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LONG RANGE ORDER IN THE "NET FRACTAL" SYSTEMS

R. Jaroszewicz, Z. Bak, W. Gruhn

Institute of Physics, Jan Dlugosz University
Al. Armii Krajowej 13/15, 42-200 Czestochowa, Poland
e-mail: z.bak@ajd.czest.pl

We define a specific class of fractals as "net fractals" and prove that in the logarithmic scale they are isomorphic with some bulk crystals. This offers possibility to describe physical phenomena in the way that is reminiscent of conventional formalism developed for crystalline systems. With the use of logarithmic coordinates, we study fracton excitations, magnetic interactions and phase transitions in the "net fractal" systems. Furthermore, we prove that in the "net fractal" magnetic system the indirect exchange, mediated by itinerant electrons, can be presented in the form that is reminiscent of the RKKY interaction. Finally, we prove that both RKKY and phase transitions are determined by the value of the effective spectral dimension.

Key words: fractals, fractons, RKKY exchange.

A low-dimensional system is one where the motion of particles or propagation of interactions is restricted from exploring the full three dimensions. There are many systems, of great interest in physics and technology, such as superlattices, fiber networks or high-temperature superconductors, in which the effects of reduced dimensionality are assumed to be essential. Potential device applications and unusual physical properties make these systems extremely interesting. The main focus of theoretical studies is understanding the combined effect of reduced-geometry and interactions. In low-dimensional systems the confinement affects the dynamics of mobile entities that travel over the system, and it is the source of new physics underlying the intriguing phenomena in confined many-body systems. The fractality of physical system can be generated two-ways, it can arise due to the fractality of underlying medium or due to the fractality of the process. We limit our study to systems with a hereditary fractality due to the geometrical structure. Below we define a specific class of fractals referred to as the "net fractals", and basing on a specific form of their self-similarity we prove that they are isomorphic with some bulk crystals [1]. This allows us to describe physical phenomena in the spirit of the solid state theory. With the use of logarithmic coordinates, we study fracton excitations, magnetic interactions and phase transitions in the "net fractal" systems. We show that there can exist two types of fracton modes: the acoustic-like and the optical-like ones. Furthermore, we prove that in the "net fractal" magnetic system the indirect exchange, can be presented in the form that is reminiscent of the RKKY interaction characteristic for a system of fractional spectral dimension [2].

A self-similar symmetry of a fractal is a transformation that leaves the system invariant, in the sense that, taken as a whole it looks the same after transformation as it did before, although individual points of the pattern may be moved by the transformation. We say that $K \subset R^n$ satisfies the scaling law S, or is a self-similar fractal, if $S : K = K$. Let us limit our considerations to fractals in which the self-similarity can be realized only via linear maps, i.e., transformations which point $r=(x_1, x_2, x_3) \in K \subset R^3$ transform into point $r' = (x_1', x_2', x_3')$ according to the formula $x_i' = S_{i1} x_1 + S_{i2} x_2 + S_{i3} x_3$; where $i = 1, 2, 3$. The vector form of the linear self-similar transformation can be written as $r'=S{:}r$, where S is the matrix of the linear self-similar transformation. If we orient coordinate axes along the eigenvectors of matrix S (i.e., $x = (x_1, x_2, x_3) \to (\varepsilon, \eta, \rho)$), then the linear self-similar mapping reduces to the transformation S: $(\varepsilon, \eta, \rho) \to (\lambda_1 \varepsilon, \lambda_2 \eta, \lambda_3 \rho)$. In the case of infinite-size fractals also the inverse S^{-1} mapping fulfills the self-similarity conditions S^{-1}: $K = K$ and for any $x \in K$, we have:

$$S^{-1} : x = S_1^{-1} \cdot S_2^{-1} \cdot S_3^{-1} : x = \left(\lambda_1^{-1}\varepsilon, \lambda_2^{-1}\eta, \lambda_3^{-1}\rho\right). \quad (1)$$

Consider a more general transformations of the type $S^{(m,n,l)} = (S_1)^n \cdot (S_2)^m \cdot (S_3)^l$, where $(S_i)^n$ denotes n-tuple superposition of transformation S_i, and define a class of infinite "net fractals" G_{nf}, for which the relation $S^{(m,n,l)}: G_{nf} = G_{nf}$ is valid. Action of $S^{(m,n,l)}$ transforms any point $x \in R^3$ according to the formula $S^{(m,n,l)}$: $x = (\lambda_1^n \varepsilon, \lambda_1^m \eta, \lambda_3^l \rho)$, where m, n, l are arbitrary (negative or positive) integers. In view of this relation we have that $S^{(m,n,l)} : G_{nf} \subset G_{nf}$, i.e. $S^{(m,n,l)}$ are the injective scaling mappings. For any linear S_1 and $F_1 \subset R$ by definition we have S_1: $F_1 = F_1$ and for any $x_o \in F_1$ we have S_1: $x_o = \lambda_1 x_o$, consequently $(S_1)^m$: $x_o = \lambda_1^m$: x_o. Using the logarithmic scale we have $\log(x_m/x_o) = m \ln \lambda_1$ ($m = \pm1, \pm2, \ldots$). This is nothing but a 1D crystal lattice with the lattice spacing given by $a_1 = \ln \lambda_1$. Using the multi-logarithmic scale we can see that the family of mappings $S^{(m,n,l)}$ is isomorphic with a 3D crystal lattice. This means that the isomorphism $S^{(m,n,l)} \leftrightarrow (ma_1, na_2, la_3)$ holds. The very same that refers to the placement of its characteristic building blocks. To show the crystal structure of self-similar fractal in the log-scale let us consider the triadic Cantor set (CS). The Cantor set is created by repeatedly deleting the open middle thirds of the interval [0, 1]. The construction of the CS starts by deleting the open middle third leaving the two segments $[0, 1/3] \cup [2/3, 1]$. In the next step the open middle third of each remaining segments is left behind. The process is continued ad infinitum. Let us now picture this procedure in the logarithmic, $\log_3$ -scale coordinates. The initial interval [0, 1] in the $\log_3$-scale is mapped onto the half-line $[-\infty, 0]$. We can illustrate this mapping in the following way, the interval [0,1] can be presented as an infinite sum of disjoint subsets $T_n = [3^{-n-1}, 3^{-n}]$. In the $\log_3$-scale each T_n is transformed into the interval $t_n = [-n - 1, -n]$ being the unit cell of the half-infinite, log-scale crystal. The set obtained after the first step of the Cantor procedure (pictured in the $\log_3$-scale) is the union of two intervals $[-\infty, -1] \cup [\log_3 2 - 1, 0]$. After the second step the log picture of Cantor procedure is given by the union of intervals $[-\infty,-2] \cup [\log_3 2,-2] \cup [\log_3 2-1, \log_3 7-2] \cup [\log_3 8-2,0]$ etc. We can see, at every step k the number of segments is doubled, and the picture of points that belong to the interval $[-n-1,-n]$ is identical with the picture of those points of the CS that belong to the preceding unit interval $[-n,-n+1]$ at the preceding stage of Cantor construction. The only exception is the appearance of subset $CS_{\log}^{[01]}$, i.e., of the points of the CS set that, in the $\log_3$-scale, fall into the $[-1, 0]$ interval. At every step of the Cantor procedure, there arise essential changes in the appearance of the $CS_{\log}^{[01]}$, subset, which always gains a novel, more complicated structure. The $\log_3$ picture of the first few steps of the $CS_{\log}$ construction is presented in the fig. 1.

We can summarize the log-scale construction as follows. After every step of the Cantor construction there arises new structure of the $CS_{log}^{[01]}$ subset, while the remaining part of the picture is identical with that obtained after preceding step of the Cantor procedure, although moved to the left by unit segment. This means that each unit interval [-*n*-1,-*n*] undergoes the same reduction scheme (but with some delay with respect to the number of steps). Consequently, when the Cantor procedure is continued ad infinitum each unit segment [-*n*-1,-*n*] becomes identical. As an effect of that the Cantor set (in the log scale) is mapped onto semi-infinite 1D crystal lattice. Evidently, if we consider the infinite-size CS_{∞} in the $\log_3$ scale picture it covers the unlimited 1D crystal lattice. We should point out here, that the "unit cell" of this lattice has a complex, Cantor like, structure.

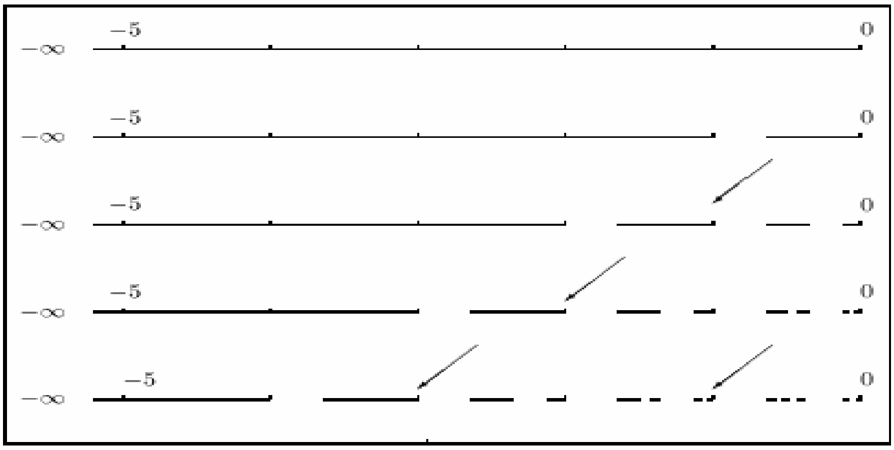

Fig. 1. The first few steps in constructing Cantor set pictured in the $\log_3$-scale

The purpose of this paper is to study the vibrations in a system deformable over a fractal subset. Most theoretical studies of the vibrations of a fractal limit considerations to a universal level without referring to the specific physical model. In our study we focus our considerations on a specific model which, we believe, describes the behavior of some real systems. Consider a "net fractal" cluster as defined above, consisting of N atoms with unit mass and linear springs connecting nearest-neighbour sites. The equations of motion of the atoms are [3]:

$$\ddot{u}_n(t) + \sum_m k_{n,m} u_n(t) = 0\,, \tag{2}$$

where the sum goes over all nearest neighbours sites of the fractal site *n*.

It is natural to assume, that the self-similarity of the fractal is reflected also in the dilation symmetry. Assuming that ω is the eigenfrequency of the fracton oscillations, we can find that the force constants k_i scale as $k_i = m_k \omega^2_k \propto (r/a)^d\ \omega^2_k$. In view of the latter relation, from here on, we assume that the forces which tend to restore the equilibrium positions of species, are linear (as regards to the coordinates of the excited fractal system). However, contrary to the conventional solid the elastic constants are not homogenous and depend on coordinates. Let us assume that elastic forces follow σ the common power law scaling with the separation $\sigma(\lambda x) = \lambda^{-\alpha}\,\sigma(x)$ [4]. As we have shown above, when presented in the logarithmic coordinates, the mass density of such a fractal becomes uniform; the same refers to the elastic constants. Suppose the fractal is perturbed locally (e.g. in the vicinity of the equilibrium position x_o, with the energy ε_o) and consider the amplitude of this excitation. In real space the amplitude of local fluctuation has the form $u_n = |x^o_n - x_n|$, while in the log coordinates we have $\zeta_n = |\xi^o_n - \xi_n|$, where $\zeta_n = \ln x_n$. Consider first a somewhat unrealistic case when there are no broken bonds in the log-scale picture. In this case (in the log scale) we have a homogenous system with uniform mass and elastic constant distribution.

Under conditions above, application of the continuous medium approximation is justified. Thus, when perturbed, the log coordinates ζ_n and the local displacement $\zeta_n (x,t)$ should satisfy the classical wave-equation $\nabla^2\zeta - (1/c^2)(\partial^2\zeta/\partial t^2) = 0$ with the plane wave solution $\zeta_i = \zeta_i^o \exp(ik_i\zeta_i - i\omega t) = \zeta_i^o (x_i)^{ik_i} e^{i\omega t}$, when the relation $\zeta_n = \ln x_n$ is taken into account. As we can see from above, the fracton appears to be the log scale phonon. When transformed to physical space, the log-scale phonon solution displays power law scaling with purely imaginary scaling exponent. The extensive discussion of the systems with complex scaling factors was given by Sornette [5], who has proved that this type of scaling results in the log-periodic oscillations of the physical quantities.

Evidently along with a conventional fracton vibrations (counterparts of optic phonons in solids), which are associated with the non-primitive character of the “unit cell” of the log-scale crystal there “acoustic“ fractons being counterparts of acoustic phonons in conventional solids. The “acoustic”, log-phonon represents this kind of the crystal lattice where all points of the elementary cell (i.e., of the subset $t_n \subset [-n-1,-n] \in CS_{log}$) move (vibrate) coherently as the whole. Contrary to the latter displacements associated with the optic phonons are always localized and limited to the “elementary cell” of the crystal lattice. Since each “elementary cell” of the log scale crystal is associated with an order of self-similar scaling present in the fractal (see fig. 1), the “optic fracton modes” transformed to real scale describe identical fractons with the amplitudes scaled by the Λ_n.

Results obtained above refer to the “net fractals”, which are ideal generalizations of the some real fractals. Most real fractals consist of backbone and attached to it sidebranches (dead ends). Thus the real fractals differ from the discussed above “net fractals” since they show the log-periodicity at most only along the backbone. It was suggested [6] that during vibration only the fractal backbone is stressed, while those parts of the mass, which are located on the sidebranches are moved along rigidly without being strained. The arguments given above suggest that in a real, dendritic fractal at most only one cascade of the fracton eigenmode can display the log-phonon features. The very same conclusion refers to other collective excitations on “net fractals” like magnons or photons.

Indirect magnetic interaction between spins arises due scattering of the electrons on the magnetic moments of the impurity ions. Conventional RKKY model of indirect coupling is valid provided that electron density is uniform and the dopant ions are distributed randomly within a matrix. However, in many cases the dopant ions show tendency towards clustering. These spontaneously patterned structures can be assembled in various geometries. The resulting clusters immersed within the matrix often show fractal symmetry. That's why different concepts which account the mutual interplay of underlying topology and magnetic interactions are still under debate [2, 7–9].

Consider a “net fractal” cluster, consisting of N localized magnetic moments. Let us discuss the indirect interactions between magnetic moments provided that some electrons in the fractal systems delocalize. As in magnetic clusters electrons delocalized on two magnetic ions suffice to form "double exchange" coupling it is evident that real fractals are large enough to allow manifestation of the RKKY-like indirect magnetic coupling. Similarly as in the case of fractons we assume that the self-similarity of the fractal is reflected also in the symmetry of exchange interactions. This means that magnetic "net fractal" is mapped onto a crystalline-like spin lattice. As the mobility of electrons which mediate the electrons is limited by the fractal geometry their spectrum and as result of that electron density is somewhat modified. As a rule, at least in a small energy window, the effective density of electron states scales as:

$$n(\varepsilon)d\varepsilon \propto (\varepsilon-\varepsilon_o)^{\alpha/2-1}d\varepsilon\ , \tag{3}$$

where α is the value of effective spectral dimension, where α can be a fraction. The most interesting fact is that in the case of low-dimensional systems the spectral dimension α as a rule differs from the geometrical dimension. For us it is important that indirect magnetic interactions are transmitted via low-energy excitations of the free electrons (or holes). In view of the above it is evident that the indirect magnetic interactions are governed by the values of the effective spectral dimension [2]. With the use of formula (3) the analytical expression for the RKKY exchange integral in the case of arbitrary spectral dimension α can be found:

$$I(\xi)=\frac{\xi^2}{\xi^{\alpha-2}}\Big[J_{\alpha/2-2}(\kappa\xi)Y_{\alpha/2-2}(\kappa\xi)+J_{\alpha/2-1}(\kappa\xi)Y_{\alpha/2-1}(\kappa\xi)+ +J_{\alpha/2}(\kappa\xi)Y_{\alpha/2}(\kappa\xi)\Big]\ , \tag{4}$$

where $J_a(\xi)$ and $Y_b(\xi)$ are the Bessel and Neumann functions respectively, and $\xi = \ln x$. In view of the above we can conclude that when the "net fractal" system is pictured in the logarithmic coordinates calculated the indirect exchange interactions exhibits RKKY-reminiscent features. The exchange integrals show conventional sign reversal oscillatory behaviour. The leading term in the exchange integrals $J(\xi)$ decays with the interspin separation ξ (i.e. measured in the log scale) as $J(\xi) \sim \xi^{-\alpha}$. This means that the envelope of the $J(\xi)$ is governed by the spectral dimension α.

Another problem that should be discussed is the possibility of magnetic ordering on a fractal substrate. As we know the topological (geometrical) dimension the dendritic fractals as the rule is lower than two. Thermodynamics predicts no collective ordering at $T>0$ in a system having dimension $D<2$. On the other hand experiments show that ferromagnetic ordering exists in some fractal systems. For example the phase separation in the doped $Pr_{1-x}\,Ca_xMnO_3$ systems results in formation of the ferromagnetic, charge delocalized regions at the nanometer level with the fractal symmetry [9]. To explain this contradiction, let us discuss the way in which the dimensionality enters into thermodynamical formulas.

To show that, that thermal evolution is governed by the value of the effective spectral dimension it is enough to recall formulas for the thermodynamical potential. For an ideal Fermi/Bose gas the grand potential reads [10]

$$\ln\Xi=\int_0^\infty n(\varepsilon)\ln\left(1\pm e^{-\beta\varepsilon}\right)d\varepsilon\ . \tag{5}$$

From eq. (5) one can easily see that all the information about the dimensionality of the actual system enters thermodynamical formulas *via* the density of states $n(\varepsilon)$. Thus, as we can see from the definition of ε the thermodynamical evolution of any system depends on its spectral dimension. It is well known that low-dimensional systems (e.g. so called quasi-2 dimensional ones or fractals) can exhibit the spectral dimensions

$\alpha>3$ [11]. Since we can have $\alpha>3$ one would expect that at least for some fractals there can exist collective phenomena e.g., ferromagnetism at T>0 even if the topological dimension D is less than two.

We have limited our study to a specific class of fractals for which the self-similarity transformations are given by the "net fractal" formula (1). In our considerations we use the logarithmic coordinates to describe the magnetic spin or ion positions instead of real-space morphology. The log-scale approach places the symmetry of linear deterministic "net fractals" within a common framework of the solid state symmetry. This points the way to a new understanding of the very nature of the collective physical phenomena in a "net fractal" systems. Detailed analysis of the indirect magnetic interaction indicates that underlying physics uncovers its simplicity when described in the logarithmic-scale. We show that in the log-scale magnetic interactions and fractal's excitations can be described in the spirit of the solid state theory. Furthermore, we point out that collective phenomena are governed by the value of the effective spectral dimension of a fractal.

As to the practical implications of our study let us note that although the main results were derived for the special class of "net fractals" it seems plausible that predicted effects can realize in the case of more general class of fractal systems. Further we believe that our study through the derivation of analytical results can be of considerable help for experimentalists when rapid and accurate estimates of magnetic exchange integrals are needed. Finally we hope that our results can be helpful in tailoring properties of spintronic devices.

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ДАЛЬНІЙ ПОРЯДОК В “МЕРЕЖЕВИХ РЕКУРСИВНИХ” СИСТЕМАХ

Р. Ярошевіч, З. Бонк, В. Грун

Інститут Фізики, Університет Яна Длугоша
Ал. Армії Крайової 3/15, 42 200 Ченстохова, Республіка Польща

Доведено, що в логарифмічному масштабі певний клас фракталів – “мережеві рекурсиви” є ізоморфними з деякими масивними кристалами, що дає змогу описати фізичні явища в них у рамках формалізму, розвиненого для кристалічних систем. Використовуючи логарифмічні координати, досліджено фрактальні збудження, магнітні взаємодії і фазові переходи в так званих “мережевих рекурсивних” магнітних системах. Показано, що в цих системах непряма обмінна взаємодія може бути представлена у формі, подібній до RKKY взаємодії. Доведено, що RKKY та фазові переходи визначаються значенням ефективної спектральної розмірності.

Ключові слова: фрактали, фрактони, RKKY взаємодія.

ДАЛЬНИЙ ПОРЯДОК В “СЕТЕВЫХ РЕКУРСИВНЫХ” СИСТЕМАХ

Г. Ярошевич, З. Бонк, В. Грун

Институт Физики, Университет Яна Длугоша
ал. Армии Краевой 3/15, 42 200 Ченстохова, Республика Польша

Доказано, что в логарифмическом масштабе определенный класс фракталов – “сетевые рекурсивы” – является изоморфным к некоторым массивным кристаллам, что дает возможность описать физические явления в них в рамках формализма, развитого для кристаллических систем. Используя логарифмические координаты, исследовано фрактальные возбуждения, магнитные взаимодействия и фазовые переходы в так называемых “сетевых рекурсивных” магнитных системах. Показано, что в этих системах непрямое обменное взаимодействие может быть представлено в форме, подобной к RKKY взаимодействию. Доказано, что RKKY и фазовые переходы определяются значением эффективной спектральной размерности.

Ключевые слова: фракталы, фрактоны, RKKY взаимодействие.

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