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# Preparation of maximally entangled four-qubit states on a quantum computer and calculation of the geometric measure of entanglement

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We study maximally entangled four-qubit quantum states. Quantum protocols for preparation of the states and calculation of the geometric measure of entanglement are constructed and realized on IBM's quantum computer ibm-perth. The results of quantum calculations are in agreement with theoretical ones. We show that the quantum device ibm-perth can be in a maximally entangled four-qubit quantum state.

**Ключові слова:** geometric measure of entanglement, four-qubit quantum state, quantum computing

## 1. Introduction

Entanglement plays an extremely important role in quantum computing and quantum programming. Among the applications of the entanglement, it is worth noting quantum cryptography [1], and quantum teleportation [2]. Therefore much attention has been devoted to quantifying of entanglement analytically as well as with quantum programming. For instance, in paper [3] it was shown that 16-qubit IBM's quantum device ibmqx5 can be fully entangled. On the 20-qubit quantum device IBM Q Poughkeepsie the entanglement of quantum states was studied in [4].

In the present paper, we consider the geometric measure of entanglement which was proposed in [5]. The measure has a clear definition with geometric interpretation and is related to the observable quantities [6]. Namely in [6] relations of the entanglement of pure and mixed states with mean spin and spin correlations were obtained. The properties of the geometric measure of entanglement were studied in [7–11]. The geometric measure of entanglement of quantum states was calculated on IBM's quantum device in [12] Also, it is worth noting that the geometric measure of entanglement of multi-qubit quantum graph states was examined analytically and with quantum calculations in [13, 14].

The paper is organized as follows. In Section 2 we present the relation of the geometric measure of entanglement with mean spin. Section 3 is devoted to studies of the entanglement of four-qubit quantum states. Quantum protocols for preparing the maximally-entangled four-qubit quantum states and quantifying their entanglement are constructed. The results of quantum calculations of the entanglement on ibm-perth are presented. Conclusions are formulated in Section 4.

## 2. Geometric measure of entanglement of quantum states and its detection on a quantum computer

Definition of the geometric measure of entanglement is based on the idea of calculation of the minimal distance between an entangled state and a set of non-entangled states. Namely, it is defined as the minimal squared Fubiny-Study distance between a state and a set of non-entangled states

$$E(\mid \psi\rangle) = \min_{|\psi_s\rangle} d^2_{FS}(|\psi_s\rangle, |\psi\rangle), \tag{1}$$

and was proposed in [5]. Finding the minimal distance requires a large amount of computational resources. In paper [6] it was shown that the geometric measure of entanglement of a spin with another spin system in an entangled state

$$\mid \psi\rangle = a \mid 0\rangle \mid \Phi_1\rangle + b \mid 1\rangle \mid \Phi_2\rangle, \tag{2}$$

(here $a$, $b$ are constants) is related with a mean value of spin. The relation reads

$$E(\mid \psi\rangle) = \frac{1}{2}(1 - \sqrt{\langle \boldsymbol{\sigma}\rangle^2}), \tag{3}$$

where

$$\sqrt{\langle \boldsymbol{\sigma}\rangle^2} = \sqrt{\langle \sigma^x\rangle^2 + \langle \sigma^y\rangle^2 + \langle \sigma^z\rangle^2}, \tag{4}$$

and $\sigma^x$, $\sigma^y$, $\sigma^z$ are Pauli matrixes. States $\mid 0\rangle$, $\mid 1\rangle$ are states of the spin, and $\mid \Phi_1\rangle$, $\mid \Phi_2\rangle$ are states of the spin system that can be nonorthogonal $\langle \Phi_i \mid \Phi_j\rangle \neq \delta_{ij}$, $\sqrt{\langle \boldsymbol{\sigma}\rangle^2} = \sqrt{\langle \sigma^x\rangle^2 + \langle \sigma^y\rangle^2 + \langle \sigma^z\rangle^2}$, $\sigma^x$, $\sigma^y$, $\sigma^z$ are Pauli matrixes [6].

The expression (3) relates the entanglement with the observable value and lays in the basis of quantum algorithms for studies of the geometric measure entanglement on quantum devices.

On IBM's quantum devices the measurement in the standard basis can be performed. On the basis of the results of the measurements the mean value of $\sigma^z$ operator can be easily found. Namely, it reads

$$\langle \sigma^z\rangle = \langle \psi|\sigma^z|\psi\rangle = |\langle \psi|0\rangle|^2 - |\langle \psi|1\rangle|^2 \tag{5}$$

To detect the mean values of $\sigma^x$ and $\sigma^z$ operators the following identities have to be used

$$\langle\sigma^x\rangle = \langle\psi|\sigma^x|\psi\rangle = \langle\tilde{\psi}^y|\sigma^z|\tilde{\psi}^y\rangle = |\langle\tilde{\psi}^y|0\rangle|^2 - |\langle\tilde{\psi}^y|1\rangle|^2 \quad (6)$$

$$\langle\sigma^y\rangle = \langle\psi|\sigma^y|\psi\rangle = \langle\tilde{\psi}^x|\sigma^z|\tilde{\psi}^x\rangle = |\langle\tilde{\psi}^x|0\rangle|^2 - |\langle\tilde{\psi}^x|1\rangle|^2, \quad (7)$$

where

$$|\tilde{\psi}^x\rangle = RX(\pi/2)|\psi\rangle = e^{-\mathrm{i}\frac{\pi}{4}\sigma^x}|\psi\rangle, \quad (8)$$

$$|\tilde{\psi}^y\rangle = RY(-\pi/2)|\psi\rangle = e^{\mathrm{i}\frac{\pi}{4}\sigma^y}|\psi\rangle. \quad (9)$$

So, before measurement in the standard basis we have to rotate the state of qubit (spin) around y or x-axis with $RX(\pi/2) = \exp(-\mathrm{i}\pi\sigma^x/4)$, $RY(-\pi/2) = \exp(\mathrm{i}\pi\sigma^x/4)$ gates.

## 3. Quantum calculations of the geometric measure of entanglement of maximally entangled four-qubit quantum states

Four-qubit states that are maximally entangled reads

$$|\psi_1\rangle = \frac{1}{4}\left(|0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle\right), \quad (10)$$

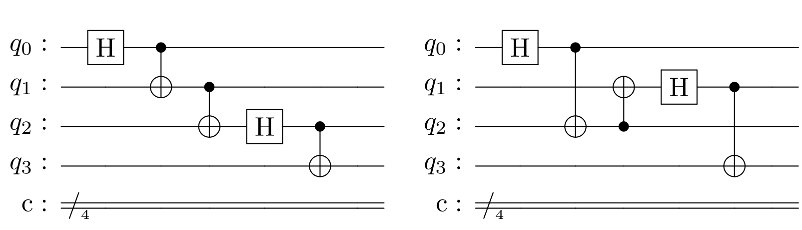
$$|\psi_2\rangle = \frac{1}{4}\left(|0000\rangle + |1010\rangle + |0101\rangle - |1111\rangle\right), \quad (11)$$

$$|\psi_3\rangle = \frac{1}{4}\left(|0000\rangle + |1001\rangle + |0110\rangle - |1111\rangle\right), \quad (12)$$

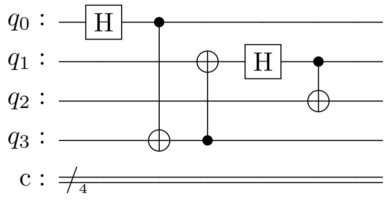
see [15]. It can be easily checked that the mean values Pauli operators in the states equal to zero.

$$\langle\psi_i|\sigma_j^\alpha|\psi_i\rangle = 0, \quad (13)$$

where $\sigma_j^\alpha$ is the $\alpha$ component of the Pauli operator of spin $j$, indexes posses values $i = (1, 2, 3)$, $j = (0, 1, 2, 3)$, $\alpha = (x, y, z)$ The states can be prepared with the following protocols




(a) (b)




$(c)$

Рис. 1: Quantum protocols for preparation of maximally entangled states (10) (a), (11) (b), (12) (c).

In the protocols Hadamard and controlled-NOT gates are used. Note, that the states $|\psi_i\rangle$, $i = (1, 2, 3)$ have similar structure. We can obtain $|\psi_2\rangle$, $|\psi_3\rangle$ after action of the SWAP gate on $|\psi_1\rangle$. Namely, we have $|\psi_2\rangle = \mathrm{SWAP}_{12}\,|\psi_1\rangle$, $|\psi_3\rangle = \mathrm{SWAP}_{13}\,|\psi_1\rangle$, and vice versa. Therefore, on a quantum device we prepare one of the states (we consider $|\psi_1\rangle$), and study the geometric measure of entanglement of each qubit with other ones in it. Quantum calculations are based on the relation of the geometric measure of entanglement with mean spin. For example, to find the entanglement of $q[0]$ with other qubits in quantum state (10) we realize quantum protocols presented on Fig.2 on IBM's quantum computer ibm-perth with number of shots 4000.

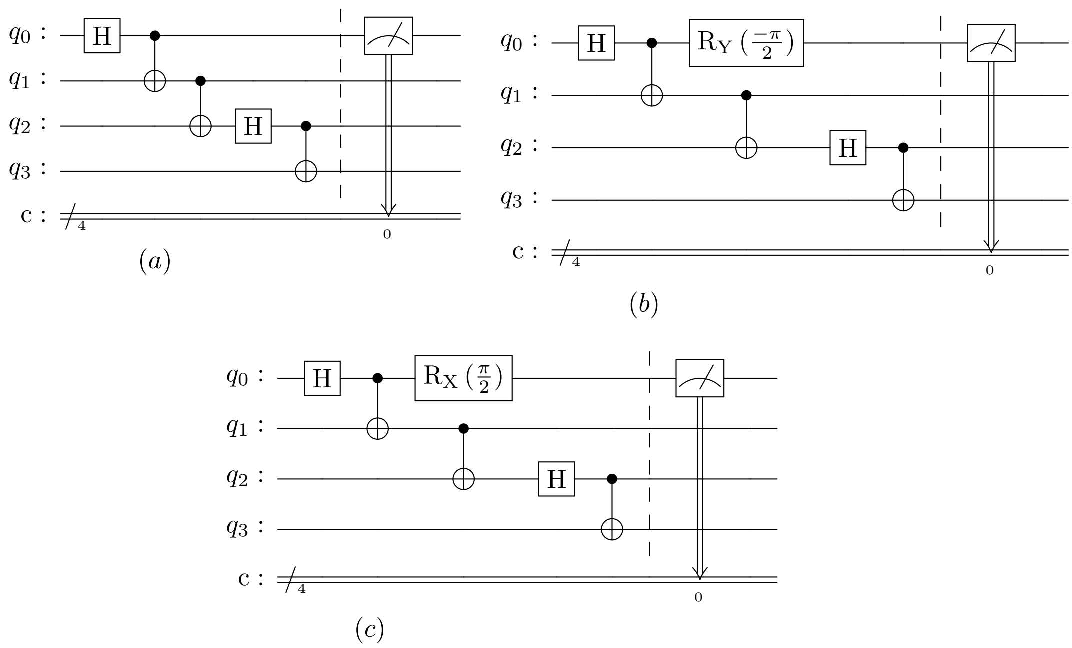



Рис. 2: Quantum protocols for calculation of the mean values $\langle\sigma_0^z\rangle$ (a), $\langle\sigma_0^x\rangle$ (b), $\langle\sigma_0^y\rangle$ (c) in state (10).

Results of quantum computing are presented in the table.

Табл. 1: Results of quantum calculations on IBM's quantum computer ibm-perth. Rows corresponds to the results for mean values $\langle\sigma_i^\alpha\rangle$ and the geometric measure of entanglement of qubit $q[i]$ with other qubits in state $|\psi_1\rangle$, $i = (0, 1, 2, 3)$.

| | $\langle\sigma^x\rangle$ | $\langle\sigma^y\rangle$ | $\langle\sigma^z\rangle$ | $E(\|\psi_1\rangle)$ |
|---|---|---|---|---|
| $q[0]$ | 0.076 | 0.061 | 0.003 | 0.498 |
| $q[1]$ | 0.069 | 0.070 | 0.003 | 0.498 |
| $q[2]$ | 0.060 | 0.063 | 0.004 | 0.498 |
| $q[3]$ | 0.068 | 0.065 | 0.004 | 0.498 |

From the table we can see that the geometric measure of entanglement of qubit $q[i]$ with other qubits in state $|\psi_1\rangle$ is close to its maximal value 0.5. So, the results of quantum calculations are in agreement with the theoretical ones.

## Conclusions

We have considered four-qubit maximally entangled quantum states and studied their geometric measure of entanglement. The studies were done on the basis of relation of the entanglemet with mean spin which was obtained in [6].

Quantum protocols for calculation of the mean values $\langle\sigma_i^x\rangle$, $\langle\sigma_i^y\rangle$, $\langle\sigma_i^z\rangle$ $(i = 0, 1, 2, 3)$ in the maximally entangled four-qubit quantum state have been constructed and run on IBM's quantum computer ibm-perth. On the basis of the results the geometric measure of entanglement of each qubit $q[i]$ with other qubits in the state has been calculated. The results of quantum calculations are in agreement with the theoretical ones. So, we show that IBM's quantum computer ibm-perth can be in four-qubit maximally entangled quantum state.

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# Приготування максимально заплутаних чотири-кубітних квантових станів на квантовому комп'ютері та обчислення геометричної міри заплутаності

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Заплутаність квантових станів є одним із найважливіших ресурсів квантових обчислень та квантового програмування. Вона, зокрема, відіграє ключову роль у квантових комунікаціях (квантова криптографія, квантова телепортація). Тому важливим є дослідження заплутаності квантових станів та її обчислення за допомогою квантового програмування. У статті розглядаються максимально заплутані чотири-кубітні квантові стани. Будуються квантові протоколи для приготування таких станів на квантовому комп'ютері. Ми розраховуємо геометричну міру заплутаності станів за допомогою квантових обчислень. Така міра заплутаності має прозору геометричну інтерпретацію, оскільки визначається як мінімальне значення квадрата відстані Фубіні-Стаді від заплутаного стану до множини незаплутаних станів [A. Shimony, Ann. N.Y. Acad. Sci. 755, 675 (1995)]. Важливо, що вона пов'язана зі спостережуваними величинами. А саме, геометрична міра заплутаності спіна з іншими спінами системи визначається його середнім значенням. Такий результат було отримано у роботі [A. M. Frydryszak, M. I. Samar, V. M. Tkachuk, Eur. Phys. J. D. 71, 233 (2017)]. Квантовий алгоритм для знаходження геометричної міри заплутаності квантових станів на квантових комп'ютерах базується на цьому зв'язку. Ми реалізували цей алгоритм на семи-кубітному квантовому комп'ютері компанії IBM ibm-perth та знайшли заплутаність кожного кубіта з іншими у максимально заплутаному чотири-кубітному квантовому стані. Для цього було пораховано середні значення операторів Паулі $\sigma_i^x$, $\sigma_i^y$, $\sigma_i^z$, які відповідають різним кубітам $i = (0, 1, 2, 3)$. Результати квантових обчислень добре узгоджуються з теоретичними розрахунками. Ми показали, що квантовий процесор ibm-perth може знаходитися у максимально заплутаному чотири-кубітному квантовому стані.

**Key words:** геометрична міра заплутаності, чотирикубітні квантові стани, квантові обчислення