

УДК 539
PACS 02.10.Ox, 02.40.Ky, 0460.Pp

DIRECTED RANDOM WALK ON THE EIGHT-BRANCHING CAYLEY TREE

A. Nazarenko

*Bogolyubov Institute for Theoretical Physics
14-b, Metrologichna Str., Kiev 03680, Ukraine
e-mail: nazarenko@bitp.kiev.ua*

In engineering of the chaotic stadiums is important to consider an ensemble of self-avoiding and directed graphs belonging to eight-branching Cayley tree generated by the Fuchsian group of a Riemann surface in genus two and embedded in the Poincaré unit disk. We give a description of two-parametric lattices and calculate the multi fractal scaling exponents for the moments of the graph lengths distribution as functions of these parameters. We show the results of numerical and statistical computations, where the latter are based on a random walk model.

Key words: directed random walk; Cayley tree

1 Introduction

Traditionally, disordered models are described using the underlying non-trivial geometrical carriers, for instance, by random environments, percolation clusters, fractals, etc. Disordered physical systems and phenomena like turbulence [1, 2], diffusion limited aggregation [3], polymers [5, 6], the particle dynamics in a random magnetic field [7, 8] are the examples. The last decades it seems natural to describe these complicated and chaotic systems by using the multifractality concept [9, 10] giving us a spectrum of critical exponents which can be connected with the results of renormalization group computations (see Ref. [11]).

Limiting by dimension two, there is a possibility to construct the infinite number of lattices, existence of which is not allowed in a flat space [12]. However, there is possibility to deal with the Lobachevsky geometry and to demonstrate that the structures, under consideration in mathematics, might pretend to describe and generate some kind of disorder, like a random walk. Similar idea to consider a directed random walk on a Cayley tree in hyperbolic space has been already exploited in Refs. [13, 14]. We also note that the models of billiards intensively use the hyperbolic Riemann surfaces as geometrical carriers [15, 16].

In our paper, we shall watch a thread of walker moving along the eight-branching and rooted Cayley tree constructed as a dual lattice to the hyperbolic octagonal lattice tiling the Poincaré disk under the action of Fuchsian group. The octagonal lattice cells, whose

opposite sides are identified, correspond topologically to the two-holed torus, geometry of which is defined here by one independent module while other two complex moduli are constrained. (Note that the Riemann surfaces of genus g are characterized by $6g - 6$ real parameters.) We characterize a walk by spectrum of lengths which result from summation of hyperbolic distances between tree sites visited and root point. Introducing a partition function and the moments of distribution function, we would like to investigate an ensemble of these lengths within the multifractality concept.

Thus, the model under consideration is deterministic by construction, and partition function is actually the truncated series over a symmetry group. To calculate this kind of series, we apply both numerical and statistical methods, where the former is needed to exhibit the basic properties of the model, which we aim to reproduce by means of the latter approach called for purely analytical description. Within the framework of statistical method, we elaborate a scheme of length spectrum computation and reduce the partition function to the Markov multiplicative chain in a random walk approximation.

2 The Model

We concentrate on the properties of Riemann surface S in genus $g = 2$, which is understood here as a compact two-dimensional orientable manifold with the Riemannian metric of the constant negative curvature. Such a surface is obtained from a *hyperbolic* simply connected octagon \mathcal{F} , embedded into the unit disk $\mathbb{D} = \{z = x + iy \in \mathbb{C} \mid |z| < 1\}$, via gluing opposite sides formed by eight geodesic arcs, whose intersections serve as vertices.

Now let us introduce a structure on the Poincaré disk. Here we aim to consider an example of lattice resulting from operation of a particular Fuchsian group Γ (stating “periodic law”) on the (complexified) Poincaré disk \mathbb{D} [12]. The associated quotient space \mathbb{D}/Γ is assumed to be a Riemann surface of genus $g = 2$, whose fundamental domain \mathcal{F} is hyperbolic octagon in \mathbb{D} , which can be viewed as the result of gluing two tori together.

Thus we should first construct the fundamental octagon \mathcal{F} and the corresponding Fuchsian group Γ by means of which we tile \mathbb{D} by the octagonal lattice. Note that an algorithm connecting \mathcal{F} for $g = 2$ with Fuchsian group Γ and vice versa is described in Ref. [17].

In our model, which has been already declared and geometrically described in [4], we assume that the vertices are at the points $a \exp(ik\pi/2)$, $b \exp[i(\alpha + k\pi/2)]$, where $0 < \alpha < \pi/2$, $0 < a, b < 1$, $k = \overline{0, 3}$ (see Fig. 1, left panel). We also require the sum of the inner angles of \mathcal{F} to be equal to 2π . This is the same as requiring $\text{area}(\mathcal{F}) = 2\pi(2g - 2) = 4\pi$ in accordance with the Gauss-Bonnet theorem [12].

The Riemann surface of genus $g = 2$ is completely determined by a set of three complex moduli corresponding to three corners of the fundamental domain \mathcal{F} (in complex plane). The fourth corner is then found to adjust the area of the fundamental domain to be $2\pi(2g - 2) = 4\pi$, according to the Gauss-Bonnet theorem [12]. Applying inversions across the origin to these four corners yield the remaining four corners of the octagon.

For the sake of simplicity, we shall focus on symmetric octagon with the *one* arbitrary module, namely, one pair of the length and angle variables denoted as (a, α) . Such an octagon (sketched in Fig. 1) can be obtained by deformation of the regular hyperbolic

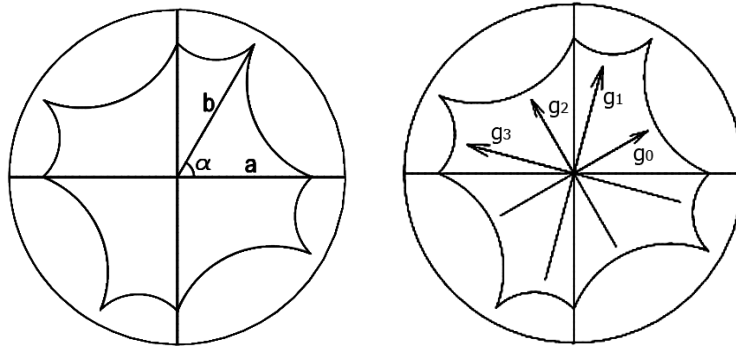


Рис. 1: Symmetric octagon with $a = 0.8$, $\alpha = \pi/3$ and generators g_k of Fuchsian group.

octagon (with $a = 2^{-1/4}$, $\alpha = \pi/4$), well studied in the context of the chaology (see, for example, Refs. [15,17] and references therein).

In the case at a hand, the boundary of octagon, $\partial\mathcal{F}$, is formed by geodesics of two kinds (labeled by “ \pm ” below). These geodesics are completely determined by the radii R_{\pm} and the angles $\phi_{\pm} + k\pi/2$ ($k = 0, 3$), defining the positions of the circle centers. Satisfying the conditions imposed above, we find that

$$R_{\pm} = \frac{1}{2a} \sqrt{T_{\pm}^2 + (1 - a^2)^2}, \quad \phi_{\pm} = \arctan \left[\left(\frac{T_{\pm}}{1 + a^2} \right)^{\pm 1} \right], \quad (1)$$

where

$$T_{\pm} = a^2 \pm \tan \left(\alpha - \frac{\pi}{4} \right). \quad (2)$$

Introducing an angle β by vertices $a \exp(ik\pi/2)$ (the angle by vertices $b \exp i(\alpha + k\pi/2)$ is then equal to $\pi/2 - \beta$),

$$\tan \beta = (1 - a^2) \frac{2a^2 \cos^2 \left(\alpha - \frac{\pi}{4} \right)}{2a^2 \cos^2 \left(\alpha - \frac{\pi}{4} \right) - 1}, \quad (3)$$

we should control the condition $0 < \beta < \pi/2$, defining the region of accessibility of parameters (a, α) :

$$\frac{1}{\sqrt{2} \cos \left(\alpha - \frac{\pi}{4} \right)} < a < 1, \quad (4)$$

which is sketched in Fig. 2.

To complete the octagon description, the parameter b , pointed out in Fig. 1, can be calculated as

$$b = \frac{1}{\sqrt{2}a \cos \left(\alpha - \frac{\pi}{4} \right)}. \quad (5)$$

Simplest way to describe changes globally consists in consideration of a perimeter of hyperbolic octagon. Within the present model, the perimeter is given by the formula:

$$P = 8 \operatorname{arccosh} \frac{1 - a^2b^2 + \sqrt{(1 - a^2)^2 + (1 - b^2)^2}}{(1 - a^2)(1 - b^2)}. \quad (6)$$

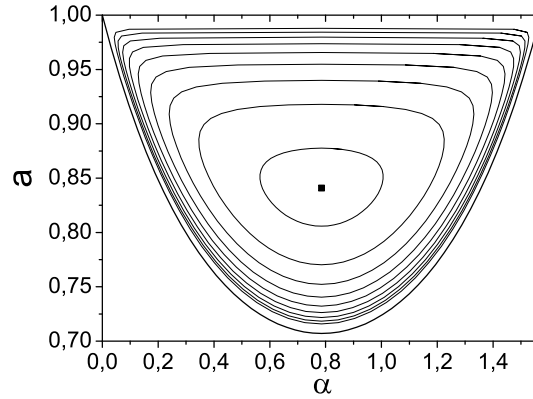


Рис. 2: The permission domain of module (a, α) .

In a some sense, this characteristic is a good candidate due to invariance of P under the octagon automorphisms and pants decomposition. It means that P can take on the same value for various values of (a, α) .

Note that the manifest dependence of the octagon parameters on moduli is necessary in the different problems where geometry of the system (carrier geometry) is not fixed. For instance, (a, α) would be the (dynamical) variables of evolution parameter in topological gravity describing an universe of genus $g = 2$; it is able to average over (a, α) in statistical physics, etc.

The Fuchsian group Γ of a Riemann surface of genus two is generated by four generators g_0, g_1, g_2, g_3 and their inverses. Let generators g_k, g_k^{-1} , acting freely as isometries of Poincaré disk, map geodesic boundary segments of octagon \mathcal{F} onto each other, thereby identifying opposite edges (see Fig. 1, right panel). This definition leads to the group relation:

$$g_0 g_1^{-1} g_2 g_3^{-1} g_0^{-1} g_1 g_2^{-1} g_3 = \text{id}. \quad (7)$$

Generally, the group of orientation-preserving isometries of (\mathbb{D}, ds^2) is presented by $PSU(1, 1) = SU(1, 1)/\{\pm 1\}$ (see textbooks like Ref. [12]), where

$$SU(1, 1) = \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \middle| u, v \in \mathbb{C}, |u|^2 - |v|^2 = 1 \right\}. \quad (8)$$

The group of isometries acts via fractional linear transformations:

$$z \mapsto \gamma[z] = \frac{uz + v}{\bar{v}z + \bar{u}}, \quad \gamma \in PSU(1, 1), \quad z \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}. \quad (9)$$

Then the Fuchsian group Γ is a discrete group such that $\Gamma \subset PSU(1, 1)$. One also adds the hyperbolicity condition, $|\text{Tr } \gamma| > 2$ for all $\gamma \in \Gamma$. Constructing the corresponding Fuchsian group for a given octagon, it is convenient to express the generators of Γ by half turns as follows.

Let p_k be the mid-point of k -th side, $k = \overline{0, 3}$. Since the opposite sides of octagon have the same lengths by construction, the generators are then written as $g_k = H(p_k)$ (see Ref. [17]), where

$$H(p) = \frac{-1}{1 - |p|^2} \begin{pmatrix} 1 + |p|^2 & 2p \\ 2\bar{p} & 1 + |p|^2 \end{pmatrix} \in PSU(1, 1). \tag{10}$$

Bar over expression, \bar{p} , means the complex conjugation.

The operation of matrices $H(p)$ consists in the half turn (rotation with angle π) of geodesic segment around the origin $z = 0$ and the half turn around point p .

In our model, all p_k are parametrized by $p_{\pm} \in \mathbb{D} \subset \mathbb{C}$: $p_0 = p_+$, $p_1 = p_-$, $p_2 = p_+e^{i\pi/2}$, and $p_3 = p_-e^{i\pi/2}$. At this time, p_{\pm} are functions of two parameters (a, α) and written down below. In principal, we could first take p_k as “input” and, using then these points, determine the octagon parameters.

To find p_k analytically, set

$$\omega_{\pm} = \frac{be^{i\alpha}(1 - a^2) + ae^{i\pi(1\mp 1)/4}(1 - b^2)}{1 - a^2b^2}, \tag{11}$$

then one has that

$$p_{\pm} = \frac{\omega_{\pm}}{1 + \sqrt{1 - |\omega_{\pm}|^2}}. \tag{12}$$

The generators g_k can be directly expressed in the terms of auxiliary variables ω_{\pm} . Therefore, $g_k = M(\omega_k)$, where

$$M(\omega) = \frac{-1}{\sqrt{1 - |\omega|^2}} \begin{pmatrix} 1 & \omega \\ \bar{\omega} & 1 \end{pmatrix} \in PSU(1, 1), \tag{13}$$

and $\omega_0 = \omega_+$, $\omega_1 = \omega_-$, $\omega_2 = \omega_+e^{i\pi/2}$, $\omega_3 = \omega_-e^{i\pi/2}$.

Now it is possible to build the octagonal lattice with a given fundamental octagon. By the construction, action of the group elements $\gamma \in \Gamma$ on \mathcal{F} produces the daughter cells tiling Poincaré disk,

$$\mathbb{D} = \bigcup_{\gamma \in \Gamma} \gamma\mathcal{F}.$$

The first eight cells $g_k^{\pm 1}\mathcal{F}$ ($k = \overline{0, 3}$) are shown in Fig. 3. We see that $g_k^{\pm 1}\mathcal{F} \cap \mathcal{F} \subset \partial\mathcal{F}$ as must be. Of course, this property is preserved for all neighboring cells (not only for fundamental polygon \mathcal{F}).

3 The Directed Random Walk

In this Section, we concern with the directed random (self-avoiding) walk (DRW) on a 8-branching Cayley tree isometrically embedded in Poincaré disk. The tree under consideration is formed by the graph connecting the centers of the neighboring octagons (see Fig. 4) and may be physically interpreted as a model of directed polymers.

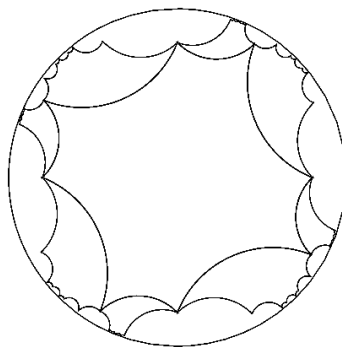


Рис. 3: The fundamental octagon with $a = 0.8$, $\alpha = \pi/3$ and the daughter cells.

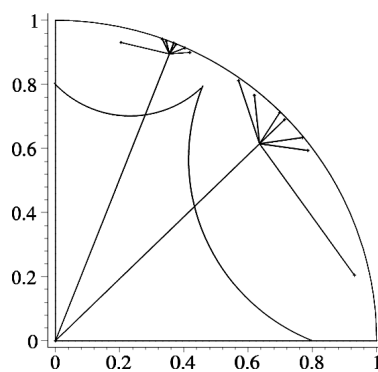


Рис. 4: The quarter of fundamental octagon with $a = 0.8$, $\alpha = \pi/3$, and the Cayley tree sites of first and second generations (connected schematically by straight lines but not geodesics).

We start from the general expression for partition function of DRW in disordered environment on two-dimensional lattice:

$$\mathcal{Z}_N(q) = \sum_{\{z\}} \prod_{t=1}^N p(z_{t-1}|z_t) \exp[qV(z_t, t)], \tag{14}$$

where N is the number of generations; z_t is a point of t -th generation of random trajectory; V is a random potential of environment; p is a transition probability; parameter q can be mapped into an inverse temperature.

In our model, we consider a deterministic motion on graphs of N -th generation, sites of which correspond to a set of numbers $\{i_1, i_2, \dots, i_N\}$, where $1 \leq i_t \leq 8, 1 \leq t \leq N$. We also assume that

$$z_t = \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_t}[0], \quad V(z_t, t) = d(0, z_t)/N, \tag{15}$$

$$p(z_{t-1}|z_t) \rightarrow p(i_{t-1}|i_t) = \begin{cases} 1, & |i_t - i_{t-1}| \neq 4 \\ 0, & |i_t - i_{t-1}| = 4 \end{cases}, \tag{16}$$

where $\{\gamma_i | i = \overline{1, 8}\} = \{g_0, g_1, g_2, g_3, g_0^{-1}, g_1^{-1}, g_2^{-1}, g_3^{-1}\}$. By construction, points z_t and z_{t-1} are the centers of neighboring cells connected by the tree branches. Definition of $p(i_{t-1}|i_t)$ (consistent with definition of generators $\gamma_i, i = \overline{1, 8}$) excludes a return to the visited sites. Hyperbolic distance $d(z, w)$ on (\mathbb{D}, ds^2) is determined by

$$\cosh d(z, w) = 1 + \frac{2|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}. \tag{17}$$

In particular case, one has

$$d(0, z) = \ln \frac{1 + |z|}{1 - |z|}. \tag{18}$$

In this way, we come to expression for partition function of our model:

$$\mathcal{Z}_N(q) = \sum_{i_1=1}^8 \dots \sum_{i_N=1}^8 \left(\prod_{t=1}^N p(i_{t-1}|i_t) \right) \exp \left[q \frac{1}{N} \sum_{t=1}^N \rho_{i_1, i_2, \dots, i_t} \right], \tag{19}$$

where $\rho_{i_1, i_2, \dots, i_t} \equiv d(0, \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_t}[0])$ is the hyperbolic (geodesic) distance between the point of t -th generation and the origin O .

The partition function defined is a sum of Boltzmann weights that depend on an “action integral” linear in the hyperbolic distance from a given root point (the origin O), i.e. the length of trajectory:

$$L(i_1, i_2, \dots, i_N) \equiv \sum_{t=1}^N \rho_{i_1, i_2, \dots, i_t}. \tag{20}$$

It is worthy to note that $\mathcal{Z}_N(q)$ is also a function of arbitrary module (a, α) , used for obtaining the generators of the group Γ . To explore the properties of a model described by $\mathcal{Z}_N(q|a, \alpha)$, we shall fix the lattice parameters.

Since the all trajectories are self-avoiding and uniquely determined, the number of graphs $Z_N(0)$ is equal to the number of end-points, $v(N) = 8 \times 7^{N-1}$, and the partition function (19) can be re-written as

$$Z_N(q) = \sum_{i=1}^{v(N)} e^{qL_i(N)/N}, \quad (21)$$

where $L_i(N)$ is the integrated hyperbolic length of i -th graph of N -th generation.

Thus, the main goal of this work is to evaluate the series (21) dependent on length spectrum $\{L_i(N)\}$. To resolve this problem, we shall first perform numerical investigations to establish basic properties of such a system. Secondly, we shall attempt to calculate the characteristics analytically with the use of statistical approach.

An example of length spectrum obtained numerically is presented in Fig. 5, where the absolute value of the number of events depends on particular choice of the width of shell. We see that the distribution of lengths L_i approaches to the Gaussian distribution,

$$\sim \exp\left(-\frac{(L - \bar{L}_N)^2}{2\sigma_N^2}\right), \quad (22)$$

with parameters defined by formulas

$$\bar{L}_N = \frac{1}{v(N)} \sum_{i=1}^{v(N)} L_i(N), \quad \sigma_N^2 = \frac{1}{v(N)} \sum_{i=1}^{v(N)} (L_i(N) - \bar{L}_N)^2, \quad (23)$$

resulting to $\bar{L}_5 \approx 41.97$, $\sigma_5^2 \approx 44.5657$.

Although the process of such a convergence is slow, it means that the structure generated by the discrete group Γ satisfactory describes a disordered environment that justifies a random walk treatment. Also note that although Fig. 5 was built for the fixed parameters of the lattice, the system reveals the same tendency for any admissible pair (a, α) .

We are giving the further analysis of the model within the framework of the concept of multifractality consisting in a scale dependence of critical exponents [9]. To do this, let us introduce the multifractal moments of the order q defined as follows

$$\mathbf{m}_N^{(q)} = \frac{Z_N(q)}{Z_N^q(1)}. \quad (24)$$

Then the scaling exponent, sometimes called as mass spectrum, is

$$\tau_N(q) = \frac{2}{\ln v(N)} \ln \mathbf{m}_N^{(q)}, \quad (25)$$

where prefactor 2 corresponds to dimension of underlying geometrical carrier (space) and $v(N)$ coincides with the number of "boxes" tilling structure under consideration.

Exponent $\tau_N(q)$ is self-averaging and therefore it is reasonable to introduce

$$\tau(q) = \lim_{N \rightarrow \infty} \tau_N(q), \quad (26)$$

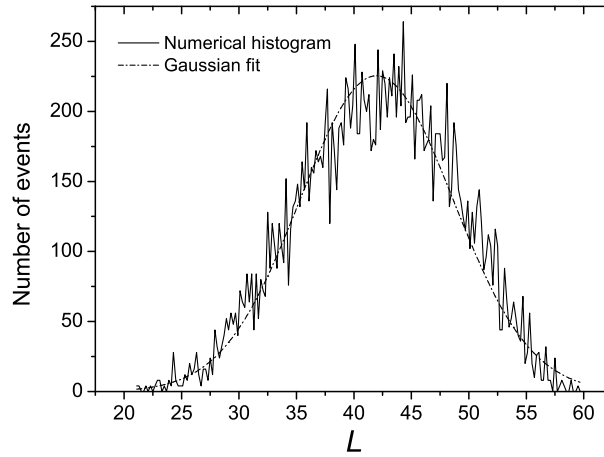


Рис. 5: The number of events as the function of the hyperbolic length $L(N = 5)$ and the Gaussian fit for lattice with $a = 0.8$, $\alpha = \pi/3$.

defining the spectrum of fractal dimensions $D_q = \tau(q)/(1 - q)$.

The result of our numerical calculations (limited to $N \leq 5$) is shown in Fig. 6. We can conclude that the multifractal behavior takes place indeed, that is reflected in non-linear dependence τ on q .

Besides the finding dependence $\tau(q)$, there is another equivalent way to describe the multifractality. It is well known that the set of exponents governing of multifractal moments of type (24) is related to the spectrum of singularities $f(\alpha)$ of fractal measure, [18] called also the spectral function, which is given by the Legendre transform:

$$f(\alpha) = q\alpha + \tau(q), \quad \alpha(q) = -\frac{d\tau(q)}{dq}, \quad (27)$$

where α is known as Lipschitz–Hölder exponent. In other words, $f(\alpha)$ denotes the dimension of the subset characterized by the singularity strength α . In the terms of thermodynamics, $\alpha(q)$ coincides with an internal energy e , while $f(\alpha)$ is interpreted as an entropy $s(e)$ [19].

The general properties of $f(\alpha)$ are as follows: it is positive on an interval $[\alpha_{\min}, \alpha_{\max}]$, where

$$\alpha_{\min} = \lim_{q \rightarrow +\infty} \tau(q)/(1 - q), \quad \alpha_{\max} = \lim_{q \rightarrow -\infty} \tau(q)/(1 - q), \quad (28)$$

and the maximum value of the spectral function gives us the (fractal) dimension of the underlying structure.

Spectral function $f(\alpha)$ obtained for $N = 5$ is given in Fig. 7. In order to compare

We preserve conventional symbol for this exponent [9]. It will always be clear from context whether α is an angle determining the geometry of octagon or the singularity strength.

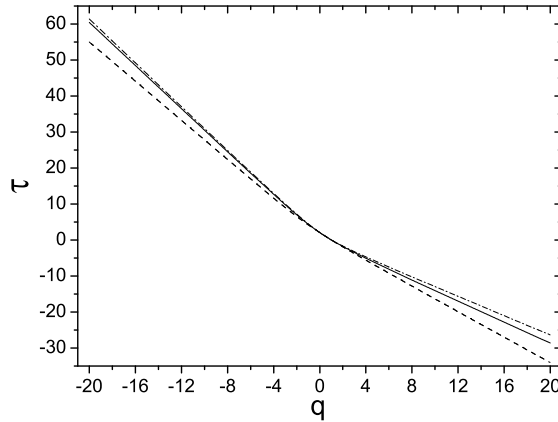


Рис. 6: Multifractal behavior of the exponent $\tau(q)$ for $N = 5$. The solid, dashed, and dashed-dotted curves correspond to the lattices with $(a = 0.8, \alpha = \pi/3)$, $(a = 2^{-1/4}, \alpha = \pi/4)$, and $(a = 0.9, \alpha = \pi/8)$, respectively.

curves in Fig. 7, we appeal to an information entropy S [9], defined as

$$S = \alpha_S = f(\alpha_S), \quad \left. \frac{df(\alpha)}{d\alpha} \right|_{\alpha_S} = 1. \tag{29}$$

The value of α_S and $f(\alpha_S)$ can be directly calculated at $q = 1$.

We find that $S(a = 0.9, \alpha = \pi/8) \approx 1.783 < S(a = 0.8, \alpha = \pi/3) \approx 1.865 < S(a = 2^{-1/4}, \alpha = \pi/4) \approx 1.95$. This relation has a simple treatment: i) due to a special symmetry, the regular octagon gives us a highest entropy which cannot exceed 2 in our model; ii) deviation from the regular symmetry leads to decreasing the entropy more and more, what says about low probable configurations.

Since the problem of random walk is mathematically the problem of addition of independent random variables, the ultimate behavior of random walkers can be deduced from a central limit theorem (CLT). Thus, another approach, applying CLT, can be also developed.

First, we need to reduce an evolution to the Markov multiplicative process. In order to achieve this, let us use the triangle rule in two-dimensional hyperbolic space:

$$\cosh d(0, z_t) = \cosh d(0, z_{t-1}) \cosh d(z_t, z_{t-1}) - \sinh d(0, z_{t-1}) \sinh d(z_t, z_{t-1}) \cos \theta_{t,t-1}, \tag{30}$$

where $\theta_{t,t-1}$ is angle opposite to side $(0, z_t)$.

Due to identities

$$d(0, \gamma^n[0]) = nd(0, \gamma[0]), \quad d(\tilde{\gamma}[0], \tilde{\gamma}\gamma[0]) = d(0, \gamma[0]), \tag{31}$$

which are valid for matrices (10), (13), we get that $d(z_t, z_{t-1}) = d(0, \gamma_{i_t}[0])$ and

$$\cosh d(0, z_t) = \cosh d(0, z_{t-1}) \cosh \ell_{i_t} - \sinh d(0, z_{t-1}) \sinh \ell_{i_t} \cos \theta_{t,t-1}, \tag{32}$$

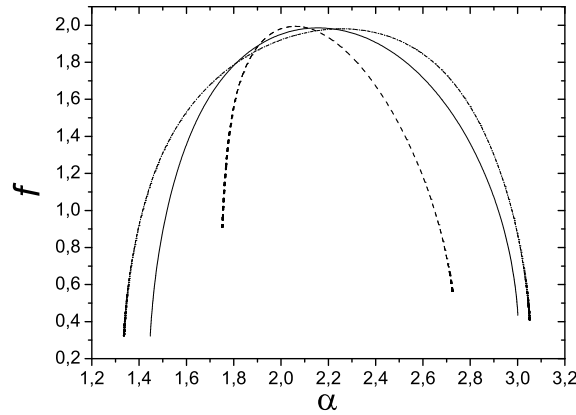


Рис. 7: The spectrum of singularities $f(\alpha)$ for $N = 5$. The solid, the dashed, and the dashed-dotted curves correspond to the lattices with $(a = 0.8, \alpha = \pi/3)$, $(a = 2^{-1/4}, \alpha = \pi/4)$, and $(a = 0.9, \alpha = \pi/8)$, respectively.

where, accounting for lattice symmetry, $\ell_i = \delta_i^+ \ell_+ + \delta_i^- \ell_-$,

$$\ell_+ = d(0, \gamma_{2k-1}[0]), \quad \ell_- = d(0, \gamma_{2k}[0]), \quad k = \overline{1, 4}, \quad (33)$$

$$\delta_i^+ = \sum_{k=1}^4 \delta_{i, 2k-1}, \quad \delta_i^- = \sum_{k=1}^4 \delta_{i, 2k}. \quad (34)$$

For relatively large $d(0, z_t)$ and $d(0, z_{t-1})$, one has

$$d(0, z_t) = d(0, z_{t-1}) + \ln(\cosh \ell_{i_t} - \sinh \ell_{i_t} \cos \theta_{t, t-1}). \quad (35)$$

Now it is useful to introduce

$$\xi_{i,j} = d(0, \gamma_i \gamma_j [0]) - \frac{\ell_i + \ell_j}{2}, \quad i, j = \overline{1, 8}, \quad (36)$$

describing a logarithmic contribution for the second generation.

Then, neglecting fluctuations of $\theta_{t, t-1}$ in higher generations, we obtain an estimation for trajectory length:

$$L(i_1, i_2, \dots, i_N) \simeq N \ell_{i_1} + \sum_{t=2}^N (N+1-t) \xi_{i_{t-1}, i_t} \quad \text{for } p(i_{t-1}|i_t) \neq 0. \quad (37)$$

Note immediately that this formula is exact for $i_1 = i_2 = \dots = i_N$ when $\theta_{t, t-1} = \pi$.

In principal, it is also possible to introduce quantities $\xi_{i_1, i_2, \dots, i_n}^{(n)} \sim d(0, \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_n} [0])$ playing a role of higher-order correlations and asymmetric by indices in contrast with symmetric matrix $\xi_{i,j} = \xi_{j,i}$. Moreover, matrices $\xi^{(n)}$ are able to account more precisely for values of angles $\theta_{t, t-1}$ or, in another word, fluctuations.

Using Eq. (37), we can easily give the theoretical estimation for the lengths spectrum of N -th generation

$$\begin{aligned} L_{\min}^{\text{th}}(N) &= N \min(\ell_+, \ell_-) + \frac{N(N-1)}{2} \min(\xi_{i,j}), \quad \text{for } p(i|j) \neq 0, \\ \bar{L}_N^{\text{th}} &= N \frac{\ell_+ + \ell_-}{2} + \frac{N(N-1)}{2} \bar{\xi}, \quad \bar{\xi} = \frac{1}{56} \sum_{i,j=1}^8 p(i|j) \xi_{i,j}, \\ L_{\max}^{\text{th}}(N) &= N \max(\ell_+, \ell_-) + \frac{N(N-1)}{2} \max(\xi_{i,j}) \\ &= \frac{N(N+1)}{2} \max(\ell_+, \ell_-). \end{aligned} \quad (38)$$

Substituting (37) in (19), we arrive at the expression for partition function as inhomogeneous Markov multiplicative chain:

$$\mathcal{Z}_N(q) \simeq \sum_{i_1=1}^8 \dots \sum_{i_N=1}^8 \exp(q \ell_{i_1}) \prod_{t=2}^N \mathcal{P}^t(i_{t-1}|i_t), \quad (39)$$

where exponent function plays a role of initial condition, and

$$\mathcal{P}^t(i_{t-1}|i_t) = p(i_{t-1}|i_t) \exp \left[q \frac{N+1-t}{N} \xi_{i_{t-1}, i_t} \right] \quad (40)$$

is transition weight dependent on t . Properties of Eq. (39) are discussed in details in Appendix C.

Thus, we would like to note that the length spectrum $\{L_i(N)\}$ and the corresponding partition function for any N are completely determined by minimal lengths ℓ_{\pm} and matrix $\xi_{i,j}$ in a given approximation.

Now, arranging the lengths of graphs in ascending order $L_1 \leq L_2 \leq \dots \leq L_s$ ($s < v(N)$), we can re-write the partition function (21) as

$$\mathcal{Z}_N(q) = v(N) \sum_{i=1}^s W_i e^{q l_i}, \quad \sum_{i=1}^s W_i = 1, \quad (41)$$

where $l_i = L_i/N$ and W_i denotes the weight of the orbit of length L_i .

Hereafter, coefficients W_i are assumed to be randomly distributed. It means that the value of W_i is independent of W_k for $i \neq k$, allowing the application of the random walk model.

Next, assuming that the CLT is valid at $N \rightarrow \infty$ for our system, let us model a distribution of the lengths l_i of N -th generation graphs by the Gaussian distribution:

$$W_N(l) = A_N \exp \left(-\frac{(l - N\bar{l})^2}{2Ns^2} \right), \quad (42)$$

where $N\bar{l} = \bar{L}_N/N$, $Ns^2 = \sigma_N^2/N^2$ when $N \rightarrow \infty$; A_N is a normalization constant. We can theoretically evaluate \bar{l} and s^2 by the use of Eqs. (37), (38).

We have already seen in Fig. 5 that the Gaussian approximation is in good agreement with numerical data. So, it is believed that an application of Gaussian distribution may simplify significantly the calculations of the series over group Γ .

Applying the approximate distribution (42), we can replace the sums (41) with integrals at large N :

$$\mathcal{Z}_N(q) \approx v(N) \int_{\ell_{\min}(N)}^{\ell_{\max}(N)} W_N(l) e^{ql} dl, \quad \int_{\ell_{\min}(N)}^{\ell_{\max}(N)} W_N(l) dl = 1, \quad (43)$$

where $\ell_{\min}(N) = L_{\min}(N)/N$ and $\ell_{\max}(N) = L_{\max}(N)/N$ are introduced in order to cut long Gaussian tails.

After simple calculations we obtain that

$$\mathcal{Z}_N(q) \approx \frac{8}{7} C_N(q) \exp N \left(\frac{1}{2} q^2 s^2 + q\bar{l} + \ln 7 \right), \quad (44)$$

where coefficient

$$C_N(q) = \frac{\operatorname{erf} \left(\frac{\ell_{\max}(N) - N\bar{l} - qNs^2}{\sqrt{2Ns^2}} \right) + \operatorname{erf} \left(\frac{N\bar{l} + qNs^2 - \ell_{\min}(N)}{\sqrt{2Ns^2}} \right)}{\operatorname{erf} \left(\frac{\ell_{\max}(N) - N\bar{l}}{\sqrt{2Ns^2}} \right) + \operatorname{erf} \left(\frac{N\bar{l} - \ell_{\min}(N)}{\sqrt{2Ns^2}} \right)} \quad (45)$$

is responsible for surface effects which are not essential in infinite Bethe lattice;

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (46)$$

Comparison of the values of $\tau(q)$ obtained numerically and calculated on the base of Eq. (44) is presented in Fig 8. We can conclude that a theoretical formula works in the limited interval of q (small $|q|$) and satisfactory describes exponent $\tau(q)$ for relatively small $1 < N < \infty$. However, we can note that there is a possibility to improve theoretical result by inclusion of higher-order correlations $\xi^{(n)}$, that leads to the processes beyond Markov theory.

4 Probability of Transition

In this Section, we would like to discuss some properties of the kernel of multiplicative chain (39) defined at small $|q|$:

$$K_{i_1, i_N}(q, N) \equiv \sum_{i_2=1}^8 \dots \sum_{i_{N-1}=1}^8 \prod_{t=2}^N \mathcal{P}^t(i_{t-1}|i_t). \quad (47)$$

First, introducing the vector

$$K_{i_1}(q, N) \equiv \sum_{i_N=1}^8 K_{i_1, i_N}(q, N), \quad (48)$$

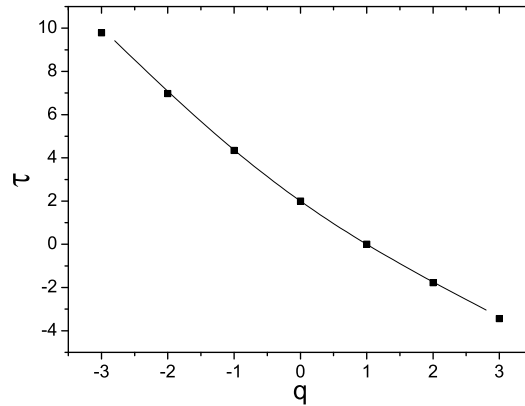


Рис. 8: Multifractal behavior of the exponent $\tau(q)$ for $N = 5$ and $(a = 0.8, \alpha = \pi/3)$. The points and the curve are obtained on the base of partition functions (21), (44), respectively.

it is worthy to note that this expression allows one the following representation:

$$K_i(q, N) = K_+(q, N)\delta_i^+ + K_-(q, N)\delta_i^-, \quad (49)$$

which is obtained on the base of semi analytical observations and reflects the symmetry of the lattice in our model; δ_i^\pm is defined by Eq. (34). Due to this expansion, Eq. (39) is re-written as

$$\mathcal{Z}_N(q) \simeq 4 \exp(q\ell_+)K_+(q, N) + 4 \exp(q\ell_-)K_-(q, N). \quad (50)$$

Next, it is useful to define the probability of transition from initial state i to final state f for N steps:

$$P_{i,f}(q, N) \equiv \frac{K_{i,f}(q, N)}{K_i(q, N)}, \quad \sum_{f=1}^8 P_{i,f}(q, N) = 1. \quad (51)$$

In principal, using $P_{i,f}(q, N)$, one can determine a more probable final state f for a given q , N , and initial state i . We can also verify that

$$P_{i,f}(0, N \rightarrow \infty) \rightarrow \frac{1}{8}. \quad (52)$$

It means that the chain (39) is ergodic at $q = 0$. The same conclusion is obtained by investigation of the mean value \bar{l} (the mean step of graphs) which is introduced in Eq. (42) and equal to

$$\bar{l} \equiv \lim_{N \rightarrow \infty} \frac{\bar{L}}{N^2} \rightarrow \frac{1}{2}\bar{\xi}. \quad (53)$$

This result is also independent on initial condition.

5 Discussion

Generally, we have concerned here with calculation of the series over Fuchsian group as group of symmetries of hyperbolic octagonal lattice with one independent complex module. To develop an analytical approach, we have considered the model of directed random walk on 8-branching Cayley tree as dual lattice for a given octagonal lattice. Then, the truncated series over symmetry group was treated as the partition function with Boltzmann weights for ensemble of lengths of tree graphs. Each of these lengths was defined by the sum of hyperbolic distances between the tree sites visited by directed walker and the origin.

As the result, we have analytically derived the spectrum of lengths (37) leading to the partition function (39) in the form of the Markov multiplicative chain allowing us to apply the central limit theorem. It says about randomizing environment, where spatial evolution is developed, and applicability of a random walk method in our calculations. Moreover, we have explored the model within the multifractality concept giving us a spectrum of scaling exponents.

In principal, the investigations performed might be mapped into the problem of calculation of the Selberg trace [20], intensively used for quantization of systems on the Riemann surfaces of genus two and requiring the knowledge of geodesics length spectrum (see Ref. [16], for instance). Also, in order to obtain the field observables on Riemann surface of the fixed genus and geometry, we can “symmetrize” these quantities under the action of corresponding group, that often coincides with the Dirichlet or Poincaré series computation. We only note a slow convergence of these series, that requires to use the accurate approximations which we have touched in this paper.

The bibliography

1. M.H. Jensen, L.P. Kadanoff, A. Libchaber, I. Procaccia, J. Stavans, Phys. Rev. Lett. **55**, 2798 (1985).
2. U. Frisch, G. Parisi, in: *Turbulence and Predictability of Geophysical Flows and Climate Dynamics*, Eds. M. Ghil, R. Benzi, G. Parisi (North-Holland, New York, 1985).
3. H. Stanley, P. Meakin, Nature **335**, 405 (1988).
4. A.V. Nazarenko, Int. J. Mod. Phys. **B25**, 3415 (2011); e-print arXiv: math-ph/1112.2278 (2011).
5. C. Vanderzande, *Lattice models of polymers* (Cambridge University Press, 1998).
6. B.D. Hughes, *Random Walks and Random Environments* **1** (Clarendon Press, Oxford, 1995).
7. C. Chamon, C. Mudry, and X.-G. Wen, Phys. Rev. Lett. **77**, 4194 (1996).
8. H.E. Castillo, C. Chamon, E. Fradkin, P.M. Goldbart, and C. Mudry, Phys. Rev. **B56**, 10668 (1997).
9. J. Feder, *Fractals* (Plenum Press, New York, 1988).
10. D. Sornette, *Critical phenomena in natural sciences: chaos, fractals, selforganization, and disorder: concepts and tools* (Springer-Verlag, Berlin, 2000).
11. V. Blavatska, W. Janke, Phys. Rev. Lett. **101**, 125701 (2008).

12. B.A. Dubrovin, S.P. Novikov, and A.T. Fomenko, *Modern Geometry. Methods and Applications* (Springer-Verlag, 1984-1990).
13. R. Voituriel, S. Nechaev, J. Phys. **A33**, 5631 (2000).
14. A. Comtet, S. Nechaev, R. Voituriel, J. Stat. Phys. **102**, 1 (2001).
15. M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
16. H. Ninnemann, Int. J. Mod. Phys. **B9**, 1647 (1995).
17. A. Aigon-Dupuy, P. Buser, M. Cibils, A.F. Kunzle, and F. Steiner, J. Math. Phys. **46**, 033513 (2005).
18. T.C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986); T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman, Phys. Rev. **A33**, 1141 (1986).
19. D. Katzen, I. Procaccia, Phys. Rev. Lett. **58**, 1169 (1987); M.H. Jensen, L.P. Kadanoff, I. Procaccia, Phys. Rev. **A36**, 1409 (1987).
20. N.E. Hurt, *Geometric Quantization in Action: Applications of Harmonic Analysis in Quantum Statistical Mechanics and Quantum Field Theory* (D. Reidel Publishing Company, 1983).

Стаття надійшла до редакції 5.11.2013
прийнята до друку 3.12.2013

СПРЯМОВАНІ ВИПАДКОВІ БЛУКАННЯ НА ВОСЬМИ-ГАЛУЖЕНОМУ ДЕРЕВІ КЕЙЛІ

А. Назаренко

*Інститут теоретичної фізики імені М.М. Боголюбова
вул. Метрологічна 14-б, 03680 Київ, Україна
e-mail: nazarenko@bitp.kiev.ua*

При моделюванні хаотичних систем важливо розглядати ансамбль спрямованих випадкових блукань та блукань із самоуніканнями на восьмигалуженому дереві Кейлі, згенерованому групою Фукса ріманової поверхні роду два, розміщеної на одиничному диску Пуанкаре. Розвинуто опис дво-параметричних ґраток і обчислено мультифрактальні показники моментів розподілу довжин графу як функцій цих параметрів. Представлено результати чисельних та статистичних досліджень, що проведені в рамках моделі випадкових блукань.

Ключові слова: спрямовані випадкові блукання; дерево Кейлі

НАПРАВЛЕННЫЕ СЛУЧАЙНЫЕ БЛУЖДЕНИЯ НА ВОСЬМИ-ВЕТВЛЕННОМ ДЕРЕВЕ КЕЙЛИ

А. Назаренко

*Институт теоретической физики имени Н.Н. Боголюбова
ул. Метрологическая 14-б, 03680 Киев, Украина
e-mail: nazarenko@bitp.kiev.ua*

При моделировании хаотических систем важно рассматривать ансамбль направленных случайных блужданий и блужданий без самопересечений на восьми-ветвленном дереве Кейли, сгенерированном группой Фукса римановой поверхности рода два, размещенной на единичном диске Пуанкаре. Развито описание дво-параметрических решеток и рассчитаны мультифрактальные показатели моментов распределения длин графа как функций этих параметров. Представлены результаты численных и статистических исследований, проведенных в рамках модели случайных блужданий.

Ключевые слова: направленные случайные блуждания; дерево Кейли