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Quantum evolution on torus for two spins with isotropic Heisenberg interaction

A.R. Kuzmak[1], V.M. Tkachuk[2]

Department for Theoretical Physics,
Ivan Franko National University of Lviv,
12 Drahomanov St., Lviv, UA-79005, Ukraine
e-mail: [1]andrijkuzmak@gmail.com, [2]voltkachuk@gmail.com

The quantum system of two-spin with isotropic Heisenberg interaction is considered. We study the quantum evolution of such system placed in the magnetic field. It is shown that this evolution happens on manifold which is torus.

Key words: quantum evolution, manifold, spins system.

1 Introduction

For understanding of the dynamic of quantum system it is useful to investigate manifolds which contain all states that can be reached during this dynamic. For instance, whole state space of two-level system (qubit) can be represented by 2D sphere, called the Bloch sphere. The trajectory of quantum evolution between two states is a curve between two points on this sphere (see, for example, [1–3]). A geometric approach to study evolution of multilevel quantum system (qudit) was developed in [4–8]. In [4–7] it was shown that for qubits system, the problem of finding of optimal quantum circuit of a unitary operation is closely related to the problem of finding of the minimal distance between two points on the Riemannian metric. A similar problem was considered for the case of n qutrits in [8]. The authors of this work showed that the optimal quantum circuit is equivalent to the shortest path between two points in a certain curved geometry of $SU\left(3^n\right)$. The geometrical properties of some well known coherent-state manifolds were studied in [9, 10]. More about geometry features of multilevel quantum systems can be found in [11–14].

In the previous paper [15] we studied evolution of two interacting spins in the framework of the quantum brachistochrone problem. In this paper we study general properties of quantum evolution of two-spin system with isotropic Heisenberg interaction in external magnetic field. We show that the final state which can be achieved for such system is defined by two real parameters, namely, the period of time of evolution and the value of the magnetic field. We obtain the Fubini-Study metric of manifold defined by this state and show that this manifold is torus.

This paper is organized as follows. In Section 2 the quantum evolution of two-spin system which is represented by the isotropic Heisenberg model is considered. The Fubini-Study metric of manifold defined by this state is obtained in Section 3. Conclusions are presented in Sec. 4.

2 The quantum evolution of two-spin system

We consider a two-spin system represented by the isotropic Heisenberg Hamiltonian. The system is placed in an external magnetic field directed along the z-axis. The Hamiltonian of the system is as follows

$$H = H_{int} + H_{mf}, \tag{1}$$

with

$$H_{int} = J\left(\sum_{i=x,y,z}\sigma_i^1\sigma_i^2 + 1\right), \tag{2}$$

$$H_{mf} = h_z\left(\sigma_z^1 + \sigma_z^2\right), \tag{3}$$

where $\sigma_i^1 = \sigma_i \otimes 1$, $\sigma_i^2 = 1 \otimes \sigma_i$, σ_i are the Pauli matrices, J is the interaction coupling which is a constant, h_z is proportional to the strength of the magnetic field. Let us consider quantum evolution of a two-spin system with this Hamiltonian in detail.

Hamiltonian (1) has four eigenvalues, namelly, $2(J+h_z)$, $2(J-h_z)$, $2J$, $-2J$ with the corresponding eigenvectors

$$|\uparrow\uparrow\rangle, \tag{4}$$

$$|\downarrow\downarrow\rangle, \tag{5}$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \tag{6}$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{7}$$

Taking into account that H_{int} commutes with H_{mf}, we can represent the evolution operator in the following form

$$U(t) = e^{-iJH_{int}t}e^{-ih_z\sigma_z^1 t}e^{-ih_z\sigma_z^2 t}, \tag{8}$$

where

$$e^{-iJH_{int}t} = \cos(2Jt) - \frac{i}{2J}\sin(2Jt)H_{int}, \tag{9}$$

Here we use the fact that $H_{int}^2 = (2J)^2$. We set $\hbar = 1$. It means that the energy is measured in frequency units. In the basis labelled as $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, the evolution operator $U(t)$ can be represented as

$$U(t) = \begin{pmatrix} e^{-i2(h_z+J)t} & 0 & 0 & 0 \\ 0 & \cos(2Jt) & -i\sin(2Jt) & 0 \\ 0 & -i\sin(2Jt) & \cos(2Jt) & 0 \\ 0 & 0 & 0 & e^{i2(h_z-J)t} \end{pmatrix}. \tag{10}$$

Let us consider the evolution of the system of two spins having started from the initial state $|\psi_i\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle$, with the normalization condition $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. The action of the evolution operator (8) on the state $|\psi_i\rangle$ is as follows

$$|\psi\rangle = U(t)|\psi_i\rangle = ae^{-i(\phi+\theta)}|\uparrow\uparrow\rangle + (b\cos\theta - ic\sin\theta)|\uparrow\downarrow\rangle$$
$$+ (-ib\sin\theta + c\cos\theta)|\downarrow\uparrow\rangle + de^{i(\phi-\theta)}|\downarrow\downarrow\rangle, \tag{11}$$

where

$$\theta = 2Jt, \quad \phi = 2h_z t. \tag{12}$$

Note that this state is defined by two real parameters θ and ϕ. An arbitrary quantum state of two qubits contains six real parameters. So we can conclude that we cannot reach an arbitrary state of two-spin system, which is represented by the Hamiltonian (1).

It is easy to see from (11) that the following equalities are satisfied

$$|\psi(\theta+\pi,\phi)\rangle = -|\psi(\theta,\phi)\rangle,$$
$$|\psi(\theta,\phi+2\pi)\rangle = |\psi(\theta,\phi)\rangle. \tag{13}$$

So, modulo a global phase this state is periodic with period π for θ and with period 2π for ϕ. This means that parameters θ and ϕ belong to the intervals $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$, respectively. In the next section in order to investigate the properties of the manifold we consider the Fubini-Study metric.

3 The Fubini-Study metric

The Fubini-Study metric is the infinitesimal distance ds between two neighbouring pure quantum states $|\psi(\xi^\alpha)\rangle$ and $|\psi(\xi^\alpha + d\xi^\alpha)\rangle$ [16–19]. It is given by the following expression

$$ds^2 = g_{\alpha\beta}d\xi^\alpha d\xi^\beta, \tag{14}$$

where ξ^α is a set of real parameters which define the state $|\psi(\xi^\alpha)\rangle$. The components of the metric tensor $g_{\alpha\beta}$ have the form

$$g_{\alpha\beta} = \gamma^2\Re\left(\langle\psi_\alpha|\psi_\beta\rangle - \langle\psi_\alpha|\psi\rangle\langle\psi|\psi_\beta\rangle\right), \tag{15}$$

where γ is an arbitrary factor which is often chosen to have value 1, $\sqrt{2}$ or 2 and

$$|\psi_\alpha\rangle = \frac{\partial}{\partial\xi^\alpha}|\psi\rangle. \tag{16}$$

Let us calculate metric of the manifold defined by state (11). This state is determined by two real parameters θ and ϕ. First of all, it is necessary to calculate the following derivatives

$$|\psi_\theta\rangle = -iae^{-i(\phi+\theta)}|\uparrow\uparrow\rangle + (-b\sin\theta - ic\cos\theta)|\uparrow\downarrow\rangle$$
$$+ (-ib\cos\theta - c\sin\theta)|\downarrow\uparrow\rangle - ide^{i(\phi-\theta)}|\downarrow\downarrow\rangle,$$
$$|\psi_\phi\rangle = -iae^{-i(\phi+\theta)}|\uparrow\uparrow\rangle + ide^{i(\phi-\theta)}|\downarrow\downarrow\rangle. \tag{17}$$

Using these results, we obtain the following scalar products

$$\langle\psi|\psi_\theta\rangle = -i\left[1-B\right], \quad \langle\psi|\psi_\phi\rangle = -iD,$$
$$\langle\psi_\theta|\psi_\theta\rangle = 1, \quad \langle\psi_\phi|\psi_\phi\rangle = A, \quad \langle\psi_\phi|\psi_\theta\rangle = D, \tag{18}$$

where

$$A = |a|^2 + |d|^2, \quad B = |b-c|^2, \quad D = |a|^2 - |d|^2. \tag{19}$$

Substituting (18) into (15), we obtain

$$ds^2 = \gamma^2\left[B\left(2-B\right)(d\theta)^2 + \left(A - D^2\right)(d\phi)^2 + 2DBd\theta d\phi\right]. \tag{20}$$

It is worth noting that the components of the metric tensor do not depend on the parameters θ an ϕ. Also, as we mentioned earlier, the state (11) is periodic modulo a global phase with period π for θ and 2ϕ for ϕ. So, we conclude that the expression (20) describes metric of the torus.

4 Conclusion

In this article we considered the quantum system of a two spins represented by the isotropic Heisenberg Hamiltonian. The quantum evolution of such system placed in external magnetic field was studied. We concluded that the evolution of the system is defined by two real parameters, namely, period of time of evolution and value of the magnetic field. Therefore, the evolution happens on two parametric manifold. We calculated the metric of this manifold and showed that the manifold is torus.

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Квантова еволюція на торі для двох спінів з ізотропною взаємодією Гайзенберґа

А. Р. Кузьмак[1], В. М. Ткачук[2]

Львівський національний університет імені Івана Франка,
кафедра теоретичної фізики,
Україна, Львів, вул. Драгоманова 12, 79005
e-mail: [1]*andrijkuzmak@gmail.com,* [2]*voltkachuk@gmail.com*

Розглядається квантова система двох спінів з ізотропною взаємодією Гайзенберґа. Ми досліджуємо квантову еволюцію цієї системи у магнітному полі. Показано, що ця еволюція відбувається на многовиді, який є тором.

Ключові слова: квантова еволюція, многовид, спінова система

Квантовая эволюция на торе для двух спинов с изотропным взаимодействием Гейзенберга

А. Р. Кузьмак[1], В. М. Ткачук[2]

Львовский национальный университет имени Ивана Франко,
кафедра теоретической физики,
Украина, Львов, ул. Драгоманова 12, 79005
e-mail: [1]*andrijkuzmak@gmail.com,* [2]*voltkachuk@gmail.com*

Рассматривается квантовая система двух спинов с изотропным взаимодействием Гейзенберга. Мы исследуем квантовую эволюцию этой системы в магнитном поле. Показано, что эта эволюция происходит на многообразии, которое является тором.

Ключевые слова: квантовая эволюция, многообразие, спиновая система.