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USING TWO–COMPONENT SUSY FIELD IN PHASE TRANSITION THEORY

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On the basis of Langevin equation the optimal SUSY field scheme is formulated to describe a non–equilibrium thermodynamic system. The cases of two–component Grassmannian fields are analysed with the second components being the most probable fluctuation and the conjugate field. It is shown that the inherent in self–organized system a four–component SUSY field transforms into a two–component form with the passage to the thermodynamic system.

Key words: SUSY field, order parameter, conjugate field, fluctuation.

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I. INTRODUCTION

During the last years significant interest has been given to the development of the microscopic theory of non–equilibrium thermodynamic systems that experience the ergodicity breaking and exhibit memory effects. The most popular examples are known to be systems type of the spin glass [1] and random heteropolymers [2]. Usually for their description the replica method is used that is based on the mathematical trick. However, it appears that within the framework of the Sherrington–Kirkpatrick model the replica approach is reduced to the supersymmetry (SUSY) method [3]. The latter is based on using Grassmannian variables that represent, roughly speaking, the square root of the number 0. So, the introduction of the SUSY field, that is a combination of usual and Grassmannian fields, are represented mathematically as the common transition from real fields to complex ones.

For the SUSY scheme formulation it is necessary to take into account that, being the gauge field, the SUSY field can be reduced to irreducible components: analogously as the electromagnetic field is split into vector and scalar components, the 4–component SUSY field can be reduced to the couple of chiral components that consist of both the ordinal and Grassmannian constituents [4]. In Sect. III the irreducible SUSY field will be constructed as well, but it turns out that the separated component represents rather a 2–component Grassmannian field than the proper SUSY field. The advantage of the former is that its components possess the explicit physical meaning such as the order parameter and conjugate field (or its fluctuation).

The work is organized as follows. In Sect. II the formulation of the simplest SUSY field scheme is fulfilled on the basis of the 2–component Grassmannian fields whose second component is either fluctuation or conjugate force (see subsections A, B respectively). Sect. III is devoted to the presentation of the above–mentioned scheme of reduction of the 4–component proper SUSY field to different 2–component forms. Appendices contain technical details of the SUSY formalism.

II. THE SIMPLEST FIELD SCHEMES

Let us start with the Langevin equation [6]

$$\dot{\eta}(\mathbf{r},t) - D\nabla^2\eta = -\gamma\partial V/\partial\eta^* + \zeta(\mathbf{r},t), \qquad (1)$$

that defines the time–spatial dependence $\eta(\mathbf{r},t)$ of the complex order parameter. Here the overpoint denotes differentiation on time, $\nabla \equiv \partial/\partial\mathbf{r}$, D is the inhomogeneity parameter type of diffusion coefficient, γ is kinetic coefficient, $V(\eta)$ is synergetic potential (Landau free energy), $\zeta(\mathbf{r},t)$ is stochastic source, normalized by conditions of white noise

$$\langle\zeta(\mathbf{r},t)\rangle_0 = 0, \qquad (2)$$
$$\langle(\zeta^*(\mathbf{r},t)\zeta(\mathbf{0},0) + \zeta(\mathbf{r},t)\zeta^*(\mathbf{0},0))\rangle_0 = 2\gamma T\delta(\mathbf{r})\delta(t).$$

The angular brackets with index 0 mean averaging over bare distribution of value ζ, T is intensity of noise being the temperature for thermodynamic system. Further, it is convenient to introduce the measure units $t_0 \equiv (\gamma T)^2/D^3$, $r_0 \equiv \gamma T/D$, $V_0 \equiv T$, $\zeta_0 \equiv D^3/(\gamma T)^2$ for time t, coordinate $\mathbf{r}$, synergetic potential V, and stochastic variable ζ, correspondingly. As a result, the motion equation (1) takes the canonical form

$$\dot{\eta}(\mathbf{r},t) = -\delta V/\delta\eta^* + \zeta(\mathbf{r},t), \qquad (3)$$

where short notation is used for the variation derivative

$$\delta V/\delta\eta^* \equiv \delta V\{\eta\}/\delta\eta^* = \partial V(\eta)/\partial\eta^* - \nabla^2\eta, \qquad (4)$$
$$V\{\eta\} \equiv \int V(\eta)\mathrm{d}\mathbf{r}.$$

Moreover, one has to put the coefficient $\gamma T = 1$ in

Eq. (3). Then the distribution of variable ζ acquires the Gaussian form

$$P_0\{\zeta\} = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}\int |\zeta(\mathbf{r},t)|^2 \mathrm{d}\mathbf{r}\mathrm{d}t\right). \quad (5)$$

The basis for construction of the field scheme is the generating functional [7]

$$Z\{u(\mathbf{r},t)\} = \int Z\{\eta\} \exp\left[\frac{1}{2}\int (u^*\eta + u\eta^*)\mathrm{d}\mathbf{r}\mathrm{d}t\right] \mathrm{D}\eta, \quad (6)$$

$$Z\{\eta(\mathbf{r},t)\} = \left\langle \prod_{(\mathbf{r},t)} \delta\left\{\dot\eta + \frac{\delta V}{\delta\eta^*} - \zeta\right\} \det\left|\frac{\delta\zeta}{\delta\eta}\right|\right\rangle_0, \quad (7)$$

whose variation on an auxiliary field $u(\mathbf{r},t)$ gives correlators of observable values. Obviously, $Z\{u\}$ represents the functional Laplace transformation of the dependence $Z\{\eta\}$, appearance of δ–function reflects the condition (3), the determinant provides transition from continual integration over ζ to one over η.

A. Fluctuation as Grassmannian component

Further development of the field scheme is determined by the type of connection between stochastic variable ζ and order parameter η. For the thermodynamic system where the thermostat state does not depend on the value of η, this connection results in a constant value of the determinant in Eq. (7) that can be chosen as unity. Then, using integral representation of the δ–function

$$\delta\{x(\mathbf{r},t)\} = \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \exp\left(-\frac{1}{2}\int (\varphi^* x + \varphi x^*)\,\mathrm{d}\mathbf{r}\mathrm{d}t\right) D\varphi \quad (8)$$

and averaging over distribution (5), we reduce the functional (7) to standard form

$$Z\{\eta(\mathbf{r},t)\} = \int \exp\left[-S\{\eta(\mathbf{r},t),\varphi(\mathbf{r},t)\}\right] D\varphi, \quad (9)$$

where the action $S = \int \mathcal{L}\mathrm{d}\mathbf{r}\mathrm{d}t$ is measured in units $S_0 = \gamma^2(T/D)^3$ to be determined by the Lagrangian

$$\mathcal{L}(\eta,\varphi) = \frac{1}{2}\left(\varphi^*\dot\eta + \varphi\dot\eta^* - |\varphi|^2\right) + \frac{1}{2}\left(\frac{\delta V}{\delta\eta}\varphi + \frac{\delta V}{\delta\eta^*}\varphi^*\right). \quad (10)$$

To obtain a canonical form of the Lagrangian (10) let us introduce the Grassmannian field

$$\Phi = \eta + \chi\varphi, \quad (11)$$

where the nilpotent coordinate χ possesses the usual properties

$$\chi^2 = 0, \qquad \int \mathrm{d}\chi = 0, \qquad \int \chi\mathrm{d}\chi = 1. \quad (12)$$

As is shown in Appendix A, the first bracket of Lagrangian (10) takes the form

$$K = \frac{1}{2}\int \Phi^* D\Phi \mathrm{d}\chi \quad (13a)$$

inherent of the kinetic energy in the Dirac field scheme [7]. Here the Hermite operator D is defined by equality

$$D = -\frac{\partial}{\partial\chi} + \left(1 - 2\chi\frac{\partial}{\partial\chi}\right)\frac{\partial}{\partial t} \quad (14)$$

and possesses the property (A.6). On the other hand, the nilpotent properties (12) of Grassmannian coordinate χ allow to write down the second bracket in Eq. (10) in standard form of potential energy (see Appendix A)

$$\Pi = \frac{1}{2}\int V(\Phi)\mathrm{d}\chi. \quad (13b)$$

As a result, the Lagrangian (10) of the Euclidean field theory is expressed in Grassmannian form

$$\mathcal{L} = K + \Pi = \int \Lambda\ \mathrm{d}\chi, \quad (15)$$

$$\Lambda(\Phi) = \frac{1}{2}\left(\Phi^* D\Phi + V(\Phi)\right).$$

According to Appendix A, this expression is invariant with respect to Grassmannian transformation (A.7) given by operator $e^{i\varepsilon D}$, $\varepsilon \to 0$. So, operator D is the generator of the Grassmannian group.

Giving infinitesimal increment $\delta\Phi^*$ to the field Φ^*, it is easy to see, that the action

$$S\{\Phi(z)\} = \int \Lambda(\Phi(z))\mathrm{d}z, \qquad z \equiv \{\mathbf{r},t,\chi\} \quad (16)$$

gets the addition $\delta S = 0$ if one satisfies the condition

$$D\frac{\delta\Lambda}{\delta D\Phi^*} + \frac{\delta\Lambda}{\delta\Phi^*} = 0, \quad (17)$$

playing a role of the Euler equation. Substituting here the latter expression (16), we find the motion equation

$$D\Phi + \delta V/\delta\Phi^* = 0. \quad (18)$$

Projecting along the axes the usual and Grassmannian variables gives a system of equations

$$\dot{\eta} = -\delta V/\delta\eta^* + \varphi, \tag{19}$$

$$\dot{\varphi} = \frac{\delta^2 V}{\delta\eta\delta\eta^*}\varphi + \frac{\delta^2 V}{\delta\eta^*\delta\eta^*}\varphi^*, \tag{20}$$

that determines the kinetic of phase transition. Being obtained according to the extremum condition for Lagrangian (10) these equations determine the maximum value of the probability distribution

$$P\left\{\eta(\mathbf{r},t),\varphi(\mathbf{r},t)\right\} = Z^{-1}\exp\left(-\int \mathcal{L}(\eta,\varphi)\mathrm{d}\mathbf{r}\mathrm{d}t\right), \tag{21}$$

that specifies the partition function $Z \equiv Z\{u = 0\}$ in Eq. (6). The comparison of expression (19) with Langevin equation (3) results in the conclusion that the quantity φ determines the most probable value of the fluctuation ζ of the field conjugated order parameter. On the other hand, this means transformation of the initial one–modal distribution (5) to the final two–modal form (21).

The kinetic equations (19), (20) can be obtained immediately from the Lagrangian (10), but they do not take into account a dissipation process. Introducing the latter calls for the insertion of the dissipation function $R = (1/2)\int|\dot{\eta}|^2\mathrm{d}\mathbf{r}$ into the Euler equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\mathcal{L}}{\delta\dot{\eta}^*} + \frac{\delta\mathcal{L}}{\delta\eta^*} = \frac{\delta R}{\delta\dot{\eta}^*}. \tag{22}$$

Then the equation (20) takes the form of

$$\dot{\varphi} = -\varphi + \frac{\delta V}{\delta\eta^*} + \left(\frac{\delta^2 V}{\delta\eta\delta\eta^*}\varphi + \frac{\delta^2 V}{\delta\eta^*\delta\eta^*}\varphi^*\right). \tag{23}$$

The system of differential equations (19), (23) allows to analyse the kinetic behaviour of the order parameter $\eta(t)$ and most probable fluctuation $\varphi(t)$ at given potential $V\{\eta\}$.

B. Conjugate field as Grassmannian component

It follows that expression (11) is not unique two–component Grassmannian field that allows to develop a consistent field scheme. Indeed, let us introduce the field $f(\mathbf{r},t)$ defined by the relation

$$\dot{\eta} = f + \varphi. \tag{24}$$

Then the Lagrangian (10) takes the form of

$$\mathcal{L}(\eta,f) = \frac{1}{2}\left(|\dot{\eta}|^2 - |f|^2\right) - \frac{1}{2}\left(\frac{\delta V}{\delta\eta}f + \frac{\delta V}{\delta\eta^*}f^*\right) + \frac{1}{2}\left(\frac{\delta V}{\delta\eta}\dot{\eta} + \frac{\delta V}{\delta\eta^*}\dot{\eta}^*\right). \tag{25}$$

Being the total time derivative $\mathrm{d}V\{\eta\}/\mathrm{d}t$ the latter bracket gives the usual expression

$$Z = \int \exp\left(-\frac{V\{\eta_f\} - V\{\eta_i\}}{2T}\right) D\eta_i D\eta_f \tag{26}$$

for the partition function as an integral over both the initial and final fields $\eta_i(\mathbf{r},t)$, $\eta_f(\mathbf{r},t)$ (here we return to the dimensional magnitude of the potential V). The remainder of Lagrangian (25) results in the Euler equations

$$\ddot{\eta} = -\left(\frac{\delta^2 V}{\delta\eta^*\delta\eta}f + \frac{\delta^2 V}{\delta\eta^*\delta\eta^*}f^*\right), \tag{27}$$

$$f = -\delta V/\delta\eta^*. \tag{28}$$

It is easily to show that these equations are equivalent to the system of Eqs.(19), (20). Indeed, differentiation of Eq. (19) over time gives Eq. (27) if Eqs. (20), (24) are taken into account. As regards Eq. (28), it is just a definition of the field $f(\mathbf{r},t)$ conjugated to the order parameter $\eta(\mathbf{r},t)$.

Similarly to definition (11) let us introduce now another Grassmannian field [8]

$$\Phi = \eta + \chi f, \tag{29}$$

whose components are the order parameter η and the force f, Eq. (28). As is shown in Appendix A, the replacement of the second component φ in Eq. (11) by f in Eq. (29) does not change the form of the Lagrangian (16). However, the generator of the Grassmannian group takes on quite a different form

$$D = -\left(\frac{\partial}{\partial\chi} + \chi\frac{\partial^2}{\partial t^2}\right) \tag{30}$$

and possesses the other properties (A.9) than (A.7).

Within the framework of the presented two–component fields using one of expressions (11), (29) is just equivalent. The mathematical reason is that the fields $\Phi_\varphi(\chi) \equiv \eta + \chi\varphi$, $\Phi_f(\chi) \equiv \eta + \chi f$ are connected by the relations

$$G_-^2\Phi_\varphi(\chi) = \Phi_f(-\chi), \qquad G_-^2\Phi_f(\chi) = \Phi_\varphi(-\chi), \tag{31}$$

obtained after the transformation given by the operator (cf. Eq. (B.1)) $G_- \equiv e^{-\partial}$, $\partial \equiv (\chi/2)\partial/\partial t$ which shifts the time t on the value $-\chi/2$.

III. REDUCTION OF PROPER SUSY FIELD TO THE TWO–COMPONENT FORM

The consideration presented in the foregoing section is stated on the simplest proposition that in Eq. (7) the Jacobian of transfer from the stochastic variable ζ to the order parameter η is constant. However, in a general case the analytical representation for arbitrary matrix A

$$\det A = \int \exp\left(-\bar{\psi} A \psi\right) \mathrm{d}^2\psi, \quad \mathrm{d}^2\psi = \mathrm{d}\bar{\psi}\mathrm{d}\psi \tag{32}$$

requires to introduce Grassmannian conjugate fields $\psi(\mathbf{r},t)$, $\bar{\psi}(\mathbf{r},t)$ that satisfy the conditions type of Eqs.(12). The physical meaning of the appearance of new freedom degrees ψ, $\bar{\psi}$ is that the thermostat state turns out to be dependent on the order parameter — as it is inherent of self–organized system [9]. As a result, elongated Lagrangian (25) takes the form of

$$\mathcal{L} = \frac{1}{2}\left(|\dot{\eta}|^2 - |f|^2\right) - \frac{1}{2}\left(\frac{\delta V}{\delta \eta} f + \frac{\delta V}{\delta \eta^*} f^*\right) + \bar{\psi}\left(\frac{\partial}{\partial t} + \frac{\delta^2 V}{\delta \eta^* \delta \eta}\right)\psi, \tag{33}$$

where the total time derivative $\mathrm{d}V/\mathrm{d}t$ is dropped and Eqs.(3), (7) are taken into account.

Introducing the four–component SUSY field

$$\Phi = \eta + \bar{\theta}\psi + \bar{\psi}\theta + \bar{\theta}\theta f, \tag{34}$$

by analogy with the above SUSY Lagrangian is obtained

$$\mathcal{L} = \int \Lambda \mathrm{d}^2\theta, \tag{35}$$
$$\Lambda(\Phi) \equiv \frac{1}{2}\left(\Phi^* \bar{D}_- D_+ \Phi + V(\Phi)\right),$$
$$\mathrm{d}^2\theta \equiv \mathrm{d}\bar{\theta}\mathrm{d}\theta,$$

where θ, $\bar{\theta}$ are Grassmannian conjugate coordinates that replace χ. In comparison with Eq. (16), where the kernel Λ has the first power of the generator (30), here the product takes place for the Grassmannian conjugate generators D_+,$\bar{D}_-$ (see Eqs.(B.6)) that are obtained after the action of operators (B.1) to the original generators

$$D = \frac{\partial}{\partial \bar{\theta}} + \frac{\theta}{2}\frac{\partial}{\partial t}, \qquad \bar{D} = \frac{\partial}{\partial \theta} + \frac{\bar{\theta}}{2}\frac{\partial}{\partial t}. \tag{36}$$

They possess the properties (A.12) and represent transformations (A.13) of the proper SUSY group. The Euler SUSY equation reads

$$(1/2)[\bar{D}_-, D_+]\Phi + \delta V/\delta \Phi^* = 0, \tag{37}$$

where the square brackets denote the commutator. Projection along the SUSY axes 1, $\bar{\theta}$, θ, $\bar{\theta}\theta$ gives the motion equations

$$\ddot{\eta} = -\left(\frac{\delta^2 V}{\delta \eta^* \delta \eta} f + \frac{\delta^2 V}{\delta \eta^* \delta \eta^*} f^*\right) + \frac{\delta^3 V}{|\delta \eta|^2 \delta \eta *}\bar{\psi}\psi, \tag{38a}$$

$$f = -\delta V/\delta \eta^*, \tag{38b}$$

$$\dot{\psi} - \nabla^2\psi = \frac{\partial^2 V}{\partial \eta \partial \eta^*}\psi, \tag{38c}$$

$$\dot{\bar{\psi}} + \nabla^2\bar{\psi} = -\frac{\partial^2 V}{\partial \eta \partial \eta^*}\bar{\psi}, \tag{38d}$$

that transfer to the form of Eqs.(27), (28) at $\psi = \bar{\psi} = 0$. Combining the last couple of these equations, we obtain the conservation law $\dot{S} + \nabla \mathbf{j} = 0$ for the quantities

$$S = \bar{\psi}\psi, \qquad \mathbf{j} = (\nabla\bar{\psi})\psi - \bar{\psi}(\nabla\psi). \tag{39}$$

In inhomogeneous thermodynamical systems S is a density of sharp boundaries, $\mathbf{j}$ is the corresponding current [5]. In particular, the approach of the four–component SUSY field relates to the strong segregation limit within the copolymer theory [10]. For a self–organized system the quantity S gives the entropy, $\mathbf{j}$ is the probability current [9]. So, passing to the thermodynamic system where the entropy is conserved we may drop the Grassmannian fields $\psi(\mathbf{r},t)$, $\bar{\psi}(\mathbf{r},t) = const$. As a result, the four–component SUSY field (34) is reduced to the two–component form (29).

It is easily to see that such a reduction is a consequence of the SUSY condition

$$\bar{\theta} D = \bar{D}\theta, \tag{40}$$

that gives the connection

$$\bar{\theta}\psi + \bar{\psi}\theta + 2\bar{\theta}\theta f = 0, \tag{41}$$

reducing the SUSY field (34) to the form of

$$\Phi = \eta + \theta\bar{\theta} f. \tag{42}$$

Introducing nilpotent variable $\chi \equiv \theta\bar{\theta}$ with the differential $\mathrm{d}\chi \equiv \mathrm{d}\bar{\theta}\mathrm{d}\theta$, one can see that χ satisfies properties (12), but being rather commutating than anticommutating the self–conjugated variable $\chi = \bar{\chi}$ is not the strong Grassmannian quantity. Nevertheless, the SUSY field (34) provided that the SUSY condition (40) is satisfied takes the two–component form (29).

Despite the same number of components, it is necessary to have in mind the different physical meaning of the reduced SUSY field (42) and a couple of Grassmannian conjugate chiral SUSY fields (B.9) whose appearance is a consequence of SUSY gauge invariance (see Appendix B). The main distinction is that the first field contains the Bose components η, f only, whereas the chiral SUSY fields Φ_+, Φ_- are the combinations of Bose η and Fermi ψ, $\bar\psi$ components. Formally it is stipulated by the fact that for the separation of the chiral SUSY fields the conditions (B.7) of the SUSY gauge invariance are fulfilled not for the initial SUSY field Φ, satisfying conditions (40), but for components $\Phi_\pm$, resulting from Φ by operators $G_\pm$ (see (B.1)).

In order to present visually the difference between the two–component SUSY fields given by Eqs.(42) and (B.9) let us represent the SUSY field (34) as a vector in a four–dimensional SUSY space with the axes $\theta^0 = \bar\theta^0 \equiv 1$, $\bar\theta$, θ, $\bar\theta\theta \equiv -\chi$. Then conditions (40) of the SUSY gauge invariance mean that the initial SUSY field (34) is reduced to vector (29) belonging to a plane formed by axes 1, χ. Accordingly conditions (B.7) of the SUSY gauge invariance results in splitting a total SUSY space into a couple of orthogonal subspaces, the first of which possesses the axes 1, θ containing the vector Φ_-, and second one — axes 1, $\bar\theta$ and vector Φ_+. Because of the indicated SUSY subspaces are Grassmannian conjugated ($\bar\Phi_- = \Phi_+$), it is enough to use one of them, considering either the vector Φ_-, or Φ_+ (see Appendix B). Such a program was realized in Ref. [11], whereas the above used SUSY field (29) is derived by projection of the chiral vectors $\Phi_\pm$ on a plane formed by axes 1, χ. From this follows that the approach stated below and theory [11] are equivalent. The SUSY method presented in book [12] is based on the usage of the chiral SUSY fields too, however used there definitions $\Phi_- = \varphi - i\bar\psi\theta$, $\Phi_+ = \eta + \bar\theta\psi$ (compare with (B.9)) contain, besides trivial introduction of the imaginary unit, the fluctuation φ as the Bose component in the SUSY field Φ_- and the order parameter η in the SUSY field Φ_+.

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APPENDIX A

For SUSY presentation let us rewrite the Lagrangian (10) in the form of Euclidian field theory [7]

$$\mathcal{L} = K + \Pi, \tag{A.1}$$

where the kinetic K and potential Π energies are

$$K = \varphi^*\dot\eta + \varphi\dot\eta^* - |\varphi|^2, \tag{A.2}$$

$$\Pi = \frac{\partial V}{\partial\eta}\varphi + \frac{\partial V}{\partial\eta^*}\varphi^*. \tag{A.3}$$

In order to obtain the SUSY form (13a) of kinetic energy in the Lagrangian (10) one ought to determine the operator D. General form of the dependence on Grassmannian coordinate χ is presented by the expression

$$D = a + b(\partial/\partial\chi) + c\chi + d\chi(\partial/\partial\chi), \tag{A.4}$$

where the coefficients a, b, c, d are any functions of the time derivative operator. Taking into account properties (12), substitution of Eqs.(11), (A.4) into Eqs.(13a) results in relevant expression (A.2) if coefficients are as follows:

$$a = \partial/\partial t, \quad b = -1, \quad c = 0, \quad d = -2\partial/\partial t. \tag{A.5}$$

As a result the operator (A.4) takes the form (14).

Taking into account definitions (11), (12), (14) it is easy to see that the operator D possesses the Hermite properties. Analogously one obtains

$$D^{2n}\Phi = \partial^{2n}\Phi/\partial t^{2n}, \qquad n = 1, 2... \tag{A.6}$$

The infinitesimal operator $\delta \equiv e^{-\mathrm{i}\varepsilon D} - 1 \simeq -\mathrm{i}\varepsilon D$ with parameter $\varepsilon \to 0$ acts to the values t, χ and Φ to results in the addings

$$\begin{aligned} &\delta t = -\mathrm{i}\varepsilon, \qquad &&\delta\chi = \mathrm{i}\varepsilon; \\ &\delta\eta = -\mathrm{i}\varepsilon\left(\dot\eta - \varphi\right), \quad &&\delta\varphi = \mathrm{i}\varepsilon\dot\varphi. \end{aligned} \tag{A.7}$$

Accordingly, the SUSY transformation gives for time and Grassmannian coordinates the imagine additions of an opposite sign, whereas the change of the order parameter is proportional to a difference between fluctuation and speed of change of the order parameter, whereas change of the fluctuation is proportional to its speed.

To prove the equivalence of the term (A.3) in Lagrangian (A.1) and the SUSY potential energy (13b) let us carry out the formal expansion of thermodynamic potential over powers of the component $\chi\varphi$ in Eq. (11):

$$\Pi = \int \left[V(\eta, \eta^*) + \left(\frac{\delta V}{\delta\eta}\varphi + \frac{\delta V}{\delta\eta^*}\varphi^*\right)\chi\right]\mathrm{d}\chi. \tag{A.8}$$

Here all terms of powers more then 1 are dropped according to the nilpotent condition. Using the integration properties (12), we obtain immediately Eq. (A.3) as it needs.

In the case of SUSY field (29) the consideration is fulfilled in analogous manner. For shortness, let us point out the difference with above–considered case (11) only. The corresponding infinitesimal transformation $\delta \simeq -\mathrm{i}\varepsilon D$ results in the additions (cf. Eqs.(A.7))

$$\begin{aligned} &\delta t = -\mathrm{i}\varepsilon, \qquad &&\delta\chi = \mathrm{i}\varepsilon; \\ &\delta\eta = -\mathrm{i}\varepsilon f, \quad &&\delta f = \mathrm{i}\varepsilon\ddot f. \end{aligned} \tag{A.9}$$

Within frequency representation the variation of the kinetic energy being the first bracket in Eq. (25) is given by expression

$$\delta K = \omega^2 \left(\eta^* \delta\eta + \eta\delta\eta^*\right) - \left(f^*\delta f + f\delta f^*\right), \tag{A.10}$$

ω is frequency. Inserting here two last equations (A.9) we obtain at once $\delta K = 0$. Correspondingly, the potential energy in the Lagrangian (25) has variation

$$\begin{aligned}\delta\Pi = &-\mathrm{i}\varepsilon\left(\ddot{\eta} + \frac{\partial^2 V}{\partial\eta^*\partial\eta}f + \frac{\partial^2 V}{\partial\eta^{*2}}f^*\right)f^* \\ &+\mathrm{i}\varepsilon\left(\ddot{\eta}^* + \frac{\partial^2 V}{\partial\eta\partial\eta^*}f^* + \frac{\partial^2 V}{\partial\eta^2}f\right)f,\end{aligned} \tag{A.11}$$

where Eqs.(A.9), (28) are taken into account. From this, according to the motion equation (27) we obtain $\delta\Pi = 0$. So, the transformation $e^{-i\varepsilon D}$ with generator (30) belongs to SUSY group. It is important, that parameter $i\varepsilon$ of this transformation is pure imaginary.

Finally, four–component SUSY field (34) corresponds to the couple of generators (36) that satisfy to conditions:

$$\{\bar{D}, D\} = \partial/\partial t, \qquad [\bar{D}, D]^2 = \partial^2/\partial t^2, \tag{A.12}$$

where curly and square brackets denote anticommutator and commutator respectively. Transformations $\delta \simeq \bar{\varepsilon}D$, $\bar{\delta} \simeq \varepsilon\bar{D}$ give

$$\begin{aligned}&\delta\chi = 0, \quad \delta\bar{\chi} = \bar{\varepsilon}, \quad \delta t = \bar{\varepsilon}\chi/2;\\ &\bar{\delta}\chi = \varepsilon, \quad \bar{\delta}\bar{\chi} = 0, \quad \bar{\delta}t = \varepsilon\bar{\chi}/2;\\ &\delta\eta = \bar{\varepsilon}\psi, \quad \delta\psi = 0, \quad \delta\bar{\psi} = \bar{\varepsilon}(\dot{\eta}/2 + f), \quad \delta f = -\bar{\varepsilon}\dot{\psi}/2;\\ &\bar{\delta}\eta = -\varepsilon\bar{\psi}, \quad \bar{\delta}\psi = \varepsilon(\dot{\eta}/2 - f), \quad \bar{\delta}\bar{\psi} = 0, \quad \bar{\delta}f = -\varepsilon\dot{\bar{\psi}}/2.\end{aligned} \tag{A.13}$$

The variation of Lagrangian (34) results in zero if this transformations and motion equations (38) take into account. The SUSY transformations bring to the Ward identity [7]

$$\langle\bar{\psi}\psi\rangle = \langle\eta\varphi^*\rangle. \tag{A.14}$$

To prove it let us take into account that the left–hand side is equal to $\left(\delta^2 V/\delta\eta^*\delta\eta\right)^{-1}$ in accordance with Eqs.(21), (34). On the other hand, using the fluctuation–dissipation theorem and Eqs.(24), (28) we have $\langle\eta\varphi^*\rangle = \delta\eta/\delta\varphi = (\delta\varphi/\delta\eta)^{-1} = (\delta^2 V/\delta\eta^*\delta\eta)^{-1}$, that needs proving. Lastly, the equation

$$\int V(\Phi)\mathrm{d}^2\theta = -\left(\frac{\delta V}{\delta\eta}f + \frac{\delta V}{\delta\eta^*}f^*\right) + \bar{\psi}\frac{\delta^2 V}{\delta\eta\delta\eta^*}\psi \tag{A.15}$$

is obtained in analogy with Eq. (A.8) to represent in SUSY form the terms in Eq. (34) that contain the potential $V\{\eta\}$. It is worthwhile to mention that contrary to the two–component case the transformation operators $e^{\bar{\varepsilon}D}$, $e^{\varepsilon\bar{D}}$ of the genuine SUSY group possess the parameters ε, $\bar{\varepsilon}$ that are Grassmannian conjugated, but can be chosen pure real.

APPENDIX B

Following standard field scheme [4], let us show how the four-component SUSY field (34) is split into a couple of chiral two–component fields $\Phi_\pm$ being Grassmannian conjugated. These SUSY fields are obtained from the initial SUSY field Φ as follows:

$$\Phi_\pm = G_\pm\Phi; \quad G_\pm \equiv e^{\pm\partial}, \quad \partial \equiv (1/2)\theta\bar{\theta}(\partial/\partial t). \tag{B.1}$$

Accordingly, the generators (36) of the SUSY group take the form

$$D_\pm = G_\pm D G_\mp, \qquad \bar{D}_\pm = G_\pm\bar{D}G_\mp. \tag{B.2}$$

Taking into account the Grassmannian nature of the parameter ∂ in operators $G_\pm$, it is convenient to rewrite (B.2) in the following form:

$$D_\pm = D \pm [\partial, D], \qquad \bar{D}_\pm = \bar{D} \pm \left[\partial, \bar{D}\right], \tag{B.3}$$

where square brackets denote commutator.

According to Eq. (34) definitions (B.1) give

$$\Phi_\pm = \eta + \bar{\theta}\psi + \bar{\psi}\theta + \bar{\theta}\theta\left(f \mp \dot{\eta}/2\right), \tag{B.4}$$

where the point denotes the time derivative. Action of the operators (B.3) on this equation gets

$$\begin{aligned}D_\pm\Phi_\pm &= \psi + \theta\left(f + \dot{\eta}/2\right) - \underline{\bar{\theta}\theta\dot{\psi}},\\ -\bar{D}_\mp\Phi_\mp &= \bar{\psi} + \bar{\theta}\left(f - \dot{\eta}/2\right) + \underline{\bar{\theta}\theta\dot{\bar{\psi}}},\end{aligned} \tag{B.5}$$

where underlined terms concern only the upper indices of the left–hand parts. Taking into account Eq. (B.4), we obtain obvious expressions for the operators (B.3)

$$\begin{aligned}&D_+ = \partial/\partial\bar{\theta} + \theta(\partial/\partial t), \qquad D_- = \partial/\partial\bar{\theta};\\ &\bar{D}_+ = \partial/\partial\theta, \qquad \bar{D}_- = \partial/\partial\theta + \bar{\theta}(\partial/\partial t).\end{aligned} \tag{B.6}$$

Comparison with the initial generators (36) shows that the action of the operator $G_+(G_-)$ leads to the addition (subtraction) of the term $(\theta/2)\partial/\partial t$ to the generator D. With transfer to the conjugate generator $\bar{D}$, the operators G_+, G_- are replaced with respect to the addition $(\bar{\theta}/2)\partial/\partial t$. According to Eqs.(B.6) the transformation G_- allows to reduce generator D to the derivative

D_- on Grassmannian coordinate $\bar{\theta}$, whereas conjugate generator $\bar{D}$ is reduced to derivative $\bar{D}_+ = \partial/\partial\theta$ under action of the operator G_- (see Eqs.(B.2)).

By definition, the chiral SUSY fields are fixed by the gauge conditions [4]

$$D_-\Phi_- = 0, \qquad \bar{D}_+\Phi_+ = 0, \tag{B.7}$$

which in accordance with definitions (B.6) mean that Φ_- does not depend on $\bar{\theta}$, and Φ_+ on θ. On the other hand, taking into account Eqs.(B.5), the gauge (B.7) results in equations

$$\psi + \theta\,(f + \dot{\eta}/2) = 0,$$
$$\bar{\psi} + \bar{\theta}\,(f - \dot{\eta}/2) = 0. \tag{B.8}$$

Substituting them into Eq. (B.4), the final expressions for the chiral SUSY field are obtained:

$$\Phi_- = \eta + \bar{\psi}\theta,$$
$$\Phi_+ = \eta + \bar{\theta}\psi. \tag{B.9}$$

These equations give non–reducible representations of the SUSY field (34) provided that gauge (B.7) holds true.

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