JOURNAL OF PHYSICAL STUDIES
v. 1, No 4 (1997) p. 551–553

THE PROBLEM OF THE "CHARGE" $Q^{(4)} = \mathbf{div}\ \mathbf{v}_s$ FOR SUPERFLUID VELOCITY IN CASE OF BOSE SYSTEMS

Z. M. Galasiewicz

Institute of Theoretical Physics, University of Wrocław, Pl. Maksa Borna 9, 50–204 Wrocław, Poland

and

Institute of Low Temperatures and Structure Research of the Polish Academy of Sciences, Wrocław, Poland

(Received September 27, 1996)

The relation div $\mathbf{v}_s(t, \mathbf{r}) = Q^{(4)}(t, \mathbf{r}) \to -Q^{(4)}(-t, \mathbf{r})$ is considered for superfluid helium 4 and weakly interacting Bose gas. In respect to the latter it was shown that $Q^{(4)}$ is expressed in terms of the phase χ. It characterizes the order parameter $\langle\psi\rangle = \sqrt{\rho_c} e^{i\chi}$ as a consequence of the broken gauge symmetry. So, similarly to the Josephson effect the phase has here significance.

Key words: superfluidity, helium 4, weakly interacting Bose gas.

PACS number(s): 67.40.-w

In [1–3] it has been shown that for superfluid helium 3 (^{3}He-A,^{3}He-B) it is necessary to introduce the notion of the magnetic type "charge" $Q^{(3)}(t, \mathbf{r})$. Namely, the normalization condition $A_{ij}A^*_{ij} = 1$ for the matrix order parameter A_{ij} leads to the relation $\dot{A}_{ij}A^*_{ij} + A_{ij}\dot{A}^*_{ij} = 0$ where $\dot{A}_{ij}$ denotes the equation of motion. The presented relation is equivalent to the condition of the form div $\mathbf{v}_s(t, \mathbf{r}) = Q^{(3)}(t, \mathbf{r})$ where $\mathbf{v}_s$ denotes superfluid velocity. In ^{3}He-A the gauge symmetry and the rotation symmetry in the spin space and the orbital space are broken. The "charge" Q seems to occur because of the breaking of the rotation symmetry in the spin space (see also [4]).

Now we are interested in examining this problem in the case of superfluid helium 4. For the superfluid Bose system

$$\langle\psi(t, \mathbf{r})\rangle \neq 0 \tag{1}$$

where ψ is a Bose field operator and $\langle\ldots\rangle$ denotes averaging with the density matrix. Formula (1) which is a manifestation of the breaking of the gauge symmetry in superfluid ^{4}He ($\langle\chi\rangle$ plays a role of the parameter).

Because of the breaking of the gauge symmetry a new hydrodynamical parameter, superfluid velocity $\mathbf{v}_s$, should be introduced to the description of our system.

According to [5] we can write

$$\langle\psi(t, \mathbf{r})\rangle = \sqrt{\rho_c(t, \mathbf{r})} e^{i\chi(t,\mathbf{r})},$$
$$\langle\psi(t, \mathbf{r})\rangle\langle\psi^+(t, \mathbf{r})\rangle = \rho_c(t, \mathbf{r}). \tag{2}$$

We see that $\langle\psi\rangle$ vanishes if the density of the condensate ρ_c vanishes i.e. $\sqrt{\rho_c}$ can be treated as the order parameter. The hydrodynamic variable $\mathbf{v}_s$ is defined

$$\mathbf{v}_s(t, \mathbf{r}) \equiv \frac{\hbar}{m}\nabla\chi(t, \mathbf{r})$$
$$\to -\mathbf{v}_s(-t, \mathbf{r}) = -\frac{\hbar}{m}\nabla\chi(-t, \mathbf{r}). \tag{3}$$

The considered superfluid Bose system is described with the help of the Hamiltonian

$$\hat{H} = \frac{1}{2m}\int \nabla\psi^+(t, \mathbf{r})\nabla\psi(t, \mathbf{r})d\mathbf{r} - \lambda\int \psi^+(t, \mathbf{r})\psi(t, \mathbf{r})d\mathbf{r}$$
$$+ \frac{1}{2}\int\int V(\mathbf{r} - \mathbf{r}')\psi^+(t, \mathbf{r})\psi^+(t, \mathbf{r})d\mathbf{r}')\psi(t, \mathbf{r})\psi(t, \mathbf{r}')d\mathbf{r}d\mathbf{r}'. \tag{4}$$

We omitted here the additional term introduced in [5] in order to underline the breaking of the gauge invariance. For the order parameter $\psi(t, \mathbf{r})$ we have the following equation of motion

$$i\hbar\frac{\partial\psi(t, \mathbf{r})}{\partial t} = [\psi(t, \mathbf{r}), \hat{H}] = -\lambda\psi(t, \mathbf{r}) - \frac{\hbar^2\nabla^2}{2m}\psi(t, \mathbf{r}) + \int d\mathbf{r}' V(\mathbf{r} - \mathbf{r}')\hat{\rho}(t, \mathbf{r}')\psi(t, \mathbf{r}). \tag{5}$$

After averaging (5) and taking into account (2), (3) we have

$$\hbar\frac{\partial\chi}{\partial t} = \lambda + \frac{\hbar^2\nabla^2\sqrt{\rho_c}}{2m\sqrt{\rho_c}} - \frac{m\mathbf{v}_s^2}{2} - \frac{1}{\rho_c}\int V\mathbf{R}\mathrm{Re}X_t(\mathbf{t},\mathbf{R})d\mathbf{R}, \tag{6}$$

$$\frac{\partial\rho_c}{\partial t} + \mathrm{div}(\rho_c\mathbf{v}_s) = -2\int V\mathbf{R}\mathrm{Im}X_t(\mathbf{t},\mathbf{R})d\mathbf{R}, \tag{7}$$

$$X_t(\mathbf{r},\mathbf{r}^{'}-\mathbf{r}) = X_t(\mathbf{r},\mathbf{R}) = \langle\hat{\rho}(t,\mathbf{r}^{'})\psi(t,\mathbf{r})\rangle\langle\psi^+(t,\mathbf{r}^{'})\rangle, \tag{8}$$

$$(\rho_c)_{eq} = \rho_0, \quad \mathbf{R} = \mathbf{r}^{'} - \mathbf{r}.$$

Near the equilibrium $\rho_c \simeq \rho_0$, we get the following relation which is of interest to us

$$\mathrm{div}\mathbf{v}_s(t,\mathbf{r}) = -\frac{2}{\rho_0}\int V(\mathbf{R})\mathrm{Im}X_t(\mathbf{r},\mathbf{R})d\mathbf{R} = Q^{(4)}(t,\mathbf{r}) \to Q^{(4)}(-t,\mathbf{r}). \tag{9}$$

In the absence of correlations, as in the mean-field approach, the expected value in (8) can be decoupled and

$$\mathrm{Im}X_t = \mathrm{Im}\langle\hat{\rho}(t,\mathbf{r})\rangle\langle\psi(t,\mathbf{r})\rangle\langle\psi^+(t,\mathbf{r})\rangle) = \mathrm{Im}\langle\hat{\rho}(t,\mathbf{r})\rangle\rho_c = 0. \tag{10}$$

Thus $Q^{(4)} = 0$ because $\langle\hat{\rho}(t,\mathbf{r})\rangle$ is real.

Now we will try to get expression for div $\mathbf{v}_s$ for a simple model of weakly interacting Bose systems. They are described by the Hamiltonian (6) (λ-fixed ensemble)

$$\hat{H} = \frac{U_0 b_0^+ b_0}{2V} + \sum_{p\neq 0}\left(\frac{\hbar^2 p^2}{2m} - \lambda\right) b_p^+ b_p + \frac{U_0}{2V}\sum_{p\neq 0}[b_0^2 b_{-p}^+ b_p^+ + b_0^{+2} b_p b_{-p} + b_0^+ b_0 b_p^+ b_p]. \tag{11}$$

The Hamiltonian (11) leads to the following equations of motion

$$\begin{aligned} i\hbar\frac{\partial b_k(t)}{\partial t} &= \left(\frac{\hbar^2 p^2}{2m} - \lambda\right) b_k(t) + \frac{U_0\rho_0}{V}[b_{-k}^+(t) + 2b_k(t)], \\ -i\hbar\frac{\partial b_k^+(t)}{\partial t} &= \left(\frac{\hbar^2 p^2}{2m} - \lambda\right) b_{-k}^+(t) + \frac{U_0\rho_0}{V}[2b_{-k}^+(t) + b_k(t)]. \end{aligned} \tag{12}$$

In addition

$$\begin{aligned} i\hbar\frac{\partial b_0(t)}{\partial t} &= (U_0\rho_0 - \lambda)b_0(t) \\ &= -\lambda b_0 + U_0\rho_0(2b_0 + b_0^+) - U_0\rho_0(b_0 + b_0^+), \\ -i\hbar\frac{\partial b_0^+(t)}{\partial t} &= (U_0\rho_0 - \lambda)b_0^+(t) \\ &= -\lambda b_0^+ + U_0\rho_0(2b_0^+ + b_0) - U_0\rho_0(b_0 + b_0^+), \end{aligned} \tag{13}$$

Eqs. (12), (13) give

$$i\hbar\frac{\partial\psi(t,\mathbf{r})}{\partial t} = -\lambda\psi(t,\mathbf{r}) - \frac{\hbar^2}{2m}\nabla^2\psi(t,\mathbf{r})$$
$$+U_0\rho_0[2\psi(t,\mathbf{r}) + \psi^+(t,\mathbf{r})] - 2U_0\rho_0\sqrt{\rho_0},$$

$$i\hbar\frac{\partial\psi^+(t,\mathbf{r})}{\partial t} = -\lambda\psi^+(t,\mathbf{r}) + \frac{\hbar^2}{2m}\nabla^2\psi^+(t,\mathbf{r}) \tag{14}$$
$$-U_0\rho_0[2\psi^+(t,\mathbf{r}) + \psi(t,\mathbf{r})] + 2U_0\rho_0\sqrt{\rho_0}.$$

From formula (2) we find

$$\begin{aligned} i\hbar\frac{\partial\langle\psi\rangle}{\partial t} &= e^{i\chi}\left[\frac{i\hbar}{2\sqrt{\rho_c}}\frac{\partial\rho_c}{\partial t} - \hbar\sqrt{\rho_c}\frac{\partial\chi}{\partial r}\right], \\ i\hbar\frac{\partial\langle\psi^+\rangle}{\partial t} &= e^{-i\chi}\left[\frac{i\hbar}{2\sqrt{\rho_c}}\frac{\partial\rho_c}{\partial t} + \hbar\sqrt{\rho_c}\frac{\partial\chi}{\partial r}\right]. \end{aligned} \tag{15}$$

On the basis of eqs. (15) we can derive the equation analogous to (7). Namely

$$\frac{\partial\rho_c}{\partial t} = -i\frac{\rho_c}{\hbar}\left[e^{i\chi}i\hbar\frac{\partial\langle\psi^+\rangle}{\partial t} + e^{i\chi}i\hbar\frac{\partial\langle\psi\rangle}{\partial t}\right]. \tag{16}$$

Now we average eqs. (14) and substitute (16). We have

$$\frac{\partial \rho_c}{\partial t} + \nabla(\rho_c \mathbf{v}_s) = -\frac{U_0 \rho_c \sqrt{\rho_c}}{\hbar} \times \sin\chi(\sqrt{\rho_c}\cos\chi - \sqrt{\rho_c}). \tag{17}$$

Near the equilibrium $\rho_c \sim const$. We have

$$\mathrm{div}\mathbf{v}_s(t,\mathbf{r}) = -\frac{4U_0\rho_0}{\hbar\sqrt{\rho_c}} \times \sin\chi(\sqrt{\rho_c}\cos\chi - \sqrt{\rho_c}) = Q^{(4)}(t,\mathbf{r}). \tag{18}$$

We see that in the expression for $Q^{(4)}(t,\mathbf{r})$ the phase plays an important role similarly as it happens at the consideration of the Josephson effect. The existence of the phase is a consequence of the gauge symmetry breaking. In the case of ^{3}He-A more important is the breaking of the rotation symmetry in the spin space.

ACKNOWLEDGEMENTS

The work was supported by the KBN grant No P302 02206.

---

[1] Z. M. Galasievicz, J. Low Temp. Phys. **57**, 123 (1984).
[2] Z. M. Galasievicz, J. Low Temp. Phys. **72**, 153 (1984).
[3] Z. M. Galasievicz, Physica A **159**, 301 (1989).
[4] Z. M. Galasievicz, Physica A **231**, 461 (1996).
[5] N. N. Bogoliubov, *Lectures on Quantum Statistics* (Gordon and Breach, New–York, 1970).
[6] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).

ПРОБЛЕМА "ЗАРЯДУ" $Q^{(4)} = \mathrm{div}\ \mathbf{v}_s$ ДЛЯ НАДПЛИННОЇ ШВИДКОСТИ У ВИПАДКУ БОЗЕ СИСТЕМ

З. М. Ґаласєвіч
Інститут теоретичної фізики Вроцлавського університету,
пл. Макса Борна, 9, 50–204 Вроцлав, Польща
і
Інститут низьких температур і структурних досліджень
Польської академії наук, Вроцлав, Польща

Розглядається співвідношення div $\mathbf{v}_s(t,\mathbf{r}) = Q^{(4)}(t,\mathbf{r}) \to -Q^{(4)}(-t,\mathbf{r})$ для надплинного гелію 4 і слабко взаємодіючого Бозе газу. Для останнього випадку показано, що $Q^{(4)}$ виражається через фазу χ. Вона характеризує параметр порядку $\langle\psi\rangle = \sqrt{\rho_c}e^{i\chi}$ в результаті порушення калібрувальної симетрії. Отже, подібно до ефекту Джозефсона фаза відіграє тут важливу роль.