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COMPREHENSIVE ANALYSIS OF MAGNETIC SUPERSTRUCTURES IN ANISOTROPIC SYSTEMS

N. Gonchar†, W. Kozyrski††, H. Hajduk
Department of Mathematical Modelling, N. Bogolubov Institute for Theoretical Physics, Ukrainian National Academy of Sciences, 14b Metrolohichna Str., Kyiv UA–252143, Ukraine
†*E-mail: mgonchar@gluk.apc.org,* ††*E-mail: kozyrski@gluk.apc.org*
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New method is developed for analytical treatment of modulated superstructures in model magnetic systems near criticality. We found the conditions for paramagnet–(in)commensurate transition in the class of anisotropic Ising models with and without an external magnetic field and the critical value of the latter. We derived a new representation of specific free energy for commensurate and incommensurate configurations and found its hidden symmetry. The existence of the global solution is proved for the incommensurate configuration minimizing free energy. We found the condition for the global solution's termination. The existence of phase transition from the incommensurate phase into the ferromagnetic one is proved to depend on exchange integrals.

Key words: magnetic systems; modulated superstructures, hidden symmetry, paramagnet–(in)commensurate transition, critical field.

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I. INTRODUCTION

The importance of a comprehensive theoretical treatment of modulated structures is well known [1]. The difficulties of studying the phase diagram for such systems are caused by the need to continue the free energy per lattice site from the commensurate configurations to incommensurate ones and constructing the extremum configurations. The interest to this problem does not diminish with time [2–7]. Here we deliver a new treatment of modulated structures in anisotropic Ising system near criticality. We derive new representation for free energy of both commensurate and incommensurate configurations and find its hidden symmetry (Sec.II). In Sec.III and IV we prove the existence of phase transition from the paramagnetic state to the modulated structures with and without external magnetic field. In Sec.IV we find the extreme magnetic field for which modulated structures still exist. In Sec.V we prove the existence of the global solution for the incommensurate configuration minimizing the free energy and derive the condition for this solution to terminate. In Sec.V the existence of the phase transition from the incommensurate phase into the ferromagnetic one is proved to depend on exchange integrals. This phase transition is of the first kind.

To be definite we illustrate the method on the ANNNI model [8] but one can apply it for models with an arbitrary number of interacting neighbors. The mean-field free energy for three-dimensional ANNNI model is

$$\bar{F}_h = \sum_{i<j} J_{ij} m_i m_j + \frac{1}{2} T \sum_i \big((1+m_i)\ln(1+m_i) + (1-m_i)\ln(1-m_i)\big) + h \sum_i m_i, \tag{1.1}$$

where m_i is an average magnetic moment at i-th site and takes values on the segment $[-1,1]$. In the planes normal to the x-axis the nearest neighbors only interact ferromagnetically ($J_0<0$). Along the axis x the next-nearest neighbors interact too, moreover the interactions of the nearest neighbors (J_1) and the next-nearest ones (J_2) can be either ferromagnetic or antiferromagnetic. Just the relations between them are essential. Expression (1.1) is formal because specific free energy only has the physical sense. Let us derive the explicit expression for it.

Denote by Λ_0 the parallelepiped from $\mathcal{R}^3$ of sizes aN_1, aN_2 and aN_3, where a is lattice spacing, $N_1 \times N_2 \times N_3$ is the number of sites in Λ_0. We consider the set of commensurate configurations on the integer-valued lattice $\mathcal{Z}^3$ with the periods N_1, N_2 and N_3. Denote by $\{\Lambda_k\}$ the sequence of volumes from $\mathcal{R}^3$ with N_k sites from $\mathcal{Z}^3$ and cover the volume Λ_k by parallelepipeds congruent to Λ_0 with their number inside Λ_k being n_1^k and the one completely covering Λ_k being n_2^k. If the sequence Λ_k tends to $\mathcal{R}^3$ and $\Lambda_k \subseteq \Lambda_{k+1}$, $\bigcup_{k=1}^{\infty} \Lambda_k = \mathcal{R}^3$, then

$$\lim_{k\to\infty} \frac{\left(n_2^k - n_1^k\right) N_1 N_2 N_3}{|\Lambda_k|} = 0, \tag{1.2}$$

where $|\Lambda_k| = N_k$ is the number of sites in Λ_k.

The specific free energy for commensurate configurations in Λ of the periods N_1, N_2, N_3 with boundary condition over Λ takes the form

$$\frac{\bar{F}_{hk}}{|\Lambda_k|} = \frac{1}{2|\Lambda_k|}\sum_{j=1}^{n_1^k}\sum_{i\in\Lambda_{oj}}\Big(m_i\big(J_1\left(m_{i+1}^{ox}+m_{i-1}^{ox}\right)$$

$$+ J_2\left(m_{i+2}^{ox}+m_{i-2}^{ox}\right) + J_0\left(m_{i+1}^{oy}+m_{i-1}^{oy}\right) + J_0\left(m_{i+1}^{oz}+m_{i-1}^{oz}\right)\big)$$

$$+ 2hm_i + T\big((1+m_i)\ln(1+m_i) + (1-m_i)\ln(1-m_i)\big)\Big) + \frac{\bar{F}_1^k}{|\Lambda_k|},$$

where the contribution of covering volumes $n_2^k - n_1^k$ accounting for the boundary conditions is denoted by $\bar{F}_1^k$. $m_{i+1}^{ox}, m_{i-1}^{ox}, m_{i+1}^{oy}, m_{i-1}^{oy}$ and so on are the average magnetic moments at sites $i+1$, $i-1$ along the corresponding axes.

The thermodynamical limit of specific free energy for commensurate configurations is equal to

$$F_h = \lim_{k\to\infty}\frac{\bar{F}_{hk}}{|\Lambda_k|} = \lim_{k\to\infty}\frac{n_1^k}{2|\Lambda_k|}\sum_{i\in\Lambda_o}\Big(m_i\big(J_1\left(m_{i+1}^{ox}+m_{i-1}^{ox}\right)$$

$$+ J_2\left(m_{i+2}^{ox}+m_{i-2}^{ox}\right) + J_0\left(m_{i+1}^{oy}+m_{i-1}^{oy}\right) + J_0\left(m_{i+1}^{oz}+m_{i-1}^{oz}\right)\big) + 2hm_i$$

$$+ T\big((1+m_i)\ln(1+m_i) + (1-m_i)\ln(1-m_i)\big)\Big) + \lim_{k\to\infty}\frac{\bar{F}_1^k}{|\Lambda_k|}$$

or [9]

$$F_h = \frac{1}{2N_1N_2N_3}\sum_{i\in\Lambda_o}\Big(m_i\big(J_1\left(m_{i+1}^{ox}+m_{i-1}^{ox}\right)$$

$$+ J_2\left(m_{i+2}^{ox}+m_{i-2}^{ox}\right) + J_0\left(m_{i+1}^{oy}+m_{i-1}^{oy}\right) + J_0\left(m_{i+1}^{oz}+m_{i-1}^{oz}\right)\big)$$

$$+ 2hm_i + T\big((1+m_i)\ln(1+m_i) + (1-m_i)\ln(1-m_i)\big)\Big).$$

Actually, the quasi-one-dimensional case is of interest, i.e., the configurations with the fixed average magnetic moment at the sites of planes parallel to the plane YOZ. The latter is valid for $J_0 < 0$ and a sufficiently low temperature T. In that case specific free energy with $m_{i+1}^{oy} = m_{i-1}^{oy} = m_{i+1}^{oz} = m_{i-1}^{oz} = m_i$ takes the form of

$$F_h = \frac{1}{2N_1}\sum_{i\in\Lambda_o'}\Big(m_i\big(J_1\left(m_{i+1}+m_{i-1}\right) + J_2\left(m_{i+2}+m_{i-2}\right) + 4J_0m_i\big) \tag{1.3}$$

$$+ 2hm_i + T\big((1+m_i)\ln(1+m_i) + (1-m_i)\ln(1-m_i)\big)\Big),$$

where Λ_o' is a piece of line with N_1 sites.

II. UNDERLYING SYMMETRY OF FREE ENERGY

Here we find the hidden one-parameter symmetry group of the free energy per lattice site in the space of commensurate and incommensurate configurations. At first we construct the new representation for specific free energy (1.3) of commensurate configurations and extend it onto the set of incommensurate ones. To do this we introduce the set of vectors $X = \{x_i\}_{i=-\infty}^{+\infty}$, where $i \in \mathcal{Z}_1$ and x_i takes values in the set of real numbers, and the scalar product

$$\langle x_1, x_2\rangle = \lim_{\varepsilon\to 0} \frac{\sum\limits_{i=-\infty}^{+\infty} e^{-\varepsilon|i|} x_i^{(1)} x_i^{(2)}}{\sum\limits_{i=-\infty}^{+\infty} e^{-\varepsilon|i|}},$$

with $x \in X$, if $\| x \|^2 = \langle x, x\rangle < +\infty$. The vector is periodic of the period N_1 if $x_{i+N_1} = x_i$ for $\forall i \in \mathcal{Z}_1$. Let N_1 be such an integer that $N = [N_1/2]$, where $[c]$ is the integral part of c, and $q = m/N_1$ is irreducible quotient. Next we introduce

$$\epsilon = \{a_i\}_{i=-\infty}^{+\infty}, \qquad a_i = 1 \text{ for } \forall i,$$

$$e_1(q) = \{\cos i2\pi q\}_{i=-\infty}^{+\infty}, \qquad e_2(q) = \{\sin i2\pi q\}_{i=-\infty}^{+\infty}.$$

Vectors

$$\{\epsilon, e_1(q), \cdots, e_1(Nq), e_2(q), \cdots, e_2(Nq)\}$$

belong to the set X, form the orthogonal system and are periodic of the period N_1. Any N_1-periodic configuration x can be expressed as

$$x = \sum_{l=1}^{N} \big(a_l e_1(lq) + b_l e_2(lq)\big) + a_0\epsilon. \tag{2.1}$$

Lemma 1. *For configuration* (2.1) *the formula*

$$\langle x, \epsilon\rangle = \frac{1}{N_1}\sum_{i=1}^{N_1} x_i \tag{2.2}$$

holds, where x_i is i-th component of the vector x.

For the proof see [10].

If $g(x)$ is some continuous function of $x \in [a, b]$, then

$$g(x) = \lim_{n\to\infty} P_n(x), \tag{2.3}$$

where $P_n(x)$ is the sequence of polynomials approximating $g(x)$ on $[a, b]$ uniformly. We consider the function

$$g(x) = (1 + x)\ln(1 + x) + (1 - x)\ln(1 - x)$$

and its derivatives. If the configuration $x = \{x_i\}_{i=-\infty}^{+\infty}$ is such that

$$|x_i| < 1 \;\; \forall i = \overline{-\infty, +\infty},$$

then $g(x)$ is defined correctly.

Using representation (4.1) for any N_1-periodic configuration we calculate the averages

$$I_{s,m} = \left\langle \prod_{k=1}^{s} e_1(i_k q) \prod_{l=1}^{m} e_2(j_l q), \epsilon \right\rangle$$

for any power n of this configuration. For $q = m/N_1$ the averages $I_{s,m}$ are expressed as [10]

$$I_{s,m} = \begin{cases} 0, \quad m = 2M+1, \quad M = 0, 1, 2, \cdots, \\ (-1)^M 2^{-s-2m} \sum\limits_{r=0,\pm1,\cdots,\pm\left[\frac{l}{N_1}\right]} \sum\limits_{\substack{t_{k'} = \pm i_{k'} \\ \tau_l = \pm j_l}} \delta_{t_1+\cdots+t_s+\tau_1+\cdots+\tau_m, |r|N_1} (-1)^k, \\ \qquad\qquad\qquad\qquad\qquad m = 2M, \end{cases} \tag{2.4}$$

k is the number of negative τ_l. For irrational q the averages $I_{s,m}$ are expressed by the formula

$$I_{s,m} = \begin{cases} 0, \quad m = 2M+1, \quad M = 0, 1, 2, \cdots, \\ (-1)^M 2^{-s-2m} \sum\limits_{\substack{t_{k'} = \pm i_{k'} \\ \tau_l = \pm j_l}} \delta_{t_1+\cdots+t_s+\tau_1+\cdots+\tau_m, 0} (-1)^k, \; m = 2M. \end{cases} \tag{2.5}$$

Due to Lemma 1 the specific free energy (1.3) has the form $F_h = \langle \epsilon, f(m)\rangle/2$, where $f(m) = \{f(m_i)\}_{i=-\infty}^{+\infty}$ with

$$f(m_i) = m_i\Big(J_1(m_{i+1} + m_{i-1}) + J_2(m_{i+2} + m_{i-2}) + 4J_0 m_i + 2h\Big)$$
$$+ T\Big((1 + m_i)\ln(1 + m_i) + (1 - m_i)\ln(1 - m_i)\Big).$$

For

$$m=\sum_{l=1}^{N}\big(a_l e_1(lq)+b_l e_2(lq)\big)+a_0\epsilon$$

we have

$$F_h=-\frac{1}{2}\langle m,Am\rangle+h\langle\epsilon,m\rangle+\frac{1}{2}T\langle\epsilon,g(m)\rangle=-\frac{1}{2}\Bigg(\frac{1}{2}\sum_{l=1}^{N}\Big(\Big(1+\delta_{l,[\frac{N_1+1}{2}]}\Big)a_l^2+\Big(1-\delta_{l,[\frac{N_1+1}{2}]}\Big)b_l^2\Big)\Phi(lq)$$

$$+a_0^2\Phi(0)\Bigg)+ha_0+\frac{1}{2}T\left\langle\epsilon,g\left(\sum_{l=1}^{N}(a_l e_1(lq)+b_l e_2(lq))+a_0\epsilon\right)\right\rangle, \tag{2.6}$$

where A is the operator defined as

$$(Am)_i=-\big(J_1(m_{i+1}+m_{i-1})+J_2(m_{i+2}+m_{i-2})+4J_0 m_i\big),$$

and the function g is

$$g(m)=(1+m)\ln(1+m)+(1-m)\ln(1-m)=\lim_{n\to\infty}2\left(\frac{m^2}{2}+\frac{m^4}{3\cdot 4}+\cdots+\frac{m^{2n}}{2n\cdot(2n-1)}\right)$$

provided that

$$\sum_{l=1}^{N}\big(|a_l|+|b_l|\big)+|a_0|<1.$$

For the ANNNI model

$$\Phi(q)=-2(J_1\cos 2\pi q+J_2\cos 4\pi q+2J_0)$$

and for the case when all the neighbors interact

$$\Phi(q)=-2\left(\sum_{i=1}^{+\infty}J_i\cos 2\pi iq+2J_0\right).$$

The free energy (2.6) is defined on the set of commensurate configuratons. The following Lemma defines specific free energy on the set of incommensurate configurations.

Lemma 2. *The thermodynamical limit of the free energy exists for irrational q on the set of incommensurate configurations*

$$m=\sum_{l=1}^{+\infty}\big(a_l e_1(lq)+b_l e_2(lq)\big)+a_0\epsilon$$

provided that

$$\sum_{l=1}^{+\infty}\big(|a_l|+|b_l|\big)+|a_0|<1.$$

For the proof see [10].

As a result free energy of configuration (2.1) with irrational q and arbitrary N is

$$F_h=-\frac{1}{2}\left(\frac{1}{2}\sum_{l=1}^{N}(a_l^2+b_l^2)\Phi(lq)+a_0^2\Phi(0)\right)+ha_0+\frac{1}{2}T\left\langle\epsilon,g\left(\sum_{l=1}^{N}\big(a_l e_1(lq)+b_l e_2(lq)\big)+a_0\epsilon\right)\right\rangle. \tag{2.7}$$

Let us find the symmetry group of free energy (2.6) for commensurate configurations of the period N_1. To do this we apply the previous representation of free energy (1.3). Due to N_1-periodicity of $m = \{m_i\}_{i=-\infty}^{+\infty}$

$$F_h = \frac{1}{2N_1}\sum_{i=1}^{N_1}\Big(m_i\big(J_1\,(m_{i+1}+m_{i-1}) + J_2\,(m_{i+2}+m_{i-2}) + 4J_0 m_i\big)$$
$$+\,2hm_i + Tg(m_i)\Big) = \frac{1}{2N_1}\sum_{i=1}^{N_1}\Big(m_{i+s}\big(J_1(m_{i+s+1}+m_{i+s-1})$$
$$+\,J_2(m_{i+s+2}+m_{i+s-2}) + 4J_0 m_{i+s}\big) + 2hm_{i+s} + Tg(m_{i+s})\Big) = F_h^s. \tag{2.8}$$

Exploiting this invariance in the new representation we convert expression (2.8) into

$$F_h^s = -\frac{1}{2}\left\langle \sum_{l=1}^{N}\big(a_l e_1^s(lq) + b_l e_2^s(lq)\big) + a_0\epsilon,\ \sum_{l=1}^{N}\big(a_l e_1^s(lq) + b_l e_2^s(lq)\big)\Phi(lq) + a_0\Phi(0)\epsilon\right\rangle \tag{2.9}$$
$$+\,h\left\langle \epsilon, \sum_{l=1}^{N}\big(a_l e_1^s(lq) + b_l e_2^s(lq)\big) + a_0\epsilon\right\rangle + \frac{1}{2}T\left\langle \epsilon, g\left(\sum_{l=1}^{N}\big(a_l e_1^s(lq) + b_l e_2^s(lq)\big) + a_0\epsilon\right)\right\rangle.$$

Using explicitly

$$\begin{aligned} e_1^s(lq) &= e_1(lq)\cos 2\pi lsq - e_2(lq)\sin 2\pi lsq,\\ e_2^s(lq) &= e_1(lq)\sin 2\pi lsq + e_2(lq)\cos 2\pi lsq \end{aligned}$$

in (2.9) we have

$$F_h = F_h^s = -\frac{1}{2}\left\langle \sum_{l=1}^{N}\big(a_l' e_1^s(lq) + b_l' e_2^s(lq)\big) + a_0\epsilon,\ \sum_{l=1}^{N}\big(a_l' e_1^s(lq) + b_l' e_2^s(lq)\big)\Phi(lq) + a_0\Phi(0)\epsilon\right\rangle \tag{2.10}$$
$$+\,h\left\langle \epsilon, \sum_{l=1}^{N}\big(a_l' e_1^s(lq) + b_l' e_2^s(lq)\big) + a_0\epsilon\right\rangle + \frac{1}{2}T\left\langle \epsilon, g\left(\sum_{l=1}^{N}\big(a_l' e_1^s(lq) + b_l' e_2^s(lq)\big) + a_0\epsilon\right)\right\rangle,$$

where

$$a_l' = a_l\cos 2\pi lsq + b_l\sin 2\pi lsq, \qquad b_l' = -a_l\sin 2\pi lsq + b_l\cos 2\pi lsq, \tag{2.11}$$
$$q = m/N_1, \qquad s = 0, 1, \cdots, N_1.$$

So for rational q free energy (2.6) is invariant under discrete transformation group (2.11). Here we describe the polynomial invariants of discrete rotation group. Consider averages

$$\frac{1}{N_1}\sum_{s=1}^{N_1}\left(a_1\cos 2\pi sq + b_1\sin 2\pi sq\right)^l = P_l(a_1, b_1).$$

$P_l(a_1, b_1)$ is homogeneous polynomial invariant. Really

$$P_l(a_1', b_1') = \frac{1}{N_1}\sum_{s=1}^{N_1}\left(a_1'\cos 2\pi sq + b_1'\sin 2\pi sq\right)^l = \frac{1}{N_1}\sum_{s=1}^{N_1}\left(a_1\cos 2\pi(s_1+s)q + b_1\sin 2\pi(s_1+s)q\right)^l$$

$$= \frac{1}{N_1} \sum_{s=1}^{N_1} (a_1 \cos 2\pi sq + b_1 \sin 2\pi sq)^l = P_l(a_1, b_1).$$

After calculating the average $P_l(a_1, b_1)$ we have [11]

$$P_l(a_1, b_1) = \frac{1}{N_1} \sum_{s=1}^{N_1} (a_1' \cos 2\pi sq + b_1' \sin 2\pi sq)^l \tag{2.12}$$

$$= \frac{1}{2^l} \sum_{|r|=0}^{\left[\frac{l}{N_1}\right]} \left(a_1^2 + b_1^2\right)^{\frac{l-|r|N_1}{2}} \sum_{k=0}^{l} C_l^k \delta_{|l-2k|,|r|N_1} \sum_{m=0}^{\left[\frac{|r|N_1}{2}\right]} (-1)^m C_{|r|N_1}^{2m} a_1^{|r|N_1 - 2m} b_1^{2m}.$$

Now we find the symmetry of free energy (2.7) for irrational q. At first we derive new representation for the free energy. The polynomial coefficients

$$I_p(a_1, b_1, \cdots, a_N, b_N, a_0) = \left\langle \epsilon, \left(\sum_{l=1}^{N} (a_l e_1(lq) + b_l e_2(lq)) + a_0 \epsilon \right)^p \right\rangle$$

are given by (2.5). The direct test gives

$$\left\langle \prod_{k=1}^{s} e_1(i_k q) \prod_{l=1}^{m} e_2(j_l q) \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \prod_{k=1}^{s} \cos i_k \phi \prod_{l=1}^{m} \sin j_l \phi,$$

from where

$$I_p(a_1, b_1, \cdots, a_N, b_N, a_0) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{l=1}^{N} (a_l \cos l\phi + b_l \sin l\phi) + a_0 \right)^p d\phi.$$

I_p is 2π-periodic function of ϕ, therefore

$$I_p(a_1', b_1', \cdots, a_N', b_N', a_0) = I_p(a_1, b_1, \cdots, a_N, b_N, a_0),$$

where

$$a_l' = a_l \cos l\phi_0 + b_l \sin l\phi_0, \qquad b_l' = -a_l \sin l\phi_0 + b_l \cos l\phi_0, \tag{2.13}$$
$$\phi_0 \in [0, 2\pi].$$

Really

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{l=1}^{N} \big(a_l \cos l(\phi + \phi_0) + b_l \sin l(\phi + \phi_0)\big) + a_0 \right)^p d\phi = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{l=1}^{N} (a_l \cos l\phi + b_l \sin l\phi) + a_0 \right)^p d\phi.$$

Therefore I_p is invariant with respect to transformations (2.13) of group G operating in a $2N$-dimensional space. Due to the arbitrariness of ϕ and convergence of free energy series for small a_l and b_l the free energy (2.7) is invariant of group G.

III. NEAR CRITICAL MODULATED STRUCTURES ($H = 0$)

The existence problem for the extremum configurations is non-trivial. Here we study it for small deviations from the critical temperature. We prove the equivalence of the existence problem for a nonzero solution to equations for the configuration minimizing free energy and that for the corresponding branching equations. We derive the branching equations and prove the existence theorem for a nonzero solution. We investigate the model with an external magnetic field and without it. The necessary conditions of free energy minimum are equations

$$\begin{aligned}
&-a_l\Phi(lq)\left(1+\delta_{l,[\frac{N_1+1}{2}]}\right)+T\langle\epsilon,g'(x)e_1(lq)\rangle=0,\\
&-b_l\Phi(lq)\left(1-\delta_{l,[\frac{N_1+1}{2}]}\right)+T\langle\epsilon,g'(x)e_2(lq)\rangle=0, \quad (3.1)\\
&-a_0\Phi(0)+h+T\langle\epsilon,g'(x)\rangle/2=0,\quad l=\overline{1,N},
\end{aligned}$$

for rational q and

$$\begin{aligned}
&-a_l\Phi(lq)+T\langle\epsilon,g'(x)e_1(lq)\rangle=0,\\
&-b_l\Phi(lq)+T\langle\epsilon,g'(x)e_2(lq)\rangle=0, \quad (3.2)\\
&-a_0\Phi(0)+h+T\langle\epsilon,g'(x)\rangle/2=0.
\end{aligned}$$

for irrational q. Since

$$\begin{aligned}
g'(x)&=\ln\frac{1+x}{1-x}\\
&=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2n+1}}{2n+1}+\cdots\right),
\end{aligned}$$

one may denote

$$g'(x)=2x+\varphi_1(x),$$

where

$$\varphi_1(x)=2\left(\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2n+1}}{2n+1}+\cdots\right),$$

$$x\in(-1,1).$$

Lemma 3. *For x being a nontrivial configuration from Hilbert space of configurations, satisfying (3.1) or (3.2) for $h=0$, and q satisfying inequality*

$$\Phi(q)>\Phi(lq),\quad l=0,2,3,4,\cdots,N-1,N,\quad \Phi(q)>0$$

critical temperature T_0 does not exceed $\Phi(q)$, i.e. $T_0\le\Phi(q)$.

For the proof see [12].

Now consider the ANNNI model for $h=0$. In that case the critical temperature is equal to

$$\begin{aligned}
T_0&=\sup_{0\le q\le 1}\Phi(q)\\
&=\sup_q(-2J_1\cos 2\pi q-2J_2\cos 4\pi q-4J_0)>0.
\end{aligned}$$

One can express T_0 in terms of J_1, J_2, J_0. From $\Phi'(q)=0$ it follows that

$$1)\quad \sin 2\pi q=0;\qquad 2)\quad J_1+4J_2\cos 2\pi q=0.$$

To be definite we assume $J_1<0$, $J_2>0$. Then in the first case $\Phi''(q)<0$, if $J_2<-J_1/4$. Since $q=0$, then the branching starts from the ferromagnetic phase for J_1, J_2 satisfying

$$T_0'=-2J_1-2J_2-4J_0.$$

In the second case

$$\begin{aligned}
&\cos 2\pi q_0=-J_1/4J_2,\quad |J_1/4J_2|<1, \quad (3.3)\\
&\Phi''(q_0)=8J_2\left(J_1^2/16J_2^2-1\right)<0,\\
&T_0=J_1^2/4J_2+2J_2-4J_0.
\end{aligned}$$

Let us consider the range of parameters for which $T_0>T_0'$. These are J_1 and J_2 satisfying the inequality $r^2>4(-2r-4)$ with $r=J_1/J_2$. The latter inequality holds for all real r except for $r=-4$. But $r<0$ and to satisfy (3.3) we suppose

$$-J_2/J_1>0.25. \quad (3.4)$$

Thus, the modulated structures can occur only when inequality (3.4) holds.

Now we prove the small branching theorems for the solutions to equations (3.1), (3.2) with $h=0$. Let q be irrational, then equations (3.2) transform into

$$\begin{aligned}
a_1&=\left(1-t^2\right)t^{-2}\left\langle 2x^3/3+\phi(x),e_1(q)\right\rangle,\\
b_1&=\left(1-t^2\right)t^{-2}\left\langle 2x^3/3+\phi(x),e_2(q)\right\rangle,\\
a_l&=\frac{(1-t^2)\Phi(q)}{\Phi(lq)-(1-t^2)\Phi(q)}\left\langle 2x^3/3+\phi(x),e_1(lq)\right\rangle, \quad (3.5)\\
b_l&=\frac{(1-t^2)\Phi(q)}{\Phi(lq)-(1-t^2)\Phi(q)}\left\langle 2x^3/3+\phi(x),e_2(q)\right\rangle,\\
a_0&=\frac{(1-t^2)\Phi(q)}{2\big(\Phi(0)-(1-t^2)\Phi(q)\big)}\left\langle 2x^3/3+\phi(x),\epsilon\right\rangle
\end{aligned}$$

with

$$\phi(x) = 2\left(\frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right),$$

$$T = T_0(1 - t^2), \quad T_0 = \Phi(q).$$

We assume solutions of equations (3.5) to be expressed as

$$a_1 = t\bar{a}_1, \quad b_1 = t\bar{b}_1, \quad a_l = t^2\bar{a}_l, \quad b_l = t^2\bar{b}_l, \quad a_0 = t^2\bar{a}_0.$$

Then the equations for configurations minimizing free energy become

$$\bar{a}_1 = F_1^{(1)}(w) = \frac{1-t^2}{2\pi t^3}\int_0^{2\pi} \varphi_1(\cdot)\cos\phi d\phi,$$

$$\bar{b}_1 = F_1^{(2)}(w) = \frac{1-t^2}{2\pi t^3}\int_0^{2\pi} \varphi_1(\cdot)\sin\phi d\phi, \tag{3.6}$$

$$\bar{a}_l = F_l^{(1)}(w) = \frac{A_l}{2\pi t^2}\int_0^{2\pi} \varphi_1(\cdot)\cos\phi d\phi,$$

$$\bar{b}_l = F_l^{(2)}(w) = \frac{A_l}{2\pi t^2}\int_0^{2\pi} \varphi_1(\cdot)\sin\phi d\phi, \tag{3.7}$$

$$\bar{a}_0 = F_0^{(0)}(w) = \frac{A_0}{2\pi t^2}\int_0^{2\pi} \varphi_1(\cdot)d\phi$$

with

$$w = \{t, z, \bar{a}_1, \bar{b}_1\},\ z = \{\bar{a}, \bar{b}\} = \left(\{\bar{a}_2, \bar{b}_2\}, \cdots, \{\bar{a}_N, \bar{b}_N\}, \bar{a}_0\right),$$

$$A_l = \frac{(1-t^2)\Phi(q)}{\Phi(lq) - (1-t^2)\Phi(q)},\ l \geq 2,$$

$$A_0 = \frac{(1-t^2)\Phi(q)}{2\big(\Phi(0) - (1-t^2)\Phi(q)\big)},$$

$$\varphi_1(\cdot) = \varphi_1\left(t^2\sum_{l=2}^{N}(\bar{a}_l\cos l\phi + \bar{b}_l\sin l\phi) + t^2\bar{a}_0 + t(\bar{a}_1\cos\phi + \bar{b}_1\sin\phi)\right).$$

To prove small branching theorem for nonlinear equations (3.5), let us proceed by steps. At first we prove the existence of a unique nontrivial solution to system (3.7). Let us introduce the norm

$$|z| = \max_{2\leq l\leq N}\big(\{|\bar{a}_l|, |\bar{b}_l|\}, |\bar{a}_0|\big)$$

and the radius A ball centered at the point

$$z_0 = \big(\{\bar{a}_2^0, \bar{b}_2^0\}, \cdots, \{\bar{a}_N^0, \bar{b}_N^0\}, \bar{a}_0^0\big)$$

in the $2N-1$-dimensional space of vectors z, where

$$\bar{a}_l^0 = \frac{A_l}{2\pi t^2}\int_0^{2\pi} \varphi_1\big(t(\bar{a}_1\cos\phi + \bar{b}_1\sin\phi)\big)\cos\phi d\phi,$$

$$\bar{b}_l^0 = \frac{A_l}{2\pi t^2}\int_0^{2\pi} \varphi_1\big(t(\bar{a}_1\cos\phi + \bar{b}_1\sin\phi)\big)\sin\phi d\phi,$$

$$\bar{a}_0^0 = \frac{A_l}{2\pi t^2}\int_0^{2\pi} \varphi_1\big(t(\bar{a}_1\cos\phi + \bar{b}_1\sin\phi)\big)d\phi.$$

Let $|\{\bar{a}, \bar{b}\}| \leq A$ and $\max\{|\bar{a}_1|, |\bar{b}_1|\} \leq A$ with A be determined later. In operator form (3.7) is

$$z = F(w), \tag{3.8}$$

with the nonlinear operator F given by the r.h.s. of (3.7).

Theorem 1. *If $\max\{|\bar{a}_1|, |\bar{b}_1|\} \leq A$, then there exists $\alpha > 0$ such that $0 < t < \alpha$ and the system (3.8) is uniquely solvable by continuously differentiable function of $t, \bar{a}_1, \bar{b}_1$ in their ranges of values.*

For the proof see [10]. The similar theorem is valid for rational q.

Theorem 2. *For all irrational q that guarantee the validity of*

$$\min_l |\Phi(lq) - \Phi(q)| = \delta > 0$$

there is $\alpha_1 > 0$ *such that a nonzero solution to system* (3.5) *exists for* $0 \le t < \alpha_1$ *if* $h = 0$. *The action of group* G (2.13) *on this solution gives different solutions of the system* (3.5) *that are continuously differentiable functions of* t.

For the proof see [10].

Theorem 3. *For all rational* q *there is* $\alpha_1' > 0$ *such that there exists nonzero solution of system* (3.1) *for* $0 \le t < \alpha_1'$ *if* $h = 0$. *The action of group* (2.11) *on this solution gives different solutions to equations* (3.1) *that are continuously differentiable functions of* t.

For the proof see [11].

IV. NEAR CRITICAL MODULATED STRUCTURES ($H \neq 0$)

Here we consider the case $h \neq 0$ and display the phase transition from the paramagnetic state to the spatially inhomogeneous commensurate configuration or the incommensurate one below the critical magnetic field. In the external magnetic field systems (3.1) and (3.2) have a nonzero solution in the set of vectors $\{(a_1, b_1), \cdots, (a_N, b_N), a_0\}$ such that

$$\sum_{l=1}^{N} \big(|a_l| + |b_l|\big) + |a_0| < 1.$$

The solution has the form $\{(0,0), \cdots, (0,0), \bar{a}_0\}$ with $\bar{a}_0$ solving the equation

$$-\bar{a}_0 \Phi(0) + T g'(\bar{a}_0)/2 + h = 0. \tag{4.1}$$

We assume that

$$x = \bar{x} + \bar{a}_0 \epsilon, \quad \bar{x} = \sum_{l=1}^{N} \big(a_l e_1(lq) + b_l e_2(lq)\big) + a_0' \epsilon,$$

$$g'(x) = \sum_{k=1}^{+\infty} \frac{g^{(k)}(\bar{a}_0)}{(k-1)!} \bar{x}^{k-1} = g'(\bar{a}_0)\epsilon + g''(\bar{a}_0)\bar{x} + \varphi_2(\bar{x}),$$

$$g'(\bar{a}_0 \epsilon) = g'(\bar{a}_0)\epsilon, \quad \bar{x}\epsilon = \bar{x}, \quad \varphi_2(\bar{x}) = \sum_{k=3}^{+\infty} \frac{g^{(k)}(\bar{a}_0)}{(k-1)!} \bar{x}^{k-1}.$$

Then system (3.1) becomes

$$\begin{aligned}
&a_l \Big(T/\left(1-\bar{a}_0^2\right) - \Phi(lq)\Big) \Big(1 + \delta_{l,\left[\frac{N_1+1}{2}\right]}\Big) + T \left\langle \epsilon, \varphi_2(\bar{x}) e_1(lq) \right\rangle = 0, \\
&b_l \Big(T/\left(1-\bar{a}_0^2\right) - \Phi(lq)\Big) \Big(1 - \delta_{l,\left[\frac{N_1+1}{2}\right]}\Big) + T \left\langle \epsilon, \varphi_2(\bar{x}) e_2(lq) \right\rangle = 0, \\
&2a_0' \left(T/\left(1-\bar{a}_0^2\right) - \Phi(0)\right) + T \left\langle \epsilon, \varphi_2(\bar{x}) \right\rangle = 0, \quad l = \overline{1, N},
\end{aligned} \tag{4.2}$$

and system (3.2) becomes

$$\begin{aligned}
&a_l \left(T/(1-\bar{a}_0^2) - \Phi(lq)\right) + T \left\langle \epsilon, \varphi_2(\bar{x}) e_1(lq) \right\rangle = 0, \\
&b_l \left(T/(1-\bar{a}_0^2) - \Phi(lq)\right) + T \left\langle \epsilon, \varphi_2(\bar{x}) e_2(lq) \right\rangle = 0, \\
&2a_0' \left(T/(1-\bar{a}_0^2) - \Phi(0)\right) + T \left\langle \epsilon, \varphi_2(\bar{x}) \right\rangle = 0.
\end{aligned} \tag{4.3}$$

The spatially inhomogeneous solution to systems (3.1), (3.2) exists if systems (4.2), (4.3) have a nonzero solution. The first degeneration of the spectrum of the linearized part of the nonlinear operator (4.2) or (4.3) occurs when

$$\begin{gathered}
T/\left(1-\bar{a}_0^2\right) - \Phi(q) = 0, \\
-\bar{a}_0 \Phi(0) + T g'(\bar{a}_0)/2 + h = 0.
\end{gathered} \tag{4.4}$$

If there exists a solution to system (4.4), then zero solution for systems (4.2), (4.3) bifurcates. It means a transition from the paramagnetic phase to the modulated structure. As a result system (4.4) gives single equivalent equation

$$T = \Phi(q) - \frac{\Phi(q) h^2}{\left(\Phi(0) - \frac{1}{2} T \left(1 - T/\Phi(q)\right)^{-\frac{1}{2}} g'\left(\sqrt{1 - T/\Phi(q)}\,\right)\right)^2}. \tag{4.5}$$

From (4.5) it follows **Lemma 4.** *The critical magnetic field for phase transition from the paramagnetic state to the modulated structure is equal to* $|\Phi(0)|$.

For the proof see [12].

Having in mind that mean-field theory critical exponent β equals 1/2 we suppose the deviation from the critical temperature to be standard. Therefore

$$\lambda^2(T,h) = \Phi(q) - T/\left(1-\bar{a}_0^2\right),$$

where $\bar{a}_0$ solves equation (4.1). If $T_0(h)$ solves equation (4.5), then $\lambda^2(T_0(h),h)=0$.

In terms of the new variables system (4.3) for the irrational q takes the form of

$$\begin{aligned} &-\lambda^2 a_1 + T\langle \epsilon, \varphi_2(\bar{x}) e_1(q)\rangle = 0,\\ &-\lambda^2 b_1 + T\langle \epsilon, \varphi_2(\bar{x}) e_2(q)\rangle = 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &a_l\left(\Phi(q)-\Phi(lq)-\lambda^2\right) + T\langle \epsilon, \varphi_2(\bar{x}) e_1(lq)\rangle = 0,\\ &b_l\left(\Phi(q)-\Phi(lq)-\lambda^2\right) + T\langle \epsilon, \varphi_2(\bar{x}) e_2(lq)\rangle = 0,\\ &2a_0'\left(\Phi(q)-\Phi(0)-\lambda^2\right) + T\langle \epsilon, \varphi_2(\bar{x})\rangle = 0. \end{aligned} \tag{4.7}$$

We seek the solution of systems (4.6), (4.7) in the form of

$$a_1 = \lambda\tilde{a}_1, \quad b_1 = \lambda\tilde{b}_1, \quad a_l = \lambda^2\tilde{a}_l, \quad b_l = \lambda^2\tilde{b}_l, \quad a_0' = \lambda^2\tilde{a}_0'.$$

Then we rewrite (4.6), (4.7) as

$$\begin{aligned} \tilde{a}_1 &= F_1^{(1)}(\tilde{w}) = \frac{T}{\lambda^3}\sum_{k=3}^{+\infty}\frac{g^{(k)}(\bar{a}_0)}{(k-1)!}\lambda^{k-1}\left\langle \epsilon, (\cdot)^{k-1}\, e_1(q)\right\rangle,\\ \tilde{b}_1 &= F_1^{(2)}(\tilde{w}) = \frac{T}{\lambda^3}\sum_{k=3}^{+\infty}\frac{g^{(k)}(\bar{a}_0)}{(k-1)!}\lambda^{k-1}\left\langle \epsilon, (\cdot)^{k-1}\, e_2(q)\right\rangle, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \tilde{a}_l &= F_l^{(1)}(\tilde{w}) = \frac{T}{\lambda^2\left(\lambda^2+\Phi(lq)-\Phi(q)\right)}\sum_{k=3}^{+\infty}\frac{g^{(k)}(\bar{a}_0)}{(k-1)!}\lambda^{k-1}\left\langle \epsilon, (\cdot)^{k-1}\, e_1(lq)\right\rangle,\\ \tilde{b}_l &= F_l^{(2)}(\tilde{w}) = \frac{T}{\lambda^2\left(\lambda^2+\Phi(lq)-\Phi(q)\right)}\sum_{k=3}^{+\infty}\frac{g^{(k)}(\bar{a}_0)}{(k-1)!}\lambda^{k-1}\left\langle \epsilon, (\cdot)^{k-1}\, e_2(lq)\right\rangle,\\ \tilde{a}_0' &= F_0^{(0)}(\tilde{w}) = \frac{T}{2\lambda^2\left(\lambda^2+\Phi(0)-\Phi(q)\right)}\sum_{k=3}^{+\infty}\frac{g^{(k)}(\bar{a}_0)}{(k-1)!}\lambda^{k-1}\left\langle \epsilon, (\cdot)^{k-1}\right\rangle, \end{aligned} \tag{4.9}$$

where

$$\tilde{w} = \{\lambda, \tilde{z}, \tilde{a}_1, \tilde{b}_1\}, \quad \tilde{z} = \{\tilde{a}, \tilde{b}\} = \left(\{\tilde{a}_2, \tilde{b}_2\}, \cdots, \{\tilde{a}_N, \tilde{b}_N\}, \tilde{a}_0'\right),$$

$$\Big(\cdot\Big) = \left(\lambda\sum_{l=2}^{N}\left(\tilde{a}_l e_1(lq) + \tilde{b}_l e_2(lq)\right) + \lambda\tilde{a}_0'\epsilon + \tilde{a}_1 e_1(q) + \tilde{b}_1 e_2(q)\right).$$

Let us consider the norm

$$|\tilde{z}| = \max_{2\le l\le N}\{|\tilde{a}_l|, |\tilde{b}_l|, |\tilde{a}_0'|\}$$

and the radius A ball centered at the point

$$\tilde{z}_0 = \left(\{\tilde{a}_2^0, \tilde{b}_2^0\}, \cdots, \{\tilde{a}_N^0, \tilde{b}_N^0\}, \tilde{a}_0^{'0}\right)$$

in the set of vectors $\tilde{z}$, where

$$\tilde{a}_l^0 = \frac{T}{\lambda^2+\Phi(lq)-\Phi(q)}\sum_{k=3}^{+\infty}\frac{g^{(k)}(\bar{a}_0)}{(k-1)!}\lambda^{k-3}\left\langle \epsilon, (\cdots)^{k-1} e_1(lq)\right\rangle,$$

$$\tilde{b}_l^0 = \frac{T}{\lambda^2 + \Phi(lq) - \Phi(q)} \sum_{k=3}^{+\infty} \frac{g^{(k)}(\bar{a}_0)}{(k-1)!} \lambda^{k-3} \left\langle \epsilon, (\cdots)^{k-1} e_2(lq) \right\rangle ,$$

$$\tilde{a}_0'^0 = \frac{T}{2\left(\lambda^2 + \Phi(0) - \Phi(q)\right)} \sum_{k=3}^{+\infty} \frac{g^{(k)}(\bar{a}_0)}{(k-1)!} \lambda^{k-3} \left\langle \epsilon, (\cdots)^{k-1} \right\rangle ,$$

$$(\cdots) = (\tilde{a}_1 e_1(q) + \tilde{b}_1 e_2(q)).$$

Let $F(\tilde{w})$ be a nonlinear operator given by the right part of system (4.9), then this system turns to be

$$\tilde{z} = F(\tilde{w}) . \tag{4.10}$$

Theorem 4. *Let*

$$\min_l |\Phi(lq) - \Phi(q)| = \delta > 0, \quad l = 0, 2, 3, 4, \cdots, N-1, N,$$

and $\lambda_0 > 0$ *satisfies the inequality*

$$\max_l \frac{T}{|\lambda^2 + \Phi(lq) - \Phi(q)|} (2N-1)\lambda \times$$

$$\times \sum_{k=3}^{+\infty} \frac{|g^{(k)}(\bar{a}_0)|}{(k-2)!} \lambda^{k-3} \left((2N-1)A\lambda + 2A\right)^{k-2} < 1,$$

then for all $\lambda \in [0, \lambda_0]$ *and* $\tilde{a}_1$, $\tilde{b}_1$, h *such that*

$$\tilde{a}_1^2 + \tilde{b}_1^2 \le A^2, \qquad |h| < |\Phi(0)|,$$

there exist the critical temperature $T_0(h)$ *and an interval* $\left(T_0(h) - \lambda^2, T_0(h)\right)$ *on which the solution of nonlinear equations* (4.10) *exists being a continuously differentiable function of* $\lambda \in [0, \lambda_0]$ *and* $\tilde{a}_1$, $\tilde{b}_1$ *for* $\tilde{a}_1^2 + \tilde{b}_1^2 \le A^2$.

For the proof see [12].A similar theorem is valid for the rational q.

Remark. Continuous differentiability with respect to the variables λ and $\tilde{a}_1, \tilde{b}_1$ results from continuous differentiability of the successive approximations and their uniform convergence to the solution.

Theorem 5. *For all irrational* q *that guarantee the validity of*

$$\min_l |\Phi(lq) - \Phi(q)| = \delta > 0$$

there is such $\lambda_1 > 0$, *that system* (4.3) *is solvable for*

$$0 < \lambda < \lambda_1, \quad |h| < |\Phi(0)|.$$

The action of group G (2.13) *on this solution gives different solutions of system* (4.3) *being continuously differentiable functions of* λ.

For the proof see [12].

Theorem 6. *For those rational* q, *for which* $N_1 \ne 3$, *there is* $\lambda_0 > 0$ *such that a nonzero solution to system* (4.2) *exists for*

$$0 \le \lambda < \lambda_0, \quad |h| < |\Phi(0)|.$$

The action of group (2.11) *on this solution gives different solutions to system* (4.2) *being continuously differentiable functions of* λ.

For the proof see [13].

V. GLOBAL INCOMMENSURATE EXTREMUM

For $N = 1$ conditions of Theorems 4 and 5 hold. In this case one can construct not only a solution near the critical temperature but also the global solution. For $N = 1$ configuration (2.1) takes the form of

$$x = a_1 e_1(q) + b_1 e_2(q) + a_0 \epsilon. \tag{5.1}$$

The specific free energy for this configuration is

$$F_h = -\frac{1}{2} \left(\frac{1}{2} \left(a_1^2 + b_1^2\right) \Phi(q) + a_0^2 \Phi(0) \right)$$

$$+ h a_0 + \frac{1}{2} T \left\langle \epsilon, g(x) \right\rangle . \tag{5.2}$$

Having calculated the average

$$\left\langle \epsilon, g(x) \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} g(u \cos\varphi + a_0) \, d\varphi$$

with $u = \left(a_1^2 + b_1^2\right)^{\frac{1}{2}}$ we have [14]

$$F_h = -\frac{1}{2} \left(\frac{1}{2} u^2 \Phi(q) + a_0^2 \Phi(0) \right) + h a_0$$

$$+ \frac{T}{2} \Bigg(2 - \sqrt{(1+a_0)^2 - u^2} - \sqrt{(1-a_0)^2 - u^2} - 2\ln 2$$

$$+ (1+a_0) \ln \left(1 + a_0 + \sqrt{(1+a_0)^2 - u^2} \right)$$

$$+(1-a_0)\ln\left(1-a_0+\sqrt{(1-a_0)^2-u^2}\,\right)\right). \tag{5.3}$$

For $h=0$ the free energy is

$$F=-\frac{1}{4}u^2\Phi(q)+T\left(1-\sqrt{1-u^2}+\right.$$
$$\left.\ln\left(1+\sqrt{1-u^2}\right)-\ln 2\right). \tag{5.4}$$

The equation for the configuration minimizing the free energy (5.4) is

$$-\frac{1}{2}\Phi(q)+\frac{T}{1+\sqrt{1-u^2}}=0. \tag{5.5}$$

From here we obtain the global solution

$$u^2=1-\left(2T/\Phi(q)-1\right)^2, \tag{5.6}$$

terminating at $u^2=1$, i.e. the termination temperature is equal to $\Phi(q)/2$. Taking into account (5.6) we write the free energy of the incommensurate phase (5.4) as

$$F=T\Big(1-T/\Phi(q)+\ln\big(T/\Phi(q)\big)\Big). \tag{5.7}$$

The configuration minimizing the free energy (5.3) satisfies equations

$$\Phi(q)+T\left(\frac{1}{1+a_0+\sqrt{(1+a_0)^2-u^2}}+\frac{1}{1-a_0+\sqrt{(1-a_0)^2-u^2}}\right)=0,$$
$$-a(0)+h+\frac{T}{2}\ln\frac{1+a_0+\sqrt{(1+a_0)^2-u^2}}{1-a_0+\sqrt{(1-a_0)^2-u^2}}=0 \tag{5.8}$$

or

$$a_0=\frac{T}{\Phi(q)}\sinh\big(2T^{-1}(a_0\Phi(0)-h)\big)-\tanh\big(T^{-1}(a_0\Phi(0)-h)\big)$$
$$u^2=\left(2(1+a_0)-\frac{T}{\Phi(q)}\Big(1+\exp\big(2T^{-1}(a_0\Phi(0)-h)\big)\Big)\right)$$
$$\times\frac{T}{\Phi(q)}\Big(1+\exp\big(2T^{-1}(a_0\Phi(0)-h)\big)\Big). \tag{5.9}$$

Theorem 7. *There exists the global solution minimizing free energy* (5.3) *for a rather small h.*

For the proof see [14].

Theorem 8. *The global solution minimizing free energy* (5.3) *terminates when the amplitude of incommensurate configuration is*

$$u=\left(a_1^2+b_1^2\right)^{\frac{1}{2}}=1-|\hat{a}_0|,$$

where $\hat{a}_0$ solves the equation

$$\frac{\Phi(q)}{4}\ln\frac{1+\sqrt{|a_0|}}{1-\sqrt{|a_0|}}(1-|a_0|)-\frac{\Phi(0)|a_0|+|h|}{1+\sqrt{|a_0|}}=0. \tag{5.10}$$

For the proof see [14].

Theorem 9. *There exists phase transition of the first kind from the incommensurate phase into the ferromagnetic one. For $h\neq 0$ the existence condition of phase transition is*

$$\Phi(q)<\Big(1-\hat{a}_0-\big(1+\sqrt{\hat{a}_0}\big)\big(2\big(1+\big(1+\hat{a}_0\big)\ln\big(1+\sqrt{\hat{a}_0}\big)-\sqrt{\hat{a}_0}-\ln 2\big)$$
$$+\big(1-\hat{a}_0\big)\ln\big(1-\hat{a}_0\big)\big)\Big)^{-1}2\Big(\Phi(0)\big(1+\hat{a}_0\big)-2\hat{h}\Big),$$

where $\hat{a}_0$ and $\hat{h}$ solve equation (5.10). *For $h = 0$ this condition has the form of*

$$\Phi(q) < \Phi(0)/(\ln 2 - 1/2).$$

For the proof see [14].

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ВИЧЕРПНИЙ АНАЛІЗ МАГНЕТНИХ НАДСТРУКТУР В АНІЗОТРОПНИХ СИСТЕМАХ

М. Гончар, В. Козирський, Г. Гайдук
Відділ математичного моделювання Інституту теоретичної фізики імені М. М. Боголюбова НАН України, Україна, 252143, Київ, вул. Метрологічна, 14б

Розвинуто новий метод аналітичного розгляду модульованих надструктур у модельних магнетних системах, близьких до критичності. Встановлено умови для переходу парамагнет–(не)співмірна фаза у класі анізотропних моделей Ізінґа з зовнішнім магнетним полем та без нього. Знайдено його критичного значення. Виведено нове представлення питомої вільної енергії для співвимірних та неспіввимірних конфіґурацій та виявлено її приховану симетрію. Доведено існування глобального розв'язку для неспіввимірної конфіґурації, що мінімізує вільну енергію. Встановлені умови зникнення глобального розв'язку. Доведено, що існування фазового переходу від неспіввимірної до феромагнетної фази залежить від обмінних інтеґралів.