two relevant physical systems are 1D harmonic oscillators (with $A = 1/(\hbar\omega)$, where ω is the oscillator frequency) and free 2D particles in a box (with $A = m/(\pi^2\hbar^2\rho)$, where m is the particle mass and ρ stands for the 2D concentration). The value of $1/A$ can be used as a unit of temperature for convenience.

Let us introduce the following notations for the integrals in (10), (11), and (15) within this model:

$$g_{11} = \int\limits_0^\infty dx\, \frac{e^{x-\nu}}{(e^{x-\nu}-1)^2} = \frac{1}{e^{-\nu}-1}, \tag{16а}$$

$$g_{12} = \int\limits_0^\infty dx\, \frac{1}{e^{2(x-\nu)}-1} = -\frac{1}{2}\ln\left(1-e^{2\nu}\right), \tag{16б}$$

$$f_{11} = \int\limits_0^\infty dx\, \frac{e^{x-\nu}}{(e^{x-\nu}+1)^2} = \frac{1}{e^{-\nu}+1}, \tag{16в}$$

$$f_{12} = \int\limits_0^\infty dx\, \eta(x,T) \frac{e^{x-\nu}}{\left(e^{x-\nu}+1\right)^2}, \tag{16г}$$

$$g_{21} = \int\limits_0^\infty dx\, \frac{x\, e^{x-\nu}}{\left(e^{x-\nu}-1\right)^2} = -\ln\left(1-e^{\nu}\right), \tag{16д}$$

$$g_{22} = \int\limits_0^\infty dx\, \frac{x}{e^{2(x-\nu)}-1} = \frac{1}{4}\,\mathrm{Li}_2\, e^{2\nu}, \quad \text{where} \quad \mathrm{Li}_s\, X = \sum_{k=1}^{\infty} \frac{X^k}{k^s}, \tag{16е}$$

$$f_{21} = \int\limits_0^\infty dx\, \frac{x\, e^{x-\nu}}{\left(e^{x-\nu}+1\right)^2} = \ln\left(1+e^{\nu}\right), \tag{16ж}$$

$$f_{22} = \int\limits_0^\infty dx\, x\eta(x,T) \frac{e^{x-\nu}}{\left(e^{x-\nu}+1\right)^2}. \tag{16и}$$

For $\nu = \mu_{\mathrm{id}}/T$ we have from Eq. (7):

$$AT \int\limits_0^\infty dx\, \frac{1}{\left(e^{x-\nu}+1\right)} = AT \ln\left(e^{\nu}+1\right) = 1, \tag{17}$$

so that

$$\nu = \ln\left(e^{\frac{1}{AT}} - 1\right). \tag{18}$$

Equations (10), (11), and (15) become

$$\begin{aligned} & g_{11}\Delta\mu_{\mathrm{G}} - g_{12}\, uT = 0, \\ & f_{11}\Delta\mu_{\mathrm{F}} - f_{12} = 0, \\ & g_{21}\Delta\mu_{\mathrm{G}} - g_{22}\, uT = f_{21}\Delta\mu_{\mathrm{F}} - f_{22}, \end{aligned} \tag{19}$$

yielding

$$u = \frac{1}{T}\, \frac{g_{11}}{f_{11}}\, \frac{f_{12}f_{21} + f_{11}f_{22}}{g_{12}g_{21} + g_{11}g_{22}}. \tag{20}$$

Let the spectrum of the Fermi system be $\varepsilon + \Delta\varepsilon$ with the correction given by

$$\Delta\varepsilon = \begin{cases} 0 & \text{for } \varepsilon < \varepsilon_0, \\ b/\varepsilon & \text{for } \varepsilon \geq \varepsilon_0. \end{cases} \tag{21}$$

Such an expression resembles the first two terms in the series expansion of the ultrarelativistic spectrum $\left(m^2c^4+c^2p^2\right)^{1/2}$ with $\varepsilon = cp$ or the BCS spectrum $E_k = (\varepsilon_k + \Delta)^{1/2}$ with $\varepsilon = \varepsilon_k$ [11]. For $b < 0$, it qualitatively corresponds, e. g., to the helium-3 elementary excitation spectrum [12], see Fig. 1.

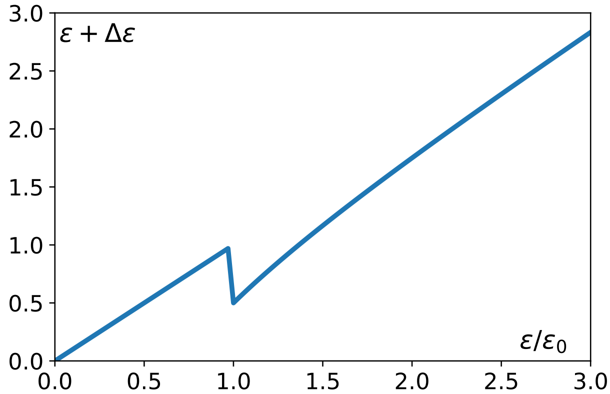

Fig. 1: Excitation spectrum corresponding to Eq. (21) with $b = -0.5$

Expression (21) for the spectrum correction was chosen for simplicity: with such a function, an analytical result can be also obtained for f_{22}:

$$f_{22} = \frac{b/T}{e^{\varepsilon_0/T-\nu}+1}, \tag{22}$$

but, unfortunately, not for f_{12}.

Results of calculation of the u parameter using Eq. (20) are shown in Fig. 2. As expected, the values of u are small thus justifying the described approach.

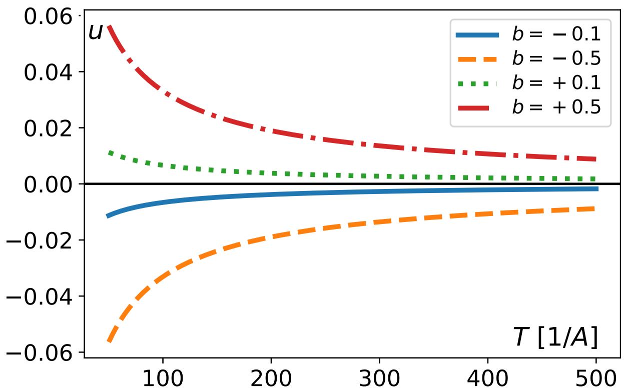

Fig. 2: Statistics parameter u as a function of temperature T (given in the units of $1/A$) for some values of b in Eq. (21)

4. Conclusion

To summarize, we have analyzed an approach that allows for mimicking a system of weakly non-ideal fermions by modifying the Gentile statistics. In the conventional Gentile statistics, the maximum level occupation is an integer. However, in this work, we set this parameter to $1+u$, where u is a small parameter. By doing so, we establish links between the deviation of the spectrum from the ideal case, the differences in chemical potentials between the Gentile and Fermi systems compared to the ideal Fermi gas, and ultimately the parameter u itself. All the relationships are derived in the linear approximation over small quantities. The analysis demonstrates that an ideal system following this modified Gentile statistics can effectively describe an interacting fermionic system with reasonable accuracy. Consideration of subsequent approximations that is relevant for non-weak non-idealities might require additional modifications in the proposed distribution.

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«Несправжній» розподіл Джентіле, що моделює слабконеідеальні ферміони

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Запропонована у 1940 році статистика Джентіле була першим відомим узагальненням квантових статистик Бозе–Айнштайна та Фермі–Дірака. В оригінальному формулюванні Джентіле максимальне заповнення рівня обмежується цілими числами M, що видається цілком природною вимогою. Зрозуміло, що $M = 1$ відповідає ферміонам, а граничний випадок $M = \infty$ описує бозони. У статті подано аналіз підходу до емуляції слабконеідеальних ферміонних систем шляхом модифікації статистики Джентіле. Ми відходимо від вимоги цілочисельності M і вводимо малий параметр u, таким чином формально встановлюючи максимальне заповнення у вигляді $M = 1 + u$, звідки й походить назва «несправжня статистика Джентіле». Використовуючи цю модифікацію статистики, ми знаходимо в лінійному наближенні за малими величинами такі зв'язки: перше рівняння пов'язує параметр u й різницю $\Delta\mu_{\mathrm{G}}$ між хімічним потенціалом системи зі статистикою Джентіле та хімічним потенціалом ідеального фермі-газу; друге рівняння пов'язує різницю $\Delta\mu_{\mathrm{F}}$ між хімічним потенціалом фермі-системи зі взаємодією та хімічним потенціалом ідеального фермі-газу з відхиленням $\Delta\varepsilon$ спектра системи від ідеального випадку; нарешті, третє рівняння, отримане з умови рівності енерґій системи зі статистикою Джентіле та ферміонів зі взаємодією, пов'язує усі згадані величини між собою. У статті отримано загальні співвідношення та продемонстровано їхнє застосування на прикладі модельної системи, де більшість виразів вдається розрахувати аналітично. Як висновок, ми показуємо, що ідеальна система, яка підкоряється запропонованій модифікації статистики Джентіле, може бути наближеною моделлю для взаємодіючих ферміонів.

Ключові слова: ферміони зі взаємодією, статистика Джентіле, дробова статистика, ефективна модель