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ON THE DYNAMIC THEORY OF SYSTEMS IN RANDOM FIELDS

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A new method of a reduced description of systems with noise is developed. This method is based on the fact that among distributions of the solution of stochastic differential equations there are ones which may be described as similar to the kinetic description of states in statistical physics. The problem of the construction of asymptotic evolution operator is resolved by summing up secular terms of the perturbation series in a random field.

Key words: stochastic differential equations, method of reduced description, asymptotic evolution operator, perturbation theory, correlation radius, integral equation.

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I. REDUCED DESCRIPTION CONCEPT FOR RANDOM PROCESSES

Let us formulate the concept of reduced description in application to random processes. The problem lies in studying the stochastic system the dynamics of which is given by differential equations [1,2,3],

$$\dot{x}_i = h^{(i)}(x,t), \quad i = 1, ..., n, \tag{1.1}$$

where $x_i(t)$ are dynamical variables and $h(x,t)$ is a random field. Otherwise speaking, the problem is to construct the distribution function of the solutions of eqs. (1.1) and to study its asymptotics at large times.

The random field $h(x,t)$ is defined with the help of a stochastic model. We define a stochastic space (Ω, σ, P), where Ω is the space of random realizations of the field $h(x,t)$, σ is the set of all possible random events for the field $h(x,t)$ (from the formal mathematical point of view σ is the σ-algebra of measurable subsets Ω, [4]) $P(\cdot)$ is a non negative function such that the equations $P(\Omega) = 1$, $P(\emptyset) = 0$ take place and for any set of A_n from σ $\{A_n \in \sigma : A_n \bigcap A_m = \emptyset, n \neq m\}$ the identity $P(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} P(A_n)$ holds.

For any physical functional $G(h)$ of the random field $h(x,t)$, the average procedure is defined by the integral

$$< G(h) >= \int G(h) dP_h. \tag{1.2}$$

It is convenient to suppose, that the space Ω is characterized by some parameter ω, i.e. $h(x,t) \equiv h^{\omega}(x,t)$. Thus the solutions of eqs. (1.1) with the initial conditions $x_i|_{t=0} = x_i(0) \equiv x(0)$ denoted as $X_i^{\omega}(t, x(0)$ depend on ω. Let us denote by $f(x(0),0)$ the distribution function of the initial data $x(0)$ normalized as follows: $\int dx(0) f(x(0),0) = 1$. The random distribution function $f_{\omega}(x,t)$ of the values x at the time t is defined by

$$f_{\omega}(x,t) = \int dx(0) f(x(0),0) \delta(x - X_{\omega}(t, x(0))). \tag{1.3}$$

Let $x = X^{-1}(t,y)$ be the solution of the system of equations $X(t,x) = y$. If we pass in eq. (1.3) from $x(0)$ to y, $y = X(t,x(0))$, then we obtain

$$f(x,t) = J(x,t) f(X^{-1}(t,x),0), \tag{1.4}$$

where $J(x,t) = ||\partial X^{-1}(x,t)/\partial x||$ is the Jacobi determinant of transition from the variables $X^{-1}(t,x)$ to the variables x.

We shall consider at first the case when the functions $h_i(x,t)$ are time independent, $h_i(x,t) = h_i(x)$. As far as, according to eqs. (1.1), we have $(d^n x_i/dt^n)_{t=0} = (h_j \partial/\partial x_j)^n x_i|_{x=x(0)}$ it is easy to obtain $X_i(t,x) = \sum\limits_{n=0}^{\infty} \frac{t^n}{n!}(h_j \partial/\partial x_j)^n x_i$ and, hence, $X_i(t,x) = \exp\{t\Lambda(x)\} x_i$, $\Lambda(x) = h_j(x)\frac{\partial}{\partial x_j}$. Besides one can easyly see that $X(t, X(t',x)) = X(t+t',x)$ and, hence, $X(tX(-t,x)) = x$. Therefore, we obtain $X^{-1}(t,x) = X(-t,x) = \exp\{-t\Lambda\}x$. Using this relation eq. (1.4) can be written as

$$f(x,t) = J(x,t) \exp\{-t\Lambda\} f(x,0).$$

Therefore, we have

$$\frac{\partial f}{\partial t} = \frac{\partial J}{\partial t} J^{-1} f - J h_j \frac{\partial}{\partial x_j} J^{-1} f. \tag{1.5}$$

Note that according to eq. (1.3) the condition $\int \frac{\partial f}{\partial t} dx = 0$ takes place. So, we have

$$\int dx f \left\{ J^{-1} \frac{\partial J}{\partial t} + J^{-1} \frac{\partial}{\partial x_j} J h_j \right\} = 0,$$

and for arbitrary f one obtains

$$\frac{\partial J}{\partial t}+\frac{\partial}{\partial x_j}Jh_j=0, J|_{t=0}=1. \tag{1.6}$$

It is possible to transform eq. (1.5) as

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x_j}(h_j f)=0. \tag{1.7}$$

Note that the formal solution of eq. (1.6) is $J(x,t)=\exp\{-t\frac{\partial}{\partial x_j}h_j...\}\cdot 1$. The Jacobi determinant satisfies the condition $J=1$ which corresponds to the canonical transformation of classical mechanics.

Let us pass now to the consideration of the general case of $h_j=h_j(x,t)$. It is easy to see that eqs. (1.1) can be rewritten as

$$\tilde{x}_i=\tilde{h}_i(x,a), \quad \dot{a}=\tilde{h}(a), \quad a|_{t=0}=a_0, \tag{1.8}$$

where we have $\tilde{h}_i(x,a(t,a_0))=h_i(x,t)$, and $a(t,a_0)$ is the solution of the equation for a. An arbitrary solution of eqs. (1.8) is $X_i(t,x(0),a_0)$, $A(t,a_0)$, $(X_i(0,x(0),a_0)=x_i(0),\ A(0,a_0)=a(0)=a_0)$. Then according to the construction we have

$$X_i(t,x,a_0)\equiv X(t,x). \tag{1.9}$$

So the solutions x and a of the equations $X_i(t,x,a_0)=y$, $A(t,a)=z$ have the form of $x=X(-t,y,z)$ and $a=A(t,z)$. Taking into account (1.9) we find $X(t,X(-t,y,A(t,a_0)))=y$ and hence

$$X^{-1}(t,y)=X(-t,y,A(t,a_0)). \tag{1.10}$$

In accordance with eq. (1.4) the function $f(x,a,t)$ of the values x and a at the time moment t is equal

$$f(x,a,t)=J(x,a,t)f(X(-t,x,a),0)\delta(A(-t,a)-a_0). \tag{1.11}$$

By defining $f(x,a,0)=f(x,0)\delta(a-a_0)$, and taking into account eq. (1.10) we obtain

$$J(x,a,t)|_{a=h(t,a_0)}=\frac{\partial(X(-t,x,a),A(-t,a))}{\partial(x,a)}$$
$$=J(x,t)\frac{\partial A(-t,a)}{\partial a}, \tag{1.12}$$

where $J(x,a,t)|_{a=h(t,a_0)}$ is the Jacobi determinant of a transformation from the variables $X(-t,x,a)$, $A(-t,a)$, to the variables x,a. It is easy to see, that $J(x,a,t)\delta(A(-t,a)-a_0)=J(x,t)\delta(a-A(t,a_0))$, where $J(x,t)$ is the Jacobi determinant in eq. (1.4). Thus, comparing eq. (1.4) and eq. (1.11) and, taking into account eq. (1.12) we obtain

$$f(x,a,t)=f(x,t)\delta(a-A(t,a_0)). \tag{1.13}$$

On the other hand, eq. (1.7) can now be expressed as

$$\frac{\partial f(x,a,t)}{\partial t}+\frac{\partial}{\partial x_j}(\tilde{h}_j(x,a)f(x,a,t)) \tag{1.14}$$
$$+\frac{\partial}{\partial a}(\tilde{h}_j(a)f(x,a,t))=0.$$

Substituting eq. (1.13), into that one yields the equation

$$\frac{\partial f_\omega(x,t)}{\partial t}+\frac{\partial}{\partial x_j}(h_j^\omega(x,t)f_\omega(x,t))=0. \tag{1.15}$$

It is the equation of motion for the random function of the distribution of solutions of eqs. (1.1). When eqs. (1.1) are Hamilton equations for momenta and coordinates $(x=p,q)$ then eq. (1.14) represents the Liouville equation for the distribution function of particles located in some random field. So, we will record eq. (1.14) in the general case in the following way

$$i\frac{\partial f_\omega(x,t)}{\partial t}=\Lambda_\omega(t)f_\omega(x,t), \tag{1.16}$$

where $\Lambda_\omega(t)$ is the "Liouville" operator, $\Lambda_\omega(t)\equiv -i\frac{\partial}{\partial x_j}h_j^\omega(x,t)$, acting on the variable x of the random function of the distribution $f_\omega(x,t)$.

It should be pointed out that eq. (1.16) is true also in the quantum case, for the random statistical operator, Λ_ω

$$\Lambda_\omega f_\omega\equiv[H_\omega,f_\omega],$$

where H_ω is the Hamilton operator of the system located in the random field. Then Λ_ω is the operator, acting not in the Hilbert space of state vectors, but in the space of statistical operators.

We suggest, that the operator $\Lambda_\omega(t)$ can be presented as

$$\Lambda_\omega(t)=\Lambda^{(0)}+\Lambda_\omega^{(1)}(t),\ \Lambda^{(1)}(t)=\int dq\varphi_\omega(q,t)a(q), \tag{1.17}$$

where $\Lambda^{(0)}$ is the time-independent part of the operator $\Lambda_\omega(t),\varphi_\omega(q,t)$ is the Fourier-component (see further) of ω — realization of the random field φ and $a(q)$ are some operators which depend on q as on a parameter. Eq. (1.17) is convenient for the construction of the perturbation series in the random field φ. Note that a representation of eq. (1.17) arises, for example, if one considers the many-particles system located in the external random field $\varphi_\omega(x,t)$ when the Hamiltonian has the form of $H_\omega = H_0 + \varphi_\omega(x,t)$, where H_0 is the Hamiltonian of free particles. Eq. (1.16), being average over the random field realizations, leads to the equation of motion for the average (true) distribution function $f(x) =< f_\omega(t) >$,

$$\partial_t f(t) = -i\Lambda^{(0)} f(t) - i < \Lambda_\omega^{(1)} f_\omega(t) >= -i\Lambda^{(0)} f + L(t).$$

The value $L(t)$ is connected with the particle-field correlation function $f_s(q_1 t_1, ..., q_s t_s; t)$

$$f_s(q_1 t_1, ..., q_s t_s; t) =< \varphi_\omega(q_1, t_1 + t)...\varphi_\omega(q_s, t_s + t) f(t) > \tag{1.18}$$

by means of the formula $L(t) = -i \int dq a(q) f_1(qt_1; t)|_{t_1=0}$.

In section III for the correlation functions f_s the principle of weakening correlations, i.e. analogy to the principle of space weakening correlations in the statistical mechanics, is formulated.

One can easily see, that the correlation functions f_s satisfy a chain of differential equations

$$\left\{ \frac{\partial}{\partial t} - \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_s} \right) \right\} f_s(q_1 t_1, ..., q_s t_s; t) + i\Lambda^{(0)} f_s \tag{1.19}$$

$$= -i \int dq_{s+1} a(q_{s+1}) f_{s+1}(q_1 t_1, ..., q_s t_s, q_{s+1} t_{s+1}; t)|_{t_{s+1}} = 0.$$

The functions $f_s(q_1 t_1, ..., q_s t_s; t)$ are similar to the s-particles distribution functions of statistical mechanics. Eq. (1.19) is analogy to the chain of BBGKY equations for many-particles system with pair interaction [5, 6].

In this work we prove, that at large times ($t \gg \tau_0$, where τ_0 is the time correlation scale of random field or chaotization time) "one-particle distribution function" $f(t)$ of the solutions of random differential equations (1.1) can be approximated by some coarse distribution function $\widetilde{f}(t)$,

$$f(t) \underset{t \gg \tau_0}{\longrightarrow} \widetilde{f}(t)) = \exp(Lt) S_0 f(0),$$

$$L(t) \underset{t \gg \tau_0}{\longrightarrow} L(\widetilde{f}(t)),$$

where L is the "collision integral" and S_0 is the memory operator.

These asymptotic relations show the possibility of a kinetic description of the random system (1.1) for the time $t \gg \tau_0$ [4,5] and the existence of a reduced description mode for such a system.

II. THE AVERAGE EVOLUTION OPERATOR. THE GAUSS δ–CORRELATED FIELD

In this section we will find the formal solution of the Cauchi problem for eq. (1.16). The expression for the average evolution operator will be obtained. It will be suggested that at the initial moment of time the function $f_\omega(t)$ is independent on ω, $f_\omega(t)|_{t=0} = f(0)$. The representation (1.17) of the operator Λ_ω permits to write the solution of eq. (1.16) as

$$f_\omega(t) = \exp\{-i\Lambda^{(0)} t\} S_\omega(t) f(0), \tag{2.1}$$

$$S_\omega(t) = T \exp \left\{ -i \int_0^t d\tau \widetilde{\Lambda}_\omega(\tau) \right\},$$

where T is the time ordering operator and

$$\widetilde{\Lambda}_\omega(\tau) = \int dq \varphi_\omega(q,\tau) a(q,\tau),$$

$$a(q,\tau) = \exp(i\Lambda^{(0)}\tau) a(q,0) \exp(-i\Lambda^{(0)}\tau).$$

In the case when the Hamiltonian of the system is defined as follows

$$H = \sum_{j=1}^{N} \left\{ \frac{p_j^2}{2m} + \varphi_\omega(x_j, t) \right\},$$

the operators $\Lambda^{(0)}$ and $\Lambda_\omega^{(1)}(t)$ are given by

$$\Lambda^{(0)} = -i \sum_{j=1}^{N} \frac{p_j}{m} \frac{\partial}{\partial x_j}, \tag{2.2}$$

$$\Lambda_{\omega}^{(1)}(t) = \int dq \varphi_{\omega}(q,t) \sum_{j=1}^{N} e^{iqx_j} q \frac{\partial}{\partial p_j},$$

where $\varphi_{\omega}(q,t) = \frac{1}{(2\pi)^n} \int dx e^{-iqx} \varphi_{\omega}(x,t)$ is the Fourier component of the random field $\varphi_{\omega}(x,t)$ (n is a dimension of space). Therefore, in the framework of classical consideration the action of one-particle operator is defined as follows

$$a(q,0)f = e^{iqx_j} q \frac{\partial}{\partial p_j} f. \tag{2.3}$$

Averaging eq. (2.1) over random field realizations, we obtain

$$f(t) = e^{-i\Lambda^{(0)}t} S(t) f(0), \tag{2.4}$$

$$S(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dq_1 \cdots dq_n T \int_0^t d\tau_1 \cdots \int_0^t d\tau_n \chi_n(q_1\tau_1 \cdots q_n\tau_n) a(q_1\tau_1) \cdots a(q_n\tau_n), \tag{2.5}$$

where

$$\chi_n(q_1\tau_1 \cdots q_n\tau_n) \equiv < \varphi_{\omega}(q_1\tau_1) \cdots \varphi_{\omega}(q_n\tau_n) > \tag{2.6}$$

is the nth order random field moment.

The main goal is the finding of asymptotic at $t \gg \tau_0$ behavior of the function $f(t)$. It should be pointed out that the problem of constructing the asymptotic evolution operator for eqs. (1.1) was formulated for the first time by van Kampen [7] by means of the cumulant expansion.

The average evolution operator $S(t)$ can be written as follows

$$S(t) = T \left\{ \exp\left(-i \int_0^t d\tau \int dq a(q,\tau) \frac{\delta}{\delta\lambda(q,\tau)} \right) F(\lambda)|_{\lambda(q,\tau)=0} \right\},$$

where

$$F(\lambda) = \sum_{m=1}^{\infty} \frac{1}{m!} \int dq_1 \cdots dq_m \int_{-\infty}^{\infty} d\tau_1 \cdots \int_{-\infty}^{\infty} d\tau_m \chi_m(q_1\tau_1 \cdots q_m\tau_m) \lambda(q_1\tau_1) \cdots \lambda(q_m\tau_m)$$

is the generating functional of the moments $\chi_m(q_1\tau_1 \cdots q_m\tau_m)$ connected with the generating functional of the random field cumulants

$$G(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \int dq_1 \cdots dq_n \int_{-\infty}^{\infty} d\tau_1 \cdots \int_{-\infty}^{\infty} d\tau_n g_n(q_1\tau_1 \cdots q_n\tau_n) \lambda(q_1\tau_1) \cdots \lambda(q_n\tau_n)$$

($g_n(q_1\tau_1 \cdots q_n\tau_n)$ is the nth order cumulant) by the relation $F(\lambda) = \exp G(\lambda)$. Taking into account that $\exp\left(-i \int_0^t d\tau \int dq a(q,\tau) \frac{\delta}{\delta\lambda(q,\tau)} \right)$ is the operator of shift over $\lambda(q,\tau)$ on the value $-i\eta_t(\tau) a(q,\tau)$, (where one has $\eta_t(\tau) = 1$ for $0 < \tau < 1$ and $\eta_t(\tau) = 1$ for $\tau < 0,\ \tau > t$), we obtain

$$S(t) = T \exp G(-i\eta_t(\tau) a(q,\tau)). \tag{2.7}$$

For the random stationary field we have $g_n(q_1\tau_1 \cdots q_n\tau_n) = g_n(q_1 \cdots q_n) \delta(\tau_1 - \tau_2) \cdots \delta(\tau_1 - \tau_n)$. Eq. (2.5) reduces to

$$S(t) = T \exp\left(\int_0^t d\tau L(\tau) \right),$$

where $L(\tau) = \exp(i\Lambda^{(0)}\tau)L\exp(-i\Lambda^{(0)}\tau)$ and

$$L = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dq_1 \cdots dq_n g_n(q_1 \cdots q_n) a(q_1 0) \cdots a(q_n 0). \tag{2.8}$$

Therefore, one obtains from eq. (2.4)

$$f(t) = \left\{ e^{-i\Lambda^{(0)}t} T \exp \int_0^t d\tau L(\tau) \right\} f(0).$$

The kinetic equation for $f(t)$ is then

$$\frac{\partial f}{\partial t} + i\Lambda^{(0)} f = Lf, \tag{2.9}$$

where the "collision integral" is defined by eq. (2.8). In particular, for the Gauss field, when $g_n(q_1\tau_1, ..., q_n\tau_n) = 0$ for $n > 2$, the "collision integral" L is given by

$$L = -i \int dq g_1(q) a(q, 0) \tag{2.10}$$

$$- \frac{1}{2} \int dq_1 \int dq_2 g_2(q_1 q_2) a(q_1, 0) a(q_2, 0).$$

Using the structure of the operators $\Lambda^{(0)}$ and $\Lambda^{(1)}_{\omega}(t)$ and eq. (2.10) for the "collision integral" it is easy to write eq. (2.9) as the Fokker – Plank equation

$$\frac{\partial f(t)}{\partial t} + \frac{p_k}{m} \frac{\partial f(t)}{\partial x_k} - \frac{\partial < \varphi(x) >}{\partial x_k} \frac{\partial f(t)}{\partial p_k} \tag{2.11}$$

$$= D_{ij}(x) \frac{\partial^2}{\partial p_i \partial p_j} f(t),$$

where $D_{ij}(x) = \frac{1}{2} \frac{\partial^2}{\partial x_{1i} \partial x_{2j}} g(x_1, x_2) | x_1 = x_2 = x$. As far as for the random δ–time correlation fields one has $\tau_0 = 0$, the evolution of the system starts from the kinetic stage.

The research of the asymptotic structure of evolution operator for a non δ–correlated random field will be done in the next sections.

III. THE CORRELATION WEAKENING PRINCIPLE FOR THE RANDOM FIELD

In this section we will study some of the special values of $\widetilde{\chi}$ (closely connected with χ) which will be necessary for the construction of the evolution operator for the time correlated processes.

Let us assume that the moments $\chi_s(\tau_1 \cdots \tau_s)$ satisfy the correlation weakening principle. Namely, when $\tau - \tau' \to \infty$, and $\tau_1 \cdots \tau_l \sim \tau$, $\tau_{l+1} \cdots \tau_s \sim \tau'$, the moments are disintegrated as (for simplicity we suppose that $\tau_i \equiv (q_i, \tau_i)$)

$$\chi_s(\tau_1 \cdots \tau_l, \tau_{l+1} \cdots \tau_s)$$

$$\underset{\tau - \tau' \to \infty}{\longrightarrow} \chi_l(\tau_1 \cdots \tau_l) \chi_{s-l}(\tau_{l+1} \cdots \tau_s).$$

We define the values $\widetilde{\chi}_l(\tau_1 \cdots \tau_l)$, connected with the moments $\chi_s(\tau_1 \cdots \tau_s)$ by means of the relation

$$\chi_s(\tau_1 \cdots \tau_s) = \sum_{l=1}^{s} \widetilde{\chi}_l(\tau_1 \cdots \tau_l) \chi_{s-l}(\tau_{l+1} \cdots \tau_s), \tag{3.1}$$

$$s = 1, 2, ..., \ \chi_0 = 1.$$

Let us study the properties of the values $\widetilde{\chi}_l$. We designate through D_l such a set of the variables $\tau_1 \cdots \tau_l$, in which one has $\tau_1 > \tau_2 > \cdots > \tau_l$. Then the following theorem is true [1].

Theorem 1. Let $\tau_1 \cdots \tau_l$, be the elements of the set D_l, $\tau_1 \cdots \tau_l \in D_l$, $\tau_1 \cdots \tau_k \sim \tau$ and $\tau_{k+1} \cdots \tau_s \sim \tau'$. If χ_l satisfies the correlation weakening principle then we have

$$\widetilde{\chi}_l(\tau_1 \cdots \tau_l) \underset{\tau - \tau' \to \infty}{\longrightarrow} 0. \tag{3.2}$$

This theorem shows that in the set D_l the value $\widetilde{\chi}_l(\tau_1 \cdots \tau_l)$ is different from zero only if $\tau_1, \tau_2, ..., \tau_l$ differ from each other less than the time correlation scale τ_0. The values of χ_l do not coincide with the cumulants g_l.

The proof. Starting from eq. (3.1) the following equation can be derived

$$\widetilde{\chi}_s(\tau_1 \cdots \tau_s) = \chi_s(\tau_1 \cdots \tau_s) \tag{3.3}$$

$$- \sum_{l=1}^{s-1} \widetilde{\chi}_l(\tau_1 \cdots \tau_s) \chi_{s-l}(\tau_l + 1 \cdots \tau_s),$$

$$s = 2, 3, ..., \widetilde{\chi}_1 = \chi_1.$$

By direct checking it is easy to obtain

$$\widetilde{\chi}_2(\tau_1 \tau_2) \underset{\tau - \tau' \to \infty}{\longrightarrow} 0.$$

We assume that eq. (3.2) is true at $l = 2, 3, ..., s$ and any $k, 1 \le k \le l - 1$. Let us prove the validity of eq. (3.2) for $l = s + 1$. Let $\tau_1 \cdots \tau_k \sim \tau$, $\tau_k + 1 \cdots \tau_l \sim \tau'$, then the correlation weakening principle for the values of χ_l leads to

$$\widetilde{\chi}_{s+1}(\tau_1 \cdots \tau_{s+1}) \underset{\tau - \tau' \to \infty}{\longrightarrow} \left\{ \chi_k - \sum_{l=1}^{k} \widetilde{\chi}_l \chi_{k-l} \right\} \chi_{s+1-k}.$$

Using definition (3.1) we prove the validity of eq. (3.2) for $l = s + 1$.

It should be pointed out that the moment $\chi_s(\tau_1 \cdots \tau_s)$ is symmetrical concerning the rearrangement of the arguments $\tau_1 \cdots \tau_s$, while the value $\widetilde{\chi}_s(\tau_1 \cdots \tau_s)$ does not have the mentioned property. The values $\widetilde{\chi}_s$ are "*ordered cumulants*" of van Kampen. However, van Kampen has given only the description of procedure for the construction of the ordered cumulants in Ref. [7]. Unlike van Kampen we propose the recurrent relation (3.3) which defines the "ordered cumulants". The property of values $\widetilde{\chi}_l$, established by Theorem 1, is a starting point in research of the evolution operator asymptotic structure.

As an example we put the expressions of values χ_s and $\widetilde{\chi}_s$ for the stationary Gauss process. According to definition (1.2), we have

$$\chi_s(\tau_1 \cdots \tau_s) = \int dP_\varphi \varphi(\tau_1) \cdots \varphi(\tau_s),$$

Where P_φ is the Gauss probability measure which is given by

$$dP_\varphi = \frac{1}{(2\pi)^{1/2}\sqrt{\det g(\tau_i - \tau_j)}} \times \tag{3.4}$$

$$\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{s} g^{-1}(\tau_i - \tau_j)\varphi(\tau_i)\varphi(\tau_j)\right\} d\varphi(\tau_1)\cdots d\varphi(\tau_s).$$

Hence, we have

$$\chi_{2l+1}(\tau_1 \cdots \tau_{2l+1}) = 0,$$

$$\chi_{2l}(\tau_1 \cdots \tau_{2l}) = \sum{}' g(\tau_1 - \tau_2)\cdots g(\tau_{2l-1} - \tau_{2l}), \tag{3.5}$$

$$l = 0, 1, 2, ...$$

The sum in eq. (3.5) is distributed on every possible split $\tau_1 \cdots \tau_{2l}$ on the pairs. The number of splits, obviously, is presented as $(2l-1)!! = \frac{(2l)!}{2^l l!}$. The function $g(\tau)$ is appreciably different from zero only at $\tau \leq \tau_0$. This property of the function $g(\tau)$ provides the feasibility of the correlation weakening principle for the values χ_{2l} determined by eq. (3.5).

The values $\widetilde{\chi}_{2l}$ could be found from recurrent relation (3.3) with the use of eq. (3.5). It is easy to see that the following relations take place

$$\widetilde{\chi}_2(\tau_1, \tau_2) = g(\tau_1 - \tau_2),$$

$$\widetilde{\chi}_4(\tau_1, \tau_2, \tau_3, \tau_4)$$

$$= g(\tau_1 - \tau_3)g(\tau_2 - \tau_4) + g(\tau_1 - \tau_4)g(\tau_2 - \tau_3).$$

In the general case one has

$$\widetilde{\chi}_{2l}(\tau_1 \cdots \tau_{2l}) = \sum{}' g(\tau_1 - \tau_k)\cdots g(\tau_{k-1} - \tau_{2l}), \tag{3.6}$$

where the summation is distributed only on such rearrangements of $\tau_1 \cdots \tau_{2l}$ for which the restriction $\tau_1 > \tau_2 > \cdots > \tau_{2l}$ takes place. The number of members P_l in eq. (3.6) can be determined by a recurrent relation

$$P_l = (2l-1)!! - \sum_{j=1}^{l-1} P_j[2(l-j)-1]!!,$$

$$(P_1 = 1, P_2 = 2, P_3 = 10, P_4 = 74, ...).$$

Expression (3.6), on the one hand, satisfies the relation (3.1), due to the condition that $g(\tau_i - \tau_j)$ is different from zero only for $|\tau_i - \tau_j| \sim \tau_0$ and, on the other hand, provides the feasibility of Theorem 1.

For the study of correlation functions we introduce the values $\widetilde{\chi}_{l,s}$, which are defined by

$$\chi_{n+s}(\tau_1 \cdots \tau_s, \tau_{s+1} \cdots \tau_{s+n}) \tag{3.7}$$

$$= \sum_{l=0}^{n} \widetilde{\chi}_{l,s}(\tau_1 \cdots \tau_{l+s})\chi_{n-l}(\tau_{l+s+1} \cdots \tau_{s+n}),$$

$$\widetilde{\chi}_{l,0} = \widetilde{\chi}_l, \; s = 0, 1, 2, ..., \; n = 0, 1, 2, ...$$

We denote by $\widetilde{D}_{l,s}$ such a set of variables $\tau_1 \cdots \tau_{l+s}$, in which $\tau_1 \sim \cdots \sim \tau_s > \tau_{s+1} > \cdots \tau_{l+s}$. Then we have [1].

Theorem 2. Let $\tau_1 \sim \cdots \sim \tau_{l+1}$ be the elements of the set $\widetilde{D}_{l,s}$, $\tau_1 \sim \cdots \sim \tau_{l+1} \in \widetilde{D}_{l,s}$, $\tau_{s+1} \cdots \tau_p \sim \tau$ and $\tau_{p+1} \cdots \tau_{l+s} \sim \tau'$. If χ_l satisfies the correlation weakening principle then we obtain

$$\widetilde{\chi}_{l,s}(\tau_1 \cdots \tau_{l+s}) \underset{\tau - \tau' \to \infty}{\longrightarrow} 0. \tag{3.8}$$

This theorem shows that in the set $\widetilde{D}_{l,s}$ the values $\widetilde{\chi}_{l,s}(\tau_1 \cdots \tau_{l+s})$ differ from zero only if $\tau_1 \cdots \tau_{l+s}$ differ from each other less than time correlation scale τ_0.

The proof. Let us rewrite eq. (3.7) as follows

$$\widetilde{\chi}_{n,s}(\tau_1 \cdots \tau_{n+s}) = \chi_{n,s}(\tau_1 \cdots \tau_{n+s}) \tag{3.9}$$

$$- \sum_{l=0}^{n-1} \widetilde{\chi}_{l,s}(\tau_1 \cdots \tau_{l+s})\chi_{n-l}(\tau_{l+s+1} \cdots \tau_{s+n}).$$

The proof is conducted by the method of mathematical induction. By direct checking one can see that $\widetilde{\chi}_{l,s}(\tau_1 \cdots \tau_s, \tau_{s+1}) \underset{\tau \equiv \tau_{s+1} \to \infty}{\longrightarrow} 0$. Let eq. (3.8) be true for $l = 1, 2, ..., n-1$ and any p, $s+1 \leq p \leq s+l-1$. We prove the validity of this relation for $l = n$. Let $\tau_{s+1} \cdots \tau_p \sim \tau$, $\tau_{p+1} \cdots \tau_{l+s} \sim \tau'$, then taking into account the correlation weakening principle for the values χ_l, we find

$$\widetilde{\chi}_{n,s}(\tau_1\cdots\tau_{n+s}) \underset{\tau-\tau'\to\infty}{\longrightarrow} \Big\{\chi_{s+p} - \chi_s\chi_p - \sum_{l=1}^{p}\widetilde{\chi}_{l,s}\chi_{p-l}\Big\}\chi_{n-p}. \tag{3.10}$$

Using eq. (3.7) we prove the relation (3.8) for $l = n$.

At least, we intend to obtain a useful relation for $\widetilde{\chi}_{l,s}$. It is easy to see that eq. (3.7) leads to

$$\sum_{l=0}^{n}\widetilde{\chi}_{l,s}(\tau_1\cdots\tau_{l+s})\chi_{n-l}(\tau_{l+s+1}\cdots\tau_{s+n}) \tag{3.11}$$
$$= \sum_{l=0}^{n-1}\widetilde{\chi}_{l,s+1}(\tau_1\cdots\tau_{l+s}\tau_{l+s+1})\chi_{n-l-1}(\tau_{l+s+2}\cdots\tau_{s+n}).$$

Let us treat this relation as identity. We present the identity as

$$\widetilde{\chi}_{0,s}\chi_n + \sum_{l=0}^{n-1}\widetilde{\chi}_{l+1,s}\chi_{n-l-1} = \sum_{l=0}^{n-1}\widetilde{\chi}_{l,s+1}\chi_{n-l-1}.$$

Using the definition (3.1), we have

$$\widetilde{\chi}_{0,s}\chi_n + \sum_{l=0}^{n-1}\left\{\widetilde{\chi}_{0,s}\widetilde{\chi}_{l+1} + \widetilde{\chi}_{l+1,s} - \widetilde{\chi}_{l,s+1}\right\}\chi_{n-l-1} = 0.$$

As far as the expression in brackets does not depend on the index n it should be equal to zero. Thus, we come to the following relation

$$\widetilde{\chi}_{0,s}\widetilde{\chi}_{l+1} + \widetilde{\chi}_{l+1,s} - \widetilde{\chi}_{l,s+1}. \tag{3.12}$$

We should point out that eq. (3.12) permits to express the two-index value $\widetilde{\chi}_{l,s}$ through the one-index values $\widetilde{\chi}_s$, χ_l as follows

$$\widetilde{\chi}_{l,s} = \chi_{l+s} - \sum_{k=0}^{l-1}\chi_{s+k}\widetilde{\chi}_{l-k},\ l = 0,1,2,...,\ s = 0,1,2,\ldots .$$

IV. SUMMING UP OF SECULAR TERMS

The procedure of summarizing the asymptotical main terms at $t\to\infty$ of the perturbation series for the statistical operator of many particles systems with pair interaction is well known. In this section the procedure of summarizing secular members will be applied to some systems in random field.

In eq. (2.4) we unite for simplification the integrals over q_i and t_i in the integrals over τ_i. Then eq. (2.4) can be written as

$$S(t) = \sum_{n=0}^{\infty}S^{(n)}(t), S^{(n)}(t) = (-i)^n\int_0^t d\tau_1\cdots\int_0^{\tau_n-1}d\tau_n\chi_n(\tau_1\cdots\tau_n)a(\tau_1)\cdots a(\tau_n). \tag{4.1}$$

Substituting χ_n from eq. (3.1) into (4.1) yields the equation

$$S^{(n)}(t) = \sum_{k=1}^{n}(-i)^k\int_0^t d\tau_1\cdots\int_0^{\tau_k-1}d\tau_k\widetilde{\chi}_k(\tau_1-\tau_2,...,\tau_1-\tau_k)a(\tau_1)\cdots a(\tau_k)S^{(n-k)}(\tau_k), \tag{4.2}$$

$$S^{(0)} = 1,\ n = 1,2,\ldots .$$

The value $\widetilde{\chi}_k$ contained in eq. (3.2) depends on the difference of time arguments due to the stationary random field. This equation will serve us as the base for the summing of secular members at the $t\to\infty$ in the expression for $S(t)$.

The value $a(\tau_1)\cdots a(\tau_n)f(0)$ in $S^{(n)}(t)f(0)$ is commutative with the momentum operator in the homogeneous case. So we have

$$a(\tau_1)\cdots a(\tau_n)f(0) = e^{-i\Lambda^{(0)}\tau}a(\tau_1)\cdots a(\tau_n)e^{+i\Lambda^{(0)}\tau}f(0) = a(\tau_1-\tau)\cdots a(\tau_n-\tau)f(0).$$

Thus, the operator $a(\tau_1)\cdots a(\tau_n)$ in eq. (4.2) actually depends only on the differences $\tau_i-\tau_j$, since this operator always acts on the space-uniform initial state $f(0)$. Hence it is natural to present the asymptotic expression at $t\to\infty$ for $S^{(n)}(t)$ in the polynomial form

$$S^{(n)}(t) \underset{t\to\infty}{\longrightarrow} \widetilde{S}^{(n)}(t) = \sum_{i=0}^{n} \alpha_i^{(n)} t^i. \tag{4.3}$$

Since the function $\widetilde{\chi}_k(\tau_1-\tau_2,\cdots,\tau_1-\tau_k)$ is different from zero only at $\tau_1 \sim \tau_2 \sim \cdots \sim \tau_k$, in the set $\tau_1 > \tau_2 > \cdots > \tau_k$ we obtain

$$S^{(n)}(t) \underset{t\to\infty}{\longrightarrow} \sum_{k=1}^{n} (-i)^k \int_0^t d\tau_1 \cdots \int_0^{\tau_k-1} d\tau_k \widetilde{\chi}_k(\tau_1-\tau_2,...,\tau_1-\tau_k) a(\tau_1)\cdots a(\tau_k) \widetilde{S}^{(n-k)}(\tau_k) + \sum_{k=1}^{n} d_k^{(n)}, \tag{4.4}$$

$$d_k^{(n)} = (-i)^k \int_0^\infty d\tau_1 \cdots \int_0^{\tau_k-1} d\tau_k \widetilde{\chi}_k(\tau_1-\tau_2,...,\tau_1-\tau_k) a(\tau_1)\cdots a(\tau_k)(S^{(n-k)}(\tau_k) - \widetilde{S}^{(n-k)}(\tau_k)).$$

Substituting in eq. (4.4) the polynomial representation (4.3) for $\widetilde{S}^{(n-k)}(\tau_k)$ and taking into account that $\tau_k^l = \sum_{i=0}^{l} \binom{l}{i} \tau_1^{l-i}(\tau_k-\tau_1)^i$ we obtain

$$S^{(n)}(t) \underset{t\to\infty}{\longrightarrow} \sum_{k=1}^{n} \sum_{l=0}^{n-k} \sum_{i=0}^{l} \int_0^t d\tau_1 a_l^{(n-k)} \binom{l}{i} \tau_1^{l-i} \Phi_k^{(i)}(\tau_1) + \sum_{k=1}^{n} d_k^{(n)} \tag{4.5}$$

where

$$\Phi_k^{(i)}(\tau_1) = (-i)^k \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_k-1} d\tau_k \widetilde{\chi}_k(\tau_1-\tau_2,...,\tau_1-\tau_k) a(\tau_1)\cdots a(\tau_k)(\tau_k-\tau_1)^i \tag{4.6}$$

$$= (-i)^k \int_{-\tau_1}^{\tau_1} d\tau_2 \cdots \int_{-\tau_1}^{\tau_k-1} d\tau_k \widetilde{\chi}_k(-\tau_2,\cdots,-\tau_k) a(\tau_2)\cdots a(\tau_k)\tau^i.$$

In order to find the asymptotic of the first term in the right hand of eq. (4.5) at $t\to\infty$ we need the following lemma.

Lemma.

$$\int_0^t d\tau_1 \tau_1^{l-i} \Phi_k^{(i)}(\tau_1) \underset{t\to\infty}{\longrightarrow} \frac{t^{l-i+1}}{l-i+1} b_k^{(i)} + c_{k,i}^{(l-i)}, \tag{4.7}$$

where

$$b_k^{(i)} = \lim_{\tau\to\infty} \Phi_k^{(i)}(\tau),\ c_{k,i}^{(m)} = \int_0^\infty d\tau_1 \tau_1^m (\Phi_k^{(i)}(\tau_1) - b_k(i)).$$

The proof. As far as the function $\widetilde{\chi}_k(-\tau_2,\cdots,-\tau_k)$ in the set $0 > \tau_2 > \cdots > \tau_k$ is different from zero only at $\tau_i \sim \tau_0$, eq. (4.5) has the limit $\tau_1 \to \infty$. Hence we get eq. (4.7) and the expressions for $b_k^{(i)}$ and $c_{k,j}^{(m)}$.

Using lemma and eqs. (4.1), (4.5) it is easy to obtain the following equation

$$\sum_{i=0}^{n} \alpha_i^{(n)} t^i = \sum_{k=1}^{n-1} \left[d_k^{(n-k)} + \sum_{l=0}^{n-k} \sum_{m=0}^{l} \binom{l}{m} \left(\frac{t^{l-m+1}}{(l-m+1) b_k^{(m)} + c_{k,i}^{(l-m)}} \right) \alpha_i^{(n-k)} \right] + c_{0,n}^{(0)} + t b_n^{(0)}.$$

With the help of this equation we find the equations for $\alpha_i^{(n)}$

$$\alpha_0^{(n)} = \sum_{k=1}^{n-1} \left[d_k^{(n-k)} + \sum_{l=0}^{n-k} \sum_{m=0}^{l} \binom{l}{m} c_{k,l}^{(l-m)} \alpha_i^{n-k} \right] + c_{0,n}^{(0)}, \tag{4.8}$$

$$\alpha_i^{(n)} = \sum_{k=1}^{n-1} \sum_{l=0}^{n-k} \sum_{m=0}^{l} \binom{l}{m} \frac{1}{i} \delta_{i,l-m+1} b_k^m \alpha_i^{n-k}, \; i = 1, 2, ..., n. \tag{4.9}$$

The asymptotic evolution operator $\widetilde{S}(t)$ in accordance with eq. (4.3) is given by

$$\widetilde{S}(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \alpha_i^{(n)} t^i = \sum_{i=0}^{n} \beta_i t^i, \; \beta_i = \sum_{n=i}^{\infty} \alpha_i^{(n)}.$$

It is easy to find from eqs. (4.8), (4.9) the recurrent relations for the operators β_i

$$\beta_0 = \sum_{n=0}^{\infty} \sum_{k=1}^{n-1} d_k^{(n-k)} + \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} c_{k,m}^{(l-m)} \beta_{l+1}, \tag{4.10}$$

$$\beta_i = \sum_{m=0}^{\infty} \frac{1}{i} \binom{i+m-1}{m} b^{(m)} \beta_{i+m-1}, \; i = 1, 2, ..., \tag{4.11}$$

$$b^{(m)} = \sum_{k=1}^{\infty} b_k^{(m)}. \tag{4.12}$$

The solution of eq. (4.11) for β_i we will find in the form

$$\beta_i = \frac{L^i}{i!} \beta_0. \tag{4.13}$$

Substituting β_i from eq. (4.13) into eq. (4.10) leads to equation for the definition of β_0

$$\beta_0 \equiv \widetilde{S}_0 = \left[1 - \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} c_{k,m}^{(l-m)} \frac{L^l}{l!} \right]^{-1} \left(1 + \sum_{m=0}^{\infty} \sum_{k=1}^{n-1} d_k^{(n-k)} \right). \tag{4.14}$$

Substituting eq. (4.13) into eq. (4.11), yields the definition of the "collisions integral" L

$$L = \sum_{m=0}^{\infty} \frac{1}{m!} b^{(m)} L^m. \tag{4.15}$$

Thus, the asymptotic structure of the evolution operator $S(t)$ at $t \to \infty$ is given by

$$S(t) = \underset{t \to \infty}{\longrightarrow} \widetilde{S}(t) = e^{Lt} \widetilde{S}_0$$

[1,2,8,9], where the values $\widetilde{S}_0$ and L are defined by eqs. (4.14) and (4.15).

So, for the distribution function we have

$$f(t) \underset{t \to \infty}{\longrightarrow} \widetilde{f}(t)) = \exp(Lt) S_0 f(0)$$

and, as a consequence, the evolution kinetic equation $\dot{\widetilde{f}}(t) = L\{\widetilde{f}(t)\}$ takes place, where $L\{\widetilde{f}(t)\} = L\widetilde{f}(t)$. The function $\widetilde{f}(t)$ is the reduced distribution function which approximates the exact distribution function $f(t)$. It is seen from these equations that "the initial distribution function" which corresponds to the kinetic stage of the evolution is the function $\widetilde{S}_0 f(0)$. The existence of memory operator $\widetilde{S}_0$ is due to the time correlation scale τ_0 of the random field. For the δ-correlated field when $\tau_0 = 0$, we have $\widetilde{S}_0 = 1$. The time τ_0 is called the chaotization time. Note that for the time independent random field the chaotization time scale will be induced by the particle dynamics in this field. Such a situation takes place when one investigates the kinetic phenomena to probe the particle moving in the field of randomly located impurities.

Thus, it is proved that for the distributions of stochastic equations solutions the reduced description method which is similar to the kinetic description of states in statistical mechanics takes place. Due to the extraction and summation of secular terms appearing in the perturbation series in the random field the asymptotic evolution operator for the reduced distribution function is constructed.

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ПРО ДИНАМІЧНУ ТЕОРІЮ СИСТЕМ У ВИПАДКОВИХ ПОЛЯХ

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Створено нову методику опису систем із шумом. Вона базується на тому, що серед розподілів розв'язків стохастичних диференційних рівнянь є такі, які можна описати подібно до кінетичного опису станів у статистичній фізиці. Задачу побудови асимптотичного оператора еволюції розв'язано сумуванням вікових членів ряду теорії збурень за випадковим полем.