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ON FREE ENERGY OF THREE–DIMENSIONAL ISING MODEL AT CRITICALITY

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Higher–order vertices at zero external momenta for the scalar field theory, describing the critical behaviour of the Ising model, are studied within the field–theoretical renormalization group (RG) approach in three dimensions. Dimensionless six–point g_6 and eight–point g_8 effective coupling constants are calculated in the three–loop approximation. Their numerical values, universal at criticality, are estimated by means of the Padé and Padé–Borel summation of the RG expansions found and by putting the renormalized quartic coupling constant equal to its universal fixed–point value known from six–loop RG calculations. The values of g_6^* obtained are compared with their analogs resulting from the ϵ–expansion, Monte Carlo simulations, the Wegner–Houghton equations and the linked cluster expansion series. The field–theoretical estimates for g_6^* are shown to be in a good agreement with each other, differing considerably from the values given by other methods.
Key words: scalar field theory, renormalization group, Ising model.

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I. INTRODUCTION

The critical thermodynamics of the three–dimensional Ising model is known to be described by Euclidean scalar field theory with the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}(m_0^2\varphi^2 + (\nabla\varphi)^2) + \frac{\lambda}{4!}\varphi^4\right], \qquad (1.1)$$

where a bare mass squared m_0^2 is proportional to $T - T_c^{(0)}$, $T_c^{(0)}$ being the phase transition temperature in the absence of the order parameter fluctuations. Taking fluctuations into account leads to renormalization of the mass $m_0^2 \to m^2$, the field $\phi \to \phi_R$, and the coupling constant $\lambda \to mg_4$, and also to the appearance of terms of the form $m^{3-n}g_{2n}\phi_R^{2n}$ with $n > 2$ in the effective action (free energy) of the system. In the critical region, where fluctuations are so strong that they completely screen out the initial (bare) interaction, the behaviour of the system becomes universal and the dimensionless vertices g_{2n} tend toward their asymptotic limits, i.e. they assume constant values which are also universal.

In this paper, we calculate numerical values of g_6 and g_8 at criticality using the field–theoretical renormalization group (RG) approach in three dimensions. Higher–order coupling constants mentioned related to corresponding vertices at zero external momenta will be found as power series in the renormalized dimensionless quartic coupling constant g_4 up to three–loop order, and the RG series will be resummed by means of the Padé and Padé–Borel techniques. Then g_4 in resummed RG expansions will be put equal to its universal value g_4^* known from the canonical six–loop RG calculations resulting in numerical estimates for g_6^* and g_8^*. The numbers obtained for g_6^* will be compared with those resulting from the ϵ–expansion for g_6/g_4^2 as well as with the values given by Monte Carlo simulations, by the approximate solution of the Wegner–Houghton equations and by the analysis of the linked cluster expansions.

II. RG EXPANSIONS FOR SEXTIC AND OCTIC EFFECTIVE INTERACTIONS

Since higher–order bare coupling constants are known to be irrelevant at criticality in the RG sense, renormalized perturbative expansions for g_6, g_8, etc. may be obtained from conventional Feynman–graph expansions of these quantities in terms of the only bare coupling constant – λ. In its turn, λ may be expressed perturbatively as a function of the renormalized dimensionless quartic coupling constant g_4. Substituting corresponding power series for λ into original Feynman–graph expansions for g_6 and g_8, we can obtain the RG series for these higher–order effective coupling constants.

Recently, one of the authors has found the sextic coupling constant g_6 in the two–loop RG approximation [1]. Thus, what we are interested in is the three–loop contribution to this quantity. As may be shown, there are 16 graphs contributing to g_6 in the three–loop order. In fact, their calculation is neither cumbersome nor lengthy since corresponding contribution may be written down as a sum of a few terms having a form of mass derivatives of some two–point and four–point graphs. So, within the three–loop approximation we get

$$g_6 = \frac{9}{\pi}\left(\frac{g_0 Z^2}{m}\right)^3 \left[1 - \frac{33}{2\pi}\frac{g_0 Z^2}{m} + 20.53966666\left(\frac{g_0 Z^2}{m}\right)^2\right], \qquad (2.1)$$

where $g_0 = \lambda/4!$ and Z relates the dressed Green function G to the renormalized one G_R in a conventional way:

$$G = ZG_R = \frac{Z}{m^2 + q^2 - \Sigma_R} = \frac{Z}{m^2 + q^2} + O(g_4^2). \quad (2.2)$$

The renormalized perturbative expansion for bare coupling constant g_0 may be obtained using the normalizing condition

$$g_0 = m g_4 \frac{Z_4}{Z^2} \quad (2.3)$$

and the RG expansion for Z_4 which has been calculated by many people:

$$Z_4 = 1 + \frac{9}{2\pi} g_4 + \frac{63}{4\pi^2} g_4^2 + O(g_4^3). \quad (2.4)$$

Combining (2.1, 2.3, 2.4) we obtain

$$g_6 = \frac{9}{\pi} g_4^3 \Big(1 - \frac{3}{\pi} g_4 + 1.389962952 g_4^2\Big). \quad (2.5)$$

The RG expansion for the eight–point effective coupling constant g_8 may be found in an analogous way although it requires more job than in the case of g_6. Two–loop and three–loop contributions to g_8 are given by 5 and 36 Feynman graphs respectively. Use of the trick mentioned above, however, considerably simplifies their calculation. The result is as follows

$$g_8 = -\frac{81}{2\pi} \left(\frac{g_0 Z^2}{m}\right)^4 \left[1 - \frac{173}{6\pi} \frac{g_0 Z^2}{m} + 54.81336082 \left(\frac{g_0 Z^2}{m}\right)^2\right]. \quad (2.6)$$

Expressing g_0 in terms of g_4 leads to the RG series for g_8:

$$g_8 = -\frac{81}{2\pi} g_4^4 \Big(1 - \frac{65}{6\pi} g_4 + 7.775001310 g_4^2\Big). \quad (2.7)$$

With the RG expansions (2.5, 2.7) in hand, we can get numerical estimates for universal critical values of g_6 and g_8.

III. RESUMMATION AND NUMERICAL ESTIMATES

Perturbative expansions in a field theory are known to be divergent, at best asymptotic. Moreover, the theory under consideration has no small parameter. That is why direct substitution of the fixed point value g_4^* to (2.5, 2.7) can not lead to satisfactory results. These expansions, however, contain important information which may be extracted provided some procedure making them convergent is applied. Here Padé and Padé–Borel methods will play roles of such procedures, i.e. we shall construct Padé approximants $[L/M]$ for the functions given by the series (2.5, 2.7) as well as for their Borel transforms which are related to the functions to be found ("sum of series") by the formula

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = \int_0^{\infty} e^{-t} F(xt) dt, \quad (3.1)$$

$$F(y) = \sum_{k=0}^{\infty} \frac{c_k}{k!} y^k, \quad (3.2)$$

and then evaluate the integral Eq. (3.1) where series Eq. (3.2) possessing nonzero radii of convergence are replaced by the corresponding Padé approximants.

Starting from the three–loop expansions available, it is possible to construct Padé approximants of the only reasonable type: [1/1]. On the other hand, one can try several different ways of resummation. Indeed, the Padé–Borel procedure may be applied not only to the series for g_6 and g_8 themselves but also to corresponding RG expansions for the ratios g_6/g_4^2 and g_8/g_4^3.

Let us construct first the Padé approximant for the six–point effective coupling constant. From Eq. (2.5) we readily obtain:

$$g_6 = \frac{9}{\pi} g_4^3 \frac{1 + 0.500636 g_4}{1 + 1.455566 g_4}. \quad (3.3)$$

Substituting into this expression the most accurate numerical estimate $g_4^* = 0.988$ for the fixed point value known from six–loop RG calculations in three dimensions [2, 3] we find for the universal value of g_6:

$$g_6^* = 1.694. \quad (3.4)$$

Apply then more sophisticated, Padé–Borel procedure to the series Eq. (2.5) which is expected to lead to more accurate estimate for g_6^*. After simple algebra we obtain

$$g_6 = 0.477465 g_4^3 \int_0^{\infty} \frac{1 + 0.052382 g_4 t}{1 + 0.291114 g_4 t} t^3 e^{-t} dt. \quad (3.5)$$

Computation of this integral for $g_4 = 0.988$ gives

$$g_6^* = 1.622. \quad (3.6)$$

The third way to get numerical estimate for g_6^* we use here is to construct the Padé–Borel approximant for

g_6/g_4^2 and to estimate the universal value of this ratio at criticality. The corresponding expression is as follows:

$$\frac{g_6}{g_4^2} = 2.86479 g_4 \int_0^\infty \frac{1 + 0.0077234 g_4 t}{1 + 0.48519 g_4 t} t e^{-t} dt. \qquad (3.7)$$

For the fixed point value of g_4 this formula leads to:

$$g_6^* = 1.577. \qquad (3.8)$$

To obtain numerical estimates for g_8^* the same procedures are applied to the series Eq. (2.7). They give

$$g_8^* = 0.68 \quad (\text{Padé}), \qquad (3.9)$$

$$g_8^* = 1.71 \quad (\text{Padé} - \text{Borel}),$$

$$g_8^* = 2.71 \quad (\text{Padé} - \text{Borel for } g_8/g_4^3).$$

IV. DISCUSSION

Although the RG expansions found in Sec. II are, in fact, rather short, numerical estimates for g_6 obtained from (2.5) by several resummation techniques are seen to be close to each other. That is why we believe that the numbers (3.4) and, in particular, (3.6) and (3.8) are also close to the exact value of g_6^*. Moreover, taking into account of the three–loop RG contribution to g_6^* turns out to change corresponding estimate by less than 10 per cent (Padé–Borel resummation of the two–loop RG expansion have led to $g_6^* = 1.50$ [1]) what may be considered as an extra argument in favour of fair numerical accuracy of the results presented above.

Let us compare our estimates for g_6^* with their counterparts obtained by different methods. The solution of the Wegner–Houghton equations within the local potential approximation presented by C.Bagnuls and C.Bervillier has yielded $(g_6/g_4^2) = 3.59$, $g_6 = 2.40$ at the non–trivial fixed point [4]. Determining by Monte Carlo simulations probability distributions for averaged magnetization of the 3D Ising model in an external magnetic field, M.M. Tsypin found from his data that $g_6^* = 2.05$ [5]. The analysis of the linked cluster expansion series performed by T.Reisz led to $g_6^* = 1.92$ [6].

All these estimates are seen to be significantly larger than ours obtained by means of perturbative RG calculations for the field–theoretical model. In such a situation, it is interesting to evaluate g_6^* using some alternative field–theoretical perturbative approach. The ϵ–expansion for the ratio g_6/g_4^2 which results from the equation of state may be used for this purpose. Up to the ϵ^3 order, it is as follows [7]:

$$\frac{g_6}{g_4^2} = 2\epsilon - \frac{20}{27}\epsilon^2 + 1.2759\epsilon^3. \qquad (4.1)$$

Resumming this expansion by Padé and Padé–Borel methods we find for $\epsilon = 1$ and $g_4 = 0.988$:

$$g_6^* = 1.687 \quad (\text{Padé}),$$

$$g_6^* = 1.653 \quad (\text{Padé} - \text{Borel}). \qquad (4.2)$$

The Padé–Borel estimate for g_6^* thus obtained is remarkably close to those given by RG calculations in three dimensions (3.6), (3.8). So, the field–theoretical RG approach in three and $(4-\epsilon)$ dimensions provides, within the three–loop approximation, the values of g_6^* which agree well to each other but differ considerably from the Monte Carlo and some other estimates mentioned above.

Let us discuss further the predictions concerning the eight–point effective coupling constant. The estimates following from the three–loop RG expansions in 3D are clearly seen to be strongly scattered being therefore numerically unreliable. It is not surprising since coefficients in the RG expansion for g_8 grow much faster than those in the RG series for g_6 making the former series less suitable for resummation than the latter. Correspondingly, much less reliable numerical estimates for g_8 at the critical point are obtained. On the other hand, all three numbers (3.9) are i) positive, i.e. have a sign opposite to that of the lowest–order (one–loop) RG estimate $g_8^* = -12.28$, and ii) much smaller than the absolute value of this one–loop estimate. It means that higher–order RG contributions to g_8 being extremely important tend to strongly diminish its universal value at the critical point with respect to the number given by the lowest–order RG approximation. This conclusion seems to be in accord with the fact that the recent analysis of relevant Monte Carlo data failed to reveal some appreciable (non–zero) value of g_8^* [5].

V. CONCLUSION

In the paper, the RG series for coefficients before M^6 and M^8 in the expansion of the free energy of the three–dimensional Ising model in powers of the order parameter M have been calculated in the three–loop approximation. Numerical estimates for universal values of g_6 and g_8 at the critical point have been found by Padé and Padé–Borel resummations of the series obtained and by putting the quartic coupling constant equal to its fixed point value 0.988. Padé–Borel procedure applied to the expansions for g_6 and g_6/g_4^2 has given $g_6^* = 1.622$ and $g_6^* = 1.577$, respectively, while analogous treatment of the ϵ–expansion for g_6/g_4^2 has been shown to result in $g_6^* = 1.653$. Being in a good agreement to each other, these field–theoretical estimates are considerably smaller than those obtained by other methods. The RG expansion for g_8 has turned out to be less suitable for resummation than that for g_6 since it possesses rapidly growing coefficients. As a result, strongly scattered numerical estimates for g_8^* lying between 0.68 and 2.71 have been found.

VI. ACKNOWLEDGMENT

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ПРО ВІЛЬНУ ЕНЕРГІЮ ТРИВИМІРНОЇ МОДЕЛІ ІЗІНҐА В КРИТИЧНІЙ ОБЛАСТІ

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Вивчаються вершини вищого порядку при нульових зовнішніх моментах скалярної теорії поля, які описують критичну поведінку моделі Ізінґа в рамках теоретико–польового ренормгрупового (РГ) підходу у трьох вимірах. Безрозмірні шести- та восьмиточкові ефективні константи зв'язку g_6 та g_8 розраховуються у триплетлевому наближенні. Їх числові величини, універсальні при критичності, визначаються шляхом пересумовування Паде та Паде–Бореля отриманих РГ розкладів та підстановки четверної константи взаємодії рівною її універсальному значенню у нерухомій точці, яка відома з шестипетлевих РГ обчислень. Отримані значення g_6^* порівнюються з їх аналогами, одержаними за допомогою ϵ-розкладу, методу Монте Карло, рівнянь Веґнера–Гоуґтона та зв'язаних кластерних розкладів. Теоретико–польові оцінки g_6^* добре узгоджуються між собою та суттєво відрізняються від значень, отриманих іншими методами.