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THE PROBLEM OF THE "CHARGE" $Q^{(4)} = \mathbf{div}\ \mathbf{v}_s$ FOR SUPERFLUID VELOCITY IN CASE OF BOSE SYSTEMS

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The relation div $\mathbf{v}_s(t,\mathbf{r}) = Q^{(4)}(t,\mathbf{r}) \to -Q^{(4)}(-t,\mathbf{r})$ is considered for superfluid helium 4 and weakly interacting Bose gas. In respect to the latter it was shown that $Q^{(4)}$ is expressed in terms of the phase χ. It characterizes the order parameter $\langle\psi\rangle = \sqrt{\rho_c}e^{i\chi}$ as a consequence of the broken gauge symmetry. So, similarly to the Josephson effect the phase has here significance.
Key words: superfluidity, helium 4, weakly interacting Bose gas.

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In [1–3] it has been shown that for superfluid helium 3 (^{3}He-A,^{3}He-B) it is necessary to introduce the notion of the magnetic type "charge" $Q^{(3)}(t,\mathbf{r})$. Namely, the normalization condition $A_{ij}A^*_{ij} = 1$ for the matrix order parameter A_{ij} leads to the relation $\dot{A}_{ij}A^*_{ij} + A_{ij}\dot{A}^*_{ij} = 0$ where $\dot{A}_{ij}$ denotes the equation of motion. The presented relation is equivalent to the condition of the form div $\mathbf{v}_s(t,\mathbf{r}) = Q^{(3)}(t,\mathbf{r})$ where $\mathbf{v}_s$ denotes superfluid velocity. In ^{3}He-A the gauge symmetry and the rotation symmetry in the spin space and the orbital space are broken. The "charge" Q seems to occur because of the breaking of the rotation symmetry in the spin space (see also [4]).

Now we are interested in examining this problem in the case of superfluid helium 4. For the superfluid Bose system

$$\langle\psi(t,\mathbf{r})\rangle \neq 0 \tag{1}$$

where ψ is a Bose field operator and $\langle\ldots\rangle$ denotes averaging with the density matrix. Formula (1) which is a manifestation of the breaking of the gauge symmetry in superfluid ^{4}He ($\langle\chi\rangle$ plays a role of the parameter).

Because of the breaking of the gauge symmetry a new hydrodynamical parameter, superfluid velocity $\mathbf{v}_s$, should be introduced to the description of our system.

According to [5] we can write

$$\langle\psi(t,\mathbf{r})\rangle = \sqrt{\rho_c(t,\mathbf{r})}e^{i\chi(t,\mathbf{r})},$$
$$\langle\psi(t,\mathbf{r})\rangle\langle\psi^+(t,\mathbf{r})\rangle = \rho_c(t,\mathbf{r}). \tag{2}$$

We see that $\langle\psi\rangle$ vanishes if the density of the condensate ρ_c vanishes i.e. $\sqrt{\rho_c}$ can be treated as the order parameter. The hydrodynamic variable $\mathbf{v}_s$ is defined

$$\mathbf{v}_s(t,\mathbf{r}) \equiv \frac{\hbar}{m}\nabla\chi(t,\mathbf{r})$$
$$\to -\mathbf{v}_s(-t,\mathbf{r}) = -\frac{\hbar}{m}\nabla\chi(-t,\mathbf{r}). \tag{3}$$

The considered superfluid Bose system is described with the help of the Hamiltonian

$$\hat{H} = \frac{1}{2m}\int \nabla\psi^+(t,\mathbf{r})\nabla\psi(t,\mathbf{r})d\mathbf{r} - \lambda\int \psi^+(t,\mathbf{r})\psi(t,\mathbf{r})d\mathbf{r}$$
$$+\frac{1}{2}\int\int V(\mathbf{r}-\mathbf{r}')\psi^+(t,\mathbf{r})\psi^+(t,\mathbf{r})d\mathbf{r}')\psi(t,\mathbf{r})\psi(t,\mathbf{r}')d\mathbf{r}d\mathbf{r}'. \tag{4}$$

We omitted here the additional term introduced in [5] in order to underline the breaking of the gauge invariance. For the order parameter $\psi(t,\mathbf{r})$ we have the following equation of motion

$$i\hbar\frac{\partial\psi(t,\mathbf{r})}{\partial t} = [\psi(t,\mathbf{r}),\hat{H}] = -\lambda\psi(t,\mathbf{r}) - \frac{\hbar^2\nabla^2}{2m}\psi(t,\mathbf{r}) + \int d\mathbf{r}'V(\mathbf{r}-\mathbf{r}')\hat{\rho}(t,\mathbf{r}')\psi(t,\mathbf{r}). \tag{5}$$

After averaging (5) and taking into account (2), (3) we have

$$\hbar\frac{\partial\chi}{\partial t}=\lambda+\frac{\hbar^2\nabla^2\sqrt{\rho_c}}{2m\sqrt{\rho_c}}-\frac{m\mathbf{v}_s^2}{2}-\frac{1}{\rho_c}\int V\mathbf{R}\mathrm{Re}X_t(\mathbf{t},\mathbf{R})d\mathbf{R}, \tag{6}$$

$$\frac{\partial\rho_c}{\partial t}+\mathrm{div}(\rho_c\mathbf{v}_s)=-2\int V\mathbf{R}\mathrm{Im}X_t(\mathbf{t},\mathbf{R})d\mathbf{R}, \tag{7}$$

$$X_t(\mathbf{r},\mathbf{r}'-\mathbf{r})=X_t(\mathbf{r},\mathbf{R})=\langle\hat\rho(t,\mathbf{r}')\psi(t,\mathbf{r})\rangle\langle\psi^+(t,\mathbf{r}')\rangle, \tag{8}$$

$$(\rho_c)_{eq}=\rho_0,\quad \mathbf{R}=\mathbf{r}'-\mathbf{r}.$$

Near the equilibrium $\rho_c\simeq\rho_0$, we get the following relation which is of interest to us

$$\mathrm{div}\mathbf{v}_s(t,\mathbf{r})=-\frac{2}{\rho_0}\int V(\mathbf{R})\mathrm{Im}X_t(\mathbf{r},\mathbf{R})d\mathbf{R}=Q^{(4)}(t,\mathbf{r})\to Q^{(4)}(-t,\mathbf{r}). \tag{9}$$

In the absence of correlations, as in the mean-field approach, the expected value in (8) can be decoupled and

$$\mathrm{Im}X_t=\mathrm{Im}\langle\hat\rho(t,\mathbf{r})\rangle\langle\psi(t,\mathbf{r})\rangle\langle\psi^+(t,\mathbf{r})\rangle)=\mathrm{Im}\langle\hat\rho(t,\mathbf{r})\rangle\rho_c=0. \tag{10}$$

Thus $Q^{(4)}=0$ because $\langle\hat\rho(t,\mathbf{r})\rangle$ is real.

Now we will try to get expression for div $\mathbf{v}_s$ for a simple model of weakly interacting Bose systems. They are described by the Hamiltonian (6) (λ-fixed ensemble)

$$\hat H=\frac{U_0b_0^+b_0}{2V}+\sum_{p\neq0}\left(\frac{\hbar^2p^2}{2m}-\lambda\right)b_p^+b_p+\frac{U_0}{2V}\sum_{p\neq0}[b_0^2b_{-p}^+b_p^++b_0^{+2}b_pb_{-p}+b_0^+b_0b_p^+b_p]. \tag{11}$$

The Hamiltonian (11) leads to the following equations of motion

$$\begin{aligned} i\hbar\frac{\partial b_k(t)}{\partial t}&=\left(\frac{\hbar^2p^2}{2m}-\lambda\right)b_k(t)+\frac{U_0\rho_0}{V}[b_{-k}^+(t)+2b_k(t)],\\ -i\hbar\frac{\partial b_k^+(t)}{\partial t}&=\left(\frac{\hbar^2p^2}{2m}-\lambda\right)b_{-k}^+(t)+\frac{U_0\rho_0}{V}[2b_{-k}^+(t)+b_k(t)]. \end{aligned} \tag{12}$$

In addition

$$\begin{aligned} i\hbar\frac{\partial b_0(t)}{\partial t}&=(U_0\rho_0-\lambda)b_0(t)\\ &=-\lambda b_0+U_0\rho_0(2b_0+b_0^+)-U_0\rho_0(b_0+b_0^+),\\ -i\hbar\frac{\partial b_0^+(t)}{\partial t}&=(U_0\rho_0-\lambda)b_0^+(t)\\ &=-\lambda b_0^++U_0\rho_0(2b_0^++b_0)-U_0\rho_0(b_0+b_0^+), \end{aligned} \tag{13}$$

Eqs. (12), (13) give

$$\begin{aligned} i\hbar\frac{\partial\psi(t,\mathbf{r})}{\partial t}&=-\lambda\psi(t,\mathbf{r})-\frac{\hbar^2}{2m}\nabla^2\psi(t,\mathbf{r})\\ &+U_0\rho_0[2\psi(t,\mathbf{r})+\psi^+(t,\mathbf{r})]-2U_0\rho_0\sqrt{\rho_0}, \end{aligned}$$

$$\begin{aligned} i\hbar\frac{\partial\psi^+(t,\mathbf{r})}{\partial t}&=-\lambda\psi^+(t,\mathbf{r})+\frac{\hbar^2}{2m}\nabla^2\psi^+(t,\mathbf{r})\\ &-U_0\rho_0[2\psi^+(t,\mathbf{r})+\psi(t,\mathbf{r})]+2U_0\rho_0\sqrt{\rho_0}. \end{aligned} \tag{14}$$

From formula (2) we find

$$\begin{aligned} i\hbar\frac{\partial\langle\psi\rangle}{\partial t}&=e^{i\chi}\left[\frac{i\hbar}{2\sqrt{\rho_c}}\frac{\partial\rho_c}{\partial t}-\hbar\sqrt{\rho_c}\frac{\partial\chi}{\partial r}\right],\\ i\hbar\frac{\partial\langle\psi^+\rangle}{\partial t}&=e^{-i\chi}\left[\frac{i\hbar}{2\sqrt{\rho_c}}\frac{\partial\rho_c}{\partial t}+\hbar\sqrt{\rho_c}\frac{\partial\chi}{\partial r}\right]. \end{aligned} \tag{15}$$

On the basis of eqs. (15) we can derive the equation analogous to (7). Namely

$$\frac{\partial\rho_c}{\partial t}=-i\frac{\rho_c}{\hbar}\left[e^{i\chi}i\hbar\frac{\partial\langle\psi^+\rangle}{\partial t}+e^{i\chi}i\hbar\frac{\partial\langle\psi\rangle}{\partial t}\right]. \tag{16}$$

Now we average eqs. (14) and substitute (16). We have

$$\frac{\partial \rho_c}{\partial t} + \nabla(\rho_c \mathbf{v}_s) = -\frac{U_0 \rho_c \sqrt{\rho_c}}{\hbar} \times \sin\chi(\sqrt{\rho_c}\cos\chi - \sqrt{\rho_c}). \tag{17}$$

Near the equilibrium $\rho_c \sim const$. We have

$$\mathrm{div}\mathbf{v}_s(t,\mathbf{r}) = -\frac{4U_0\rho_0}{\hbar\sqrt{\rho_c}} \times \sin\chi(\sqrt{\rho_c}\cos\chi - \sqrt{\rho_c}) = Q^{(4)}(t,\mathbf{r}). \tag{18}$$

We see that in the expression for $Q^{(4)}(t,\mathbf{r})$ the phase plays an important role similarly as it happens at the consideration of the Josephson effect. The existence of the phase is a consequence of the gauge symmetry breaking. In the case of ^{3}He-A more important is the breaking of the rotation symmetry in the spin space.

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ПРОБЛЕМА “ЗАРЯДУ” $Q^{(4)} = \mathrm{div}\ \mathbf{v}_s$ ДЛЯ НАДПЛИННОЇ ШВИДКОСТИ У ВИПАДКУ БОЗЕ СИСТЕМ

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Розглядається співвідношення div $\mathbf{v}_s(t,\mathbf{r}) = Q^{(4)}(t,\mathbf{r}) \to -Q^{(4)}(-t,\mathbf{r})$ для надплинного гелію 4 і слабко взаємодіючого Бозе газу. Для останнього випадку показано, що $Q^{(4)}$ виражається через фазу χ. Вона характеризує параметр порядку $\langle\psi\rangle = \sqrt{\rho_c}e^{i\chi}$ в результаті порушення калібрувальної симетрії. Отже, подібно до ефекту Джозефсона фаза відіграє тут важливу роль.