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# THE $N=4$ SUPERSYMMETRY OF ELECTRON IN THE MAGNETIC FIELD

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It is shown that the $N$=2,3,4 SUSY of Pauli Hamiltonian takes place in a three-dimensional magnetic field possessing spatial symmetry with respect to the inversion of coordinates. For example, the $N$=4 SUSY is realized in the field of magnetic octopole and the non-zero energy levels are 4-fold degenerated. We also show that Dirac equation possesses the $N$=2,3,4 SUSY in a three–dimensional magnetic field.

**Key words**: supersymmetry, quantum mechanics, magnetic field.

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## I. INTRODUCTION

The notion of supersymmetry was introduced into quantum field theory in order to unify bosons and fermions [1–3]. After the appearance of these works the idea of SUSY began to penetrate into other areas of physics and mathematics. Supersymmetrical quantum mechanics was proposed in papers [4, 5]. The motion of electrons in the magnetic field is an interesting example of quantum mechanical problem where SUSY is physical symmetry. It is known that supersymmetry is present in the case of an arbitrary two-dimensional magnetic field $B_z = B(x,y), B_x = B_y = 0$ as well as in a three-dimensional magnetic field $\mathbf{B}(-\mathbf{r}) = \pm\mathbf{B}(\mathbf{r})$ (see for example [6–8] and the references therein). The field of magnetic monopole is one of the examples where SUSY is realized in a three–dimensional case [10–12]. SUSY also takes place in the case of electron motion on the surface orthogonal to the magnetic field [9].

In the present paper we show that electron motion in a three–dimensional magnetic field with a somewhat different spatial symmetry with respect to the inversion of coordinates possesses the $N$=2,3,4 SUSY as well.

## II. SUPERSYMMETRIC QUANTUM MECHANICS

Suppose that Hamiltonian $H$ can be written in the form:

$$H = Q_0^2, \tag{2.1}$$

where $Q_0$ is a self-adjoint operator called supercharge. In addition let us postulate the existence of $n$ selfadjoint operators $T_i$ that anticommute with the supercharge:

$$\{\, Q_0, T_i\} = 0, \quad i = 1, ..., n, \tag{2.2}$$

and also fulfil the Klifford algebra:

$$\{\, T_i, T_j\} = 2\delta_{ij}. \tag{2.3}$$

As a result of (2.1) and (2.2) $T_i$ commutes with the Hamiltonian

$$[H, T_i] = 0. \tag{2.4}$$

Using the introduced operators we may construct supercharges

$$Q_j = iT_jQ_0, \quad j = 1, ..., n. \tag{2.5}$$

They fulfil $N = n+1$ superalgebra together with $Q_0$:

$$\{\, Q_i, Q_j\} = 2\delta_{i,j}H, \qquad i,j = 0,1,...,n, \tag{2.6}$$
$$[H, Q_i] = 0.$$

Note, that the method of constructing one-dimensional $N$-extended supersymmetrical quantum mechanics was suggested in [13].

The introduced operators $T_i$ are useful for the study of SUSY in real quantum mechanical systems. In sections 3 and 4 we are going to write operators $T_i$ in explicit form for Pauli and Dirac Hamiltonians.

## III. SUSY IN THE PAULI HAMILTONIAN

The Pauli Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 - \mu\boldsymbol{\sigma}\mathbf{B} \tag{3.1}$$

can be written in the form (2.1) where the supercharge is

$$Q_0 = \frac{1}{\sqrt{2m}}\boldsymbol{\sigma}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right). \tag{3.2}$$

Here $\mathbf{A}$ is an external vector potential, $\mathbf{B} = rot\mathbf{A}$ is the magnetic field.

### A. $N$=2 SUSY

Let us consider a three–dimensional magnetic field with the vector potential which possesses the following spatial symmetry with respect to the inversion of $z$:

$$\begin{aligned} A_x(x,y,-z) &= A_x(x,y,z), \\ A_y(x,y,-z) &= A_y(x,y,z), \\ A_z(x,y,-z) &= -A_z(x,y,z), \end{aligned} \tag{3.3}$$

where dependencies on $x$ and $y$ are arbitrary. The magnetic field in this case has the following properties:

$$\begin{aligned} B_x(x,y,-z) &= -B_x(x,y,z), \\ B_y(x,y,-z) &= -B_y(x,y,z), \\ B_z(x,y,-z) &= B_z(x,y,z). \end{aligned} \tag{3.4}$$

Now one can find the operator that anticommutes with the supercharge (3.2)

$$T = \sigma_z I_z. \tag{3.5}$$

Here $I_\beta$ is the operator of inversion in the direction of $\beta$ ($\beta = x, y, z$). Using the results of the previous section we obtain the $N$=2 SUSY with supercharges $Q_0$ and $Q_1$.

Note that magnetic field (3.4) covers the case of a two–dimensional field where SUSY was discovered earlier.

### B. $N$=3 SUSY

The $N$=3 SUSY is realized in a magnetic field with the vector potential which possesses the following spatial symmetry with respect to $y$ and $z$

$$\begin{aligned} A_x(x,-y,z) &= A_x(x,y,z), \\ A_y(x,-y,z) &= -A_y(x,y,z), \\ A_z(x,-y,z) &= A_z(x,y,z), \\ A_x(x,y,-z) &= A_x(x,y,z), \\ A_y(x,y,-z) &= A_y(x,y,z), \\ A_z(x,y,-z) &= -A_z(x,y,z), \end{aligned} \tag{3.6}$$

where the dependence on $x$ is arbitrary. The magnetic field has the opposite parity in comparison with $\mathbf{A}$ (3.6). Now there are two operators that satisfy (2.2) and (2.3)

$$T_y = \sigma_y I_y, \qquad T_z = \sigma_z I_z. \tag{3.7}$$

Thus, using (2.5) we shall come to the $N$=3 SUSY.

### C. $N$=4 SUSY

The $N$=4 SUSY is realized when the vector potential possesses the following spatial symmetry with respect to the inversion of $x, y$ and $z$:

$$A_\beta(-x_\beta) = -A_\beta(x_\beta), \quad \beta = x, y, z, \tag{3.8}$$

and is even with respect to other variables. The magnetic field which corresponds to the vector potential (3.8) has the opposite spatial symmetry

$$B_\beta(-x_\beta) = B_\beta(x_\beta), \tag{3.9}$$

and is odd with respect to other variables. $T$ operators here must be as follows

$$T_\beta = \sigma_\beta I_\beta, \quad \beta = x, y, z, \tag{3.10}$$

and using (2.5) we finally obtain the $N$=4 SUSY. The degeneracy of energy levels connected with the $N$ SUSY is equal to $2^{[N/2]}$, where $[N/2]$ means the integer part of the number. Thus, for the case of (3.8) energy levels are four-fold degenerated. As an interesting example of the $N$=4 SUSY system we can adduce the electron motion in the field of magnetic octopole.

To conclude this section we want to note that operators (3.10) satisfy simultaneously both (2.3) and the following algebra:

$$\begin{aligned} [T_\alpha, T_\beta] &= i2\epsilon^{\alpha\beta\gamma}\tilde{T}_\gamma, \\ [\tilde{T}_\alpha, T_\beta] &= i2\epsilon^{\alpha\beta\gamma}T_\gamma, \\ [\tilde{T}_\alpha, \tilde{T}_\beta] &= i2\epsilon^{\alpha\beta\gamma}\tilde{T}_\gamma, \end{aligned} \tag{3.11}$$

where

$$\tilde{T}_\alpha = I T_\alpha, \tag{3.12}$$

$I = I_x I_y I_z$ is the operator of full inversion. Hamiltonian commutes with $\tilde{T}_\alpha$ and $T_\alpha$. The supercharge $Q_0$ also commutes with $\tilde{T}_\alpha$. It is also interesting to note that the operator of inversion anticommutes with the supercharge $Q_0$:

$$\{ I, Q_0\} = 0, \tag{3.13}$$

but commutes with $T_\alpha$:

$$[I, T_\alpha] = 0. \tag{3.14}$$

Thus (3.11), (3.13) and (3.14) together with (2.1)–(2.4) fulfil the so-called generalized algebra of supersymmetrical quantum mechanics where commuting $T$ operators are present [14].

## IV. SUSY IN THE DIRAC HAMILTONIAN

The Dirac Hamiltonian reads:

$$H_D = c\boldsymbol{\alpha}(\mathbf{p} - \frac{e}{c}\mathbf{A}) + \beta mc^2, \tag{4.1}$$

where

$$\alpha_\gamma = \begin{pmatrix} 0 & \sigma_\gamma \\ \sigma_\gamma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

Let us introduce the matrices which are connected with the electron spin:

$$\Sigma_\gamma = \begin{pmatrix} \sigma_\gamma & 0 \\ 0 & \sigma_\gamma \end{pmatrix}. \tag{4.3}$$

The supercharge commuting with $H_D$ reads:

$$Q_0 = c\boldsymbol{\Sigma}(\mathbf{p} - \frac{e}{c}\mathbf{A}). \tag{4.4}$$

Squaring of $Q_0$ gives squared Dirac Hamiltonian

$$Q_0^2 = H_D^2 - m^2c^4 = H, \tag{4.5}$$

where $H$ can be called a new Hamiltonian.

Similarly to Pauli Hamiltonian, $H$ also possesses the $N$=2,3,4 SUSY in the case of respective fields (3.3), (3.6) and (3.8). $T$ operators for Dirac Hamiltonian read:

$$T_\alpha = \beta I_\alpha \Sigma_\alpha, \quad \alpha = x, y, z. \tag{4.6}$$

They anticommute with $Q_0$ (4.4):

$$\{\, Q_0, T_\alpha\} = 0 \tag{4.7}$$

and commute with $H_D$

$$[T_\alpha, H_D] = 0. \tag{4.8}$$

Thus $Q_0, Q_\alpha = \beta I_\alpha \Sigma_\alpha Q_0$ and $H$ also fulfil the SUSY algebra (2.6).


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## $N = 4$ СУПЕРСИМЕТРІЯ ЕЛЕКТРОНА В МАГНЕТНОМУ ПОЛІ


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Суперсиметрія була представлена вперше в квантовій теорії поля з метою об'єднати бозони та ферміони. Далі ідея суперсиметрії почала проникати в інші області фізики та математики. Суперсиметрична квантова механіка була запропонована в працях [4, 5]. Цікавим прикладом квантовомеханічної задачі, де суперсиметрія є фізичною симетрією, може бути рух електрона в магнетному полі. Добре відомо, що суперсиметрія реалізується в довільному двовимірному магнетному полі і у тривимірному полі $\mathbf{B}(-\mathbf{r}) = \pm\mathbf{B}(\mathbf{r})$. Суперсиметрія має місце при русі електрона по поверхні, ортогональній до магнетного поля. Як приклад SUSY в тривимірному полі можна також відмітити роботи, що стосуються суперсиметрії електрона в полі магнетного монополя.

В даній роботі показано, що $N$=2,3,4 SUSY реалізується в тривимірному магнетному полі, яке володіє певною просторовою симетрією відносно інверсії координат. Наприклад, $N$=4 SUSY реалізується в полі симетричного магнетного октуполя, і ненульові енергетичні рівні є 4-кратно вироджені. Показано, що рівняння Дірака теж володіє $N$=2,3,4 суперсиметрією в тривимірному магнетному полі.