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THE BOGOLIUBOV WEAKLY IMPERFECT BOSE–GAS

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It is shown that the condensation in the Bogoliubov Weakly Imperfect Bose–Gas (WIBG) may appear in two stages. If the interaction is such that the pressure of the WIBG does not coincide with the pressure of the Perfect Bose–Gas (PBG), then the WIBG may manifest two kinds of condensations: a *non–conventional* condensation in the zero–mode due to the interaction (the first stage) and a *conventional* (generalized of type I) Bose–Einstein condensation in modes next to the zero–mode due to the particle density saturation (the second stage).

Key words: Bogoliubov Weakly Imperfect Gas, non–conventional Bose–condensation, generalized (type I) condensation.

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I. INTRODUCTION

Consider a system of bosons of mass m in a cubic box $\Lambda = L\times L\times L \subset \mathbb{R}^3$ of the volume $V\equiv|\Lambda| = L^3$, with periodic boundary conditions on $\partial\Lambda$. If $\varphi(x)$ denotes an absolutely integrable two–body interaction potential and

$$v(q)=\int_{\mathbb{R}^3} d^3x\varphi(x)\,e^{-iqx},\;\; q\in\mathbb{R}^3, \tag{1.1}$$

then its second–quantized Hamiltonian acting on the boson Fock space $\mathcal{F}_\Lambda$ can be written as

$$H_\Lambda=\sum_{k\in\Lambda^*}\varepsilon_k a_k^*a_k + \frac{1}{2V}\sum_{k_1,k_2,q\in\Lambda^*} v(q)\,a^*_{k_1+q}a^*_{k_2-q}a_{k_1}a_{k_2}, \tag{1.2}$$

where all sums run over the set Λ^* defined by

$$\Lambda^*=\left\{k\in\mathbb{R}^3:\alpha=1,2,3,\; k_\alpha=\frac{2\pi n_\alpha}{L} \text{ et } n_\alpha=0,\pm1,\pm2,\dots\right\}. \tag{1.3}$$

Here $\varepsilon_k=\hbar^2k^2/2m$ is the kinetic energy, and $a_k^\#=\{a_k^*,a_k\}$ are usual boson creation and annihilation operators in the one–particle state $\psi_k(x)=V^{-\frac12}e^{ikx}, k\in\Lambda^*$, $x\in\Lambda$; for example, $a_k^*\equiv a^*(\psi_k)=\int_\Lambda dx\psi_k(x)\,a^*(x)$ where $a^\#(x)$ are basic boson operators in the Fock space $\mathcal{F}_\Lambda$ over $L^2(\Lambda)$.

Below we suppose that:
(A) $\varphi(x)=\varphi(\|x\|)$ and $\varphi\in L^1(\mathbb{R}^3)$;
(B) $v(k)$ is a real continuous function, satisfying $v(0)>0$ and $0\le v(k)\le v(0)$ for $k\in\mathbb{R}^3$.

If one expects that Bose–Einstein condensation, which occurs for the Perfect Bose–Gas (PBG) in the mode $k=0$, persists for a weak interaction $\varphi(x)$, then according to Bogoliubov [1,2] the most important terms in (1.2) should be those in which at least two operators a_0^*, a_0 appear. We are thus led to consider the following truncated Hamiltonian (the Bogoliubov Hamiltonian for a Weakly Imperfect Bose gas (WIBG), see [1,2]):

$$H_\Lambda^B=T_\Lambda+U_\Lambda^D+U_\Lambda, \tag{1.4}$$

where

$$T_\Lambda=\sum_{k\in\Lambda^*}\varepsilon_k a_k^*a_k, \tag{1.5}$$

$$U_\Lambda^D=\frac{v(0)}{V}a_0^*a_0\sum_{k\in\Lambda^*,k\neq0}a_k^*a_k + \frac{1}{2V}\sum_{k\in\Lambda^*,k\neq0}v(k)\,a_0^*a_0\left(a_k^*a_k+a_{-k}^*a_{-k}\right) + \frac{v(0)}{2V}a_0^{*^2}a_0^2, \tag{1.6}$$

$$U_\Lambda=\frac{1}{2V}\sum_{k\in\Lambda^*,k\neq0}v(k)\left(a_k^*a_{-k}^*a_0^2+a_0^{*^2}a_ka_{-k}\right). \tag{1.7}$$

Notice that the self–adjoint operator H_Λ^B is defined on a dense domain in the boson Fock space $\mathcal{F}_\Lambda\approx\mathcal{F}_{0\Lambda}\otimes\mathcal{F}'_\Lambda$ over $L^2(\Lambda)$, where $\mathcal{F}_{0\Lambda}$ and $\mathcal{F}'_\Lambda$ are the boson Fock spaces constructed out of $\mathcal{H}_{0\Lambda}$ (the one–dimensional subspace generated by $\psi_{k=0}\in L^2(\Lambda)$) and of its orthogonal complement $\mathcal{H}_{0\Lambda}^\perp$ respectively.

For any complex $c\in\mathbb{C}$, we can define in $\mathcal{F}_{0\Lambda}$ a coherent vector

$$\psi_{0\Lambda}(c) = e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V} c\right)^k (a_0^*)^k \, \Omega_0, \tag{1.8}$$

where Ω_0 is the vacuum of $\mathcal{F}_\Lambda$ and therefore $a_0 \psi_{0\Lambda}(c) = c\sqrt{V}\psi_{0\Lambda}(c)$. Using this concept of the coherent vectors in $\mathcal{F}_{0\Lambda}$, Ginibre [3] defines the *Bogoliubov approximation* to a Hamiltonian H_Λ in $\mathcal{F}_\Lambda$ as follows:

Definition 1 *The Bogoliubov approximation* $H_\Lambda(c^\#, \mu)$ *for a Hamiltonian* $H_\Lambda(\mu) \equiv H_\Lambda - \mu N_\Lambda$ *on* $\mathcal{F}_\Lambda$ *is the operator defined on* $\mathcal{F}'_\Lambda$ *by its quadratic form*

$$\left(\psi'_1, H_\Lambda(c^\#, \mu)\, \psi'_2\right)_{\mathcal{F}'_\Lambda} \equiv \left(\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda(\mu)\, \psi_{0\Lambda}(c) \otimes \psi'_2\right)_{\mathcal{F}_\Lambda}$$

for $\psi_{0\Lambda}(c) \otimes \psi'_{1,2}$ *in the form–domain of* $H_\Lambda(\mu)$*, where* $c^\# = (c, \overline{c})$ *and*

$$N_\Lambda = \sum_{k \in \Lambda^*} N_k$$

is the particle–number operator (here $N_k \equiv a_k^* a_k$ *is the occupation–number operator for the mode* k*) and* μ *is the chemical potential.*

Therefore, the *Bogoliubov approximation* in the Bogoliubov Hamiltonian for the WIBG (1.4) gets the form:

$$\begin{aligned} H_\Lambda^B(c^\#, \mu) \; = & \sum_{k \in \Lambda^*, k \neq 0} \left[\varepsilon_k - \mu + v(0)|c|^2\right] a_k^* a_k + \frac{1}{2} \sum k \in \Lambda^*, k \neq 0 v(k)\, |c|^2 \left[a_k^* a_k + a_{-k}^* a_{-k}\right] \\ & + \frac{1}{2} \sum_{k \in \Lambda^*, k \neq 0} v(k) \left[c^2 a_k^* a_{-k}^* + \overline{c}^2 a_k a_{-k}\right] - \mu |c|^2 V + \frac{1}{2} v(0) |c|^4 V. \end{aligned} \tag{1.9}$$

Then the Hamiltonian (1.9) can be diagonalized (cf. [1,2]). The pressure associated with $H_\Lambda^B(c^\#, \mu)$:

$$\widetilde{p}_\Lambda^B(\beta, \mu; c^\#) \equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}'_\Lambda} e^{-\beta H_\Lambda^B(c^\#, \mu)}, \tag{1.10}$$

(where $\theta = \beta^{-1}$ is the temperature) is well–defined for $\mu \leq v(0)|c|^2$ and has the following explicit form:

$$\widetilde{p}_\Lambda^B(\beta, \mu; c^\#) = \xi_\Lambda(\beta, \mu; x) + \eta_\Lambda(\mu; x), \tag{1.11}$$

$$\xi_\Lambda(\beta, \mu; x) = \frac{1}{\beta V} \sum_{k \in \Lambda^*, k \neq 0} \ln\left(1 - e^{-\beta E_k}\right)^{-1},$$

$$\eta_\Lambda(\mu; x) = -\frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} (E_k - f_k) + \mu x - \frac{1}{2} v(0) x^2,$$

where $x = |c|^2 \geq 0$ and

$$f_k = \varepsilon_k - \mu + x\left[v(0) + v(k)\right], \tag{1.12}$$

$$h_k = x v(k),$$

$$E_k = \sqrt{f_k^2 - h_k^2}.$$

Another observation concerns the original Hamiltonian (1.4), see [4,5].

Proposition 1 *The pressure* $p_\Lambda^B(\beta, \mu)$ *associated with the Bogoliubov Hamiltonian* H_Λ^B*, i.e.*

$$p_\Lambda^B(\beta, \mu) \equiv p_\Lambda\left[H_\Lambda^B\right] \equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}_\Lambda} e^{-\beta\left(H_\Lambda^B - \mu N_\Lambda\right)}, \tag{1.13}$$

is defined only in domain $Q = \{\mu \leq 0\} \times \{\theta \geq 0\}$ *and it is equal in the thermodynamic limit to*

$$\begin{aligned} p^B(\beta, \mu) \; &= \; \sup_{c \in \mathbb{C}} \widetilde{p}^B(\beta, \mu; c^\#) \\ &\equiv \; \lim_\Lambda \left\{\sup_{c \in \mathbb{C}} \widetilde{p}_\Lambda^B(\beta, \mu; c^\#)\right\}. \end{aligned} \tag{1.14}$$

Therefore, from the explicit form (1.11) of $\widetilde{p}_\Lambda^B(\beta, \mu; c^\#)$ we can deduce (cf. [4,5]) the following two corollaries:

Corollary 2. *Let* $v(k)$ *satisfy (A), (B) and*

$$v(0) \geq \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{v(k)^2}{\varepsilon_k}. \tag{1.15}$$

Then

$$p^B(\beta,\mu) = \sup_{c\in\mathbb{C}} \widetilde{p}^B(\beta,\mu;c^\#) = \widetilde{p}^B(\beta,\mu;0) \tag{1.16}$$
$$= p^P(\beta,\mu),$$

where

$$p^P(\beta,\mu) \equiv \lim_\Lambda p_\Lambda[T_\Lambda]$$

is the pressure of the Perfect Bose–Gas (PBG).

Corollary 3. *Let $v(k)$ satisfy (A), (B) and (C) :*

$$v(0) < \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{v(k)^2}{\varepsilon_k}. \tag{1.17}$$

Then there are $\mu_0 < 0$ and $\theta_0(\mu) > 0$ such that one has

$$p^B(\beta,\mu) = \sup_{c\in\mathbb{C}} \widetilde{p}^B(\beta,\mu;c^\#) \tag{1.18}$$
$$= \widetilde{p}^B\left(\beta,\mu;\widehat{c}^\#(\beta,\mu)\neq 0\right) > p^P(\beta,\mu),$$

for $(\theta,\mu)\in D$ defined by

$$D = \{(\theta,\mu) : \mu_0 < \mu \le 0, 0 \le \theta < \theta_0(\mu)\}, \tag{1.19}$$

and

$$p^B(\beta,\mu) = \sup_{c\in\mathbb{C}} \widetilde{p}^B(\beta,\mu;c^\#) = p^P(\beta,\mu), \tag{1.20}$$

for $(\theta,\mu)\notin \overline{D}$.

Moreover, see [4,5], D is a domain which corresponds to a *non–conventional* condensation in the mode $k=0$:

$$\rho_0^B(\theta,\mu) \equiv \lim_\Lambda \left\langle \frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda^B}(\beta,\mu) =$$
$$= \left\{ \begin{array}{c} |\widehat{c}(\beta,\mu)|^2 > 0,\ (\theta,\mu)\in D \\ 0,\ (\theta,\mu)\in Q\backslash\overline{D} \end{array} \right\}, \tag{1.21}$$

where $\widehat{c}(\beta,\mu)$ is defined by (1.18) and

$$\omega_\Lambda^B(-) \equiv \langle - \rangle_{H_\Lambda^B}(\beta,\mu) \tag{1.22}$$

represents the grand–canonical Gibbs state for the Hamiltonian H_Λ^B. The *non–conventional* Bose–condensate (1.21) undergoes a jump on the boundary ∂D, see [4,5] .

However, we have to admit that in [4,5] we study the WIBG only in the grand–canonical ensemble, i.e. by fixing the chemical potential μ. On the other hand, it is well–known that the *conventional* Bose–Einstein condensation in the PBG is parametrized by the total particle density ρ which should be higher than the saturated for $\mu=0$ particle density $\rho^P(\beta,\mu)$ in the grand canonical ensemble: $\rho > \rho_c^P(\theta) \equiv \rho^P(\theta^{-1},\mu=0)$. Thus [4,5] do not study the *conventional* Bose–Einstein condensation in the WIBG.

Notice that using the Griffiths Lemma (see [6,7]) and Proposition 1, one finds for the grand–canonical total particle density in the WIBG:

$$\rho^B(\beta,\mu) \equiv \lim_\Lambda \omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right) = \lim_\Lambda \frac{1}{V}\sum_{k\in\Lambda^*}\omega_\Lambda^B(N_k) = \lim_\Lambda \partial_\mu p_\Lambda^B(\beta,\mu) = \partial_\mu \widetilde{p}^B(\beta,\mu;0)$$
$$= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(e^{\beta(\varepsilon_k-\mu)}-1\right)^{-1} d^3k, \tag{1.23}$$

for $(\theta,\mu<0)\in Q\backslash\overline{D}$ and:

$$\rho^B(\beta,\mu) = \partial_\mu \widetilde{p}^B\left(\beta,\mu;\widehat{c}^\#(\beta,\mu)\neq 0\right) \tag{1.24}$$
$$= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left[\frac{f_k}{E_k}\left(e^{\beta E_k}-1\right)^{-1} + \frac{h_k^2}{2E_k(f_k+E_k)}\right] d^3k\bigg|_{c=\widehat{c}(\beta,\mu)} + |\widehat{c}(\beta,\mu)|^2,$$

for $(\theta,\mu<0)\in D$. Then, from (1.23) and (1.24), we see that the total density $\rho^B(\beta,\mu)$ reaches its maximal (critical) value at $\mu=0$, i.e. $\rho_c^B(\theta)\equiv\rho^B(\beta,\mu=0)$:

(*i*) for $\theta > \theta_0(\mu=0)$

$$\rho_c^B(\theta) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(e^{\beta\varepsilon_k}-1\right)^{-1} d^3k < +\infty, \tag{1.25}$$

(ii) for $\theta < \theta_0\,(\mu = 0)$

$$\rho_c^B(\theta) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left[\frac{f_k}{E_k}\left(e^{\beta E_k}-1\right)^{-1} + \frac{h_k^2}{2E_k\,(f_k+E_k)}\right] d^3k\Big|_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}} + \left|\widehat{c}\,(\beta,\mu=0)\right|^2 < +\infty. \tag{1.26}$$

By convexity of $p^B(\beta,\mu)$ with respect to the parameter μ one gets that

$$\lim_{\mu\to\mu_0(\theta)^-}\rho^B(\beta,\mu) \equiv \rho_{\inf}^B(\theta) < \lim_{\mu\to\mu_0(\theta)^+}\rho^B(\beta,\mu) \equiv \rho_{\sup}^B(\theta), \tag{1.27}$$

where $\mu_0 = \mu_0(\theta)$ is the inverse function of $\theta_0(\mu)$, and

$$\lim_{\theta\to\theta_0(0)^+}\rho_c^B(\theta) < \lim_{\theta\to\theta_0(0)^-}\rho_c^B(\theta)\,. \tag{1.28}$$

Thus the aim of the present paper is to study the thermodynamic properties of the WIBG in function of the total particle density ρ to answer the question of its thermodynamic behaviour for the densities $\rho \geq \rho_c^B(\theta)$. Our main statements are formulated in the next Section II where we explicit the existence of a *conventional* (*generalized*) Bose–Einstein condensation for $\mu = 0$ and densities $\rho > \rho_c^B(\theta)$ which occurs *after* a *non–conventional* condensation (1.21) [4,5] if $\theta \leq \theta_0(0)$, see Figure 1. Section III contains discussions and concluding remarks. Some technical statements are formulated in Appendix A.

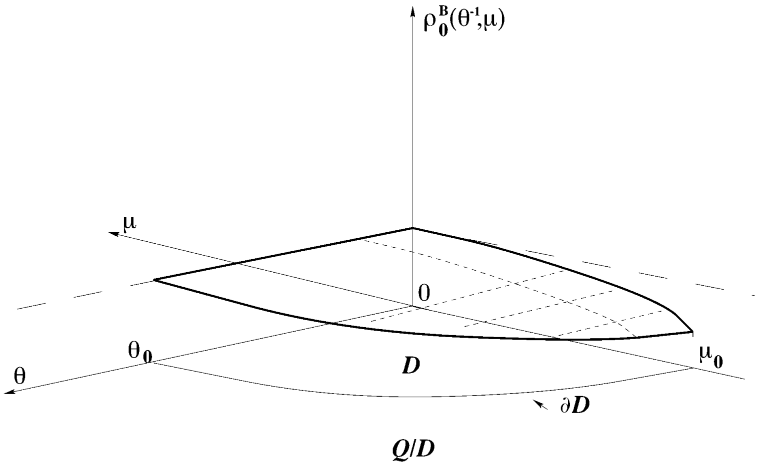

Fig. 1. Density of the non–conventional condensation in the Bogoliubov WIBG.

II. BOSE–EINSTEIN CONDENSATION IN THE WIBG

In this section we study the WIBG for temperature and the total particle density as given parameters.
Theorem 4. *Let interaction (1.1) satisfies (A) and (B). Then there exists* $\varepsilon_{\Lambda,1}$*:*

$$\varepsilon_{\Lambda,1} \in \left[\inf_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right], \widehat{\varepsilon}_{\Lambda,1} = \inf_{k\neq 0}\varepsilon_k\right],$$

such that for $\mu < \varepsilon_{\Lambda,1}$

$$p_\Lambda^B(\beta,\mu) < +\infty, \qquad \omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right) < +\infty, \tag{2.1}$$

and

$$\lim_{\mu\to\varepsilon_{\Lambda,1}} p_\Lambda^B(\beta,\mu)=+\infty,\qquad \lim_{\mu\to\varepsilon_{\Lambda,1}} \omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right)=+\infty. \tag{2.2}$$

Proof. Since $v(k)$ satisfies (A) and (B), by regrouping terms in (1.6), (1.7) one gets

$$H_\Lambda^B=\widetilde{H}_\Lambda+\frac{v(0)}{V}a_0^*a_0\sum_{k\in\Lambda^*,k\neq0}a_k^*a_k+\frac{1}{2V}\sum_{k\in\Lambda^*,k\neq0}v(k)\left(a_0^*a_k+a_{-k}^*a_0\right)^*\left(a_0^*a_k+a_{-k}^*a_0\right), \tag{2.3}$$

where

$$\widetilde{H}_\Lambda=\sum_{k\in\Lambda^*,k\neq0}\left(\varepsilon_k-\frac{v(k)}{2V}\right)a_k^*a_k+\frac{v(0)}{2V}\left(a_0^*a_0\right)^2-\frac{1}{2}\varphi(0)\,a_0^*a_0. \tag{2.4}$$

Thus from (2.3), (2.4) we obtain

$$H_\Lambda^B\geq\widetilde{H}_\Lambda. \tag{2.5}$$

By straightforward calculations one gets

$$p_\Lambda\left[\widetilde{H}_\Lambda\right]=\frac{1}{\beta V}\sum_{k\in\Lambda^*,k\neq0}\ln\left\{1-e^{-\beta\left[\varepsilon_k-\left(\mu+\frac{v(k)}{2V}\right)\right]}\right\}^{-1}+\frac{1}{\beta V}\ln\sum_{n_0=0}^{+\infty}e^{\beta V\left[\left(\mu+\frac{1}{2}\varphi(0)\right)\frac{n_0}{V}-\frac{v(0)}{2V}\left[\frac{n_0}{V}\right]^2\right]},$$

which together with (2.5) implies

$$p_\Lambda^B(\beta,\mu)\leq p_\Lambda\left[\widetilde{H}_\Lambda\right]<+\infty \tag{2.6}$$

for $\mu<\inf_{k\neq0}\left[\varepsilon_k-\frac{v(k)}{2V}\right]$. Since

$$\omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right)=\partial_\mu p_\Lambda^B(\beta,\mu),$$

by (2.6) and by convexity of the pressure $p_\Lambda^B(\beta,\mu)$ in parameter μ we deduce that

$$\omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right)<+\infty$$

for $\mu<\inf_{k\neq0}\left[\varepsilon_k-\frac{v(k)}{2V}\right]$. Moreover by the Bogoliubov inequality (see e.g. [8,9]), one gets:

$$\frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^B}\leq p_\Lambda\left[H_\Lambda^{BD}\right]-p_\Lambda\left[H_\Lambda^B\right]\leq\frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^{BD}}, \tag{2.7}$$

where $H_\Lambda^{BD}\equiv T_\Lambda+U_\Lambda^D$ is the diagonal part of the Bogoliubov Hamiltonian with T_Λ and U_Λ^D defined respectively by (1.5) and (1.7). Since $\langle U_\Lambda\rangle_{H_\Lambda^{BD}}=0$, we deduce from (2.7) that

$$p_\Lambda^B(\beta,\mu)\geq p_\Lambda\left[H_\Lambda^{BD}\right].$$

Combining this with the estimate (cf. [4,5])

$$p_\Lambda\left[H_\Lambda^{BD}\right]\geq\frac{1}{\beta V}\sum_{k\in\Lambda^*,k\neq0}\ln\left[\left(1-e^{[-\beta(\varepsilon_k-\mu)]}\right)^{-1}\right]$$

we get

$$\lim_{\mu\to\inf_{k\neq0}\varepsilon_k}p_\Lambda\left[H_\Lambda^{BD}\right]=+\infty. \tag{2.8}$$

Therefore, by (2.6) and (2.8) we deduce that there exists $\varepsilon_{\Lambda,1}\in\left[\inf_{k\neq0}\left[\varepsilon_k-\frac{v(k)}{2V}\right],\inf_{k\neq0}\varepsilon_k\right]$ such that $p_\Lambda^B(\beta,\mu)$ and $\omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right)$ are bounded for $\mu<\varepsilon_{\Lambda,1}$ and

$$\lim_{\mu\to\varepsilon_{\Lambda,1}}p_\Lambda^B(\beta,\mu)=+\infty. \tag{2.9}$$

Notice that by convexity of $p_\Lambda^B(\beta,\mu)$ one gets

$$\frac{p_\Lambda^B(\beta,\mu)-p_\Lambda^B(\beta,0)}{\mu}\leq\partial_\mu p_\Lambda^B(\beta,\mu)=\omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right).$$

Then the limit (2.9) implies

$$\lim_{\mu\to\varepsilon_{\Lambda,1}} \omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right) = +\infty,$$

which completes the proof of (2.2). ■

Since the case of $\rho < \rho_c^B(\theta)$ (cf. (1.25), (1.26)) has been already studied in Section I, see (1.23), (1.24), below we consider the case $\rho \ge \rho_c^B(\theta)$.

Corollary 5. *By Theorem 4 for any $\rho \ge \rho_c^B(\theta)$ there is a unique value of the chemical potential $\mu_\Lambda^B(\theta,\rho) < \varepsilon_{\Lambda,1}$ (notice that in general $\mu_\Lambda^B(\theta,\rho) \gtrless 0$) such that*

$$\omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right) = \rho, \tag{2.10}$$

and

$$\lim_\Lambda \mu_\Lambda^B\left(\theta,\rho\ge\rho_c^B(\theta)\right) = 0. \tag{2.11}$$

From now on we put

$$\omega_{\Lambda,\rho}^B(-) \equiv \omega_\Lambda^B(-)\mid_{\mu=\mu_\Lambda^B(\theta,\rho)}. \tag{2.12}$$

According to [4,5] the WIBG non–conventional condensation in the mode $k=0$ is saturated for $\mu\to 0^-$ either by $|\widehat{c}(\beta,0)|^2 > 0$ (for $\theta < \theta_0(0)$), or by $|\widehat{c}(\beta,0)|^2 = 0$ (for $\theta > \theta_0(0)$), see (1.21). Therefore, by (1.23)–(1.26) and Theorem 4 the saturation of the total particle density should imply the conventional Bose–Einstein condensation in modes *next to* $k=0$. For discussion of this phenomenon of *two kinds* of condensations in the framework of simple models see e.g. recent papers [10,11].

To control the condensation in $k\neq 0$ we introduce an auxiliary Hamiltonian

$$H_{\Lambda,\alpha}^B = H_\Lambda^B - \alpha \sum_{k\in\Lambda^*, a<\|k\|<b} a_k^* a_k,$$

for a fixed $a>0$ and $b>a>0$. Then we set

$$p_\Lambda^B(\beta,\mu,\alpha) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}_\Lambda} e^{-\beta H_{\Lambda,\alpha}^B(\mu)}, \tag{2.13}$$

and

$$\omega_\Lambda^{B,\alpha}(-) \equiv \langle -\rangle_{H_{\Lambda,\alpha}^B}(\beta,\mu)$$

for the grand–canonical Gibbs state corresponding to $H_{\Lambda,\alpha}^B(\mu)$.

Recall that $\mu_0(\theta)$ is the function (inverse to $\theta_0(\mu)$) which defines a borderline of domain D, see (1.19).

Proposition 6. *[4,5] Let $\alpha\in[-\delta,\delta]$ where $0\le\delta\le\varepsilon_a/2$ and $\varepsilon_a = \inf_{\|k\|\ge a}\varepsilon_k$. Then there exists a domain $D_\delta\subset D$:*

$$D_\delta \equiv \{(\theta,\mu) : \mu_0 < \mu_0(\delta) \le \mu \le 0,\ 0\le\theta\le\theta_0(\mu,\delta) < \theta_0(\mu)\} \tag{2.14}$$

such that

$$\left| p_\Lambda^B(\beta,\mu,\alpha) - \sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B(\beta,\mu,\alpha;c^\#)\right| \le \frac{K(\delta)}{\sqrt{V}} \tag{2.15}$$

for V sufficiently large, uniformly in $\alpha\in[-\delta,\delta]$ and for:

(i) $(\theta,\mu)\in D_\delta$, *if* $\mu_\Lambda^B\left(\theta,\rho\ge\rho_c^B(\theta)\right)\le 0$; (2.16)

(ii) $(\theta,\mu)\in D_\delta\cup\left\{(\theta,\mu) : 0\le\mu\le\mu_\Lambda^B\left(\theta,\rho\ge\rho_c^B(\theta)\right),\ 0\le\theta\le\theta_0(\mu=0,\delta)\right\}$, *if* $\mu_\Lambda^B\left(\theta,\rho\ge\rho_c^B(\theta)\right)\ge 0$.

Proof. The existence of the domain D_δ follows from the proof of Theorem 3.14 [5]. This means that the estimate (2.15) is stable with respect to local perturbations of the free–particle spectrum: $\varepsilon_k\to\varepsilon_k-\alpha\chi_{(a,b)}(\|k\|)$ for $|\alpha|\le\delta\le\varepsilon_a/2$ in a reduced domain $D_\delta\subset D$. Here $\chi_{(a,b)}(\|k\|)$ is the characteristic function of $(a,b)\subset\mathbb{R}$. Extension in (2.16) (cf. Corollary 5) is due to continuity of the pressure $p_\Lambda^B(\beta,\mu,\alpha)$ and the trial pressure $\widetilde{p}_\Lambda^B(\beta,\mu,\alpha;c^\#)$ in parameters $\alpha\in[-\delta,\delta]$ and $\mu\le\mu_\Lambda^B\left(\theta,\rho\ge\rho_c^B(\theta)\right)$, see (2.11). ■

Corollary 7. *Let $\rho\ge\rho_c^B(\theta)$, see (1.25), (1.26). Then for $\theta<\theta_0(0)$ one has*

$$\lim_\Lambda \frac{1}{V}\sum_{k\in\Lambda^*, a<\|k\|<b}\omega_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3}\int_{a<\|k\|<b}\left[\frac{f_k}{E_k}\left(e^{\beta E_k}-1\right)^{-1} + \frac{h_k^2}{2E_k(f_k+E_k)}\right]d^3k\Bigg|_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}}, \tag{2.17}$$

whereas for $\theta > \theta_0(0)$ *we have*

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^*, a < \|k\| < b} \omega_{\Lambda,\rho}^{B}(N_k) = \frac{1}{(2\pi)^3} \int_{a < \|k\| < b} \left(e^{\beta \varepsilon_k} - 1\right)^{-1} d^3k. \tag{2.18}$$

Proof. Consider the sequence of functions $\left\{p_{\Lambda}^{B}\left(\beta, \mu_{\Lambda}^{B}(\theta,\rho), \alpha\right)\right\}_{\Lambda}$ (2.13), where chemical potential is defined by (2.10), (2.11) and $\alpha \in [-\delta, \delta]$. Since by (2.13)

$$\partial_{\alpha} p_{\Lambda}^{B}\left(\beta, \mu_{\Lambda}^{B}(\theta,\rho), \alpha\right) = \frac{1}{V} \sum_{k \in \Lambda^*, a < \|k\| < b} \omega_{\Lambda,\rho}^{B,\alpha}(N_k) \tag{2.19}$$

and $\left\{p_{\Lambda}^{B}\left(\beta, \mu_{\Lambda}^{B}(\theta,\rho), \alpha\right)\right\}_{\Lambda}$ are convex functions of $\alpha \in [-\delta, \delta]$, Proposition 6 and the Griffiths lemma [6,7] imply

$$\lim_{\Lambda} \partial_{\alpha} p_{\Lambda}^{B}\left(\beta, \mu_{\Lambda}^{B}(\theta,\rho), \alpha\right) = \lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^*, a < \|k\| < b} \omega_{\Lambda,\rho}^{B,\alpha}(N_k) = \partial_{\alpha} \lim_{\Lambda} \sup_{c \in \mathbb{C}} \widetilde{p}_{\Lambda}^{B}\left(\beta, \mu_{\Lambda}^{B}(\theta,\rho), \alpha; c^{\#}\right), \tag{2.20}$$

for $\alpha \in [-\delta, \delta]$. Therefore, by explicit calculations in the right–hand side of (2.20) (cf. (1.10)–(1.12)) we obtain for $\alpha = 0$ equalities (2.17) and (2.18). ■

Remark 1. *Notice that the expectation values* $\omega_{\Lambda}^{B}(N_k) = \langle N_k \rangle_{H_{\Lambda}^{B}}(\beta,\mu)$ *(and similar* $\omega_{\Lambda,\rho}^{B}(N_k) = \langle N_k \rangle_{H_{\Lambda}^{B}}\left(\beta, \mu_{\Lambda}^{B}(\theta,\rho)\right)$*) are defined on the discrete set* Λ^**. Below we denote by* $\left\{\omega_{\Lambda}^{B}(N_k)\right\}_{k \in \mathbb{R}^3}$ *a continuous interpolation of these values from the set* Λ^* *to* $\mathbb{R}^3$*.*

Now we are in position to prove the main statement of this section about the Bose–Einstein condensation manifested by the WIBG for large densities ρ at fixed temperature $\theta = \beta^{-1}$.

Theorem 8. *For* $\rho > \rho_c^B(\theta)$ *we have that*

(i)

$$\lim_{\Lambda} \omega_{\Lambda,\rho}^{B}\left(\frac{a_0^* a_0}{V}\right) = \left\{ \begin{array}{c} |\widehat{c}(\beta,0)|^2,\ \theta < \theta_0(0) \\ 0,\ \theta > \theta_0(0) \end{array} \right\}; \tag{2.21}$$

(ii) for any $k \in \Lambda^*$*, such that* $\|k\| > \frac{2\pi}{L}$*,*

$$\lim_{\Lambda} \omega_{\Lambda,\rho}^{B}\left(\frac{N_k}{V}\right) = 0; \tag{2.22}$$

(iii) for $\theta < \theta_0(0)$ *and for all* $k \in \Lambda^*$*, such that* $\|k\| > \delta > 0$

$$\lim_{\Lambda} \omega_{\Lambda,\rho}^{B}(N_k) = \left[\frac{f_k}{E_k}\left(e^{\beta E_k} - 1\right)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)}\right]_{\substack{c=\widehat{c}(\beta,0) \\ \mu=0}} \tag{2.23}$$

whereas for $\theta > \theta_0(0)$

$$\lim_{\Lambda} \omega_{\Lambda,\rho}^{B}(N_k) = \frac{1}{e^{\beta \varepsilon_k} - 1}; \tag{2.24}$$

(iv) the double limit

$$\lim_{\delta \to 0+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| \le \delta\}} \omega_{\Lambda,\rho}^{B}(N_k) = \rho - \rho_c^B(\theta), \tag{2.25}$$

which means that the WIBG manifests a conventional generalized Bose–Einstein condensation in the 2d modes next to the zero–mode due to particle density saturation.

Proof. *(i)* Since by (2.11) we have

$$\lim_{\Lambda} \mu_{\Lambda}^{B}(\theta,\rho) = 0, \tag{2.26}$$

the thermodynamic limit (2.21) results from Theorem 4.4 and Corollary 4.8 of [5], see also (1.21) for $\mu = 0$.

(ii) Since $\|k\| > \frac{2\pi}{L}$ and $\Lambda = L \times L \times L$, which excludes a generalized Bose–Einstein condensation due to anisotropy [12], the thermodynamic limit (2.22) follows from Lemma 10.

(iii) Let us consider $g_{\theta}(k)$ defined for $k \in \mathbb{R}^3$, $\|k\| > \delta > 0$ by

$$g_{\theta}(k) \equiv \lim_{\Lambda} \omega_{\Lambda,\rho}^{B}(N_k), \tag{2.27}$$

where (cf. (2.12)) the state $\omega_{\Lambda,\rho}^{B}(-)$ stands for $\omega_{\Lambda}^{B}(-)$ with $\mu = \mu_{\Lambda}^{B}(\theta,\rho)$. Notice that by Lemma 10 and the fact that

$$\mu_\Lambda^B(\theta,\rho) < \varepsilon_{\Lambda,1} < \inf_{k\neq 0}\varepsilon_k = \varepsilon_{\|k\|=\frac{2\pi}{L}}$$

the thermodynamic limit (2.27) exists and it is informally bounded for $\|k\| > \delta > 0$. Moreover, for any interval $(a > \delta, b)$ we have

$$\lim_\Lambda \frac{1}{V}\sum_{k\in\Lambda^*,\|k\|\in(a,b)} \omega_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3}\int_{\|k\|>\delta} g_\theta(k)\,\chi_{(a,b)}(\|k\|)\,d^3k,$$

where $\chi_{(a,b)}(\|k\|)$ is the characteristic function of (a,b). Then Corollary 7 implies that

$$\frac{1}{(2\pi)^3}\int_{\|k\|>\delta} g_\theta(k)\,\chi_{(a,b)}(\|k\|)\,d^3k = \frac{1}{(2\pi)^3}\int_{\|k\|>\delta} f_\theta(k)\,\chi_{(a,b)}(\|k\|)\,d^3k, \tag{2.28}$$

where $f_\theta(k)$ is a continuous function on $k\in\mathbb{R}^3$ defined by (2.17), (2.18), i.e.

$$f_\theta(k) \equiv \frac{1}{(2\pi)^3}\left[\frac{f_k}{E_k}\left(e^{\beta E_k}-1\right)^{-1} + \frac{h_k^2}{2E_k(f_k+E_k)}\right]_{\substack{c=\hat{c}(\beta,0)\\ \mu=0}}, \tag{2.29}$$

for $\theta < \theta_0(0)$ and

$$f_\theta(k) \equiv \frac{1}{(2\pi)^3}\left(e^{\beta\varepsilon_k}-1\right)^{-1}, \tag{2.30}$$

for $\theta > \theta_0(0)$. Since the relation (2.28) is valid for any interval $(a>\delta, b)\subset\mathbb{R}$ one gets

$$g_\theta(k) = f_\theta(k),\ k\in\mathbb{R}^3,\ \|k\|>\delta>0$$

from which by (2.27), (2.29) and (2.30) we deduce (2.23) and (2.24).

(iv) Since the total density ρ is fixed, we have

$$\frac{1}{V}\sum_{\{k\in\Lambda^*,0<\|k\|\leq\delta\}} \omega_{\Lambda,\rho}^B(N_k) = \rho - \omega_{\Lambda,\rho}^B\left(\frac{a_0^*a_0}{V}\right) - \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|>\delta\}} \omega_{\Lambda,\rho}^B(N_k). \tag{2.31}$$

Using Corollary 7 for $a=\delta$ and $b\to+\infty$ we obtain for $\theta<\theta_0(0)$

$$\lim_\Lambda \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|>\delta\}} \omega_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3}\left[\int_{\|k\|>\delta} \frac{f_k}{E_k}\left(e^{\beta E_k}-1\right)^{-1} + \frac{h_k^2}{2E_k(f_k+E_k)}d^3k\right]_{\substack{c=\hat{c}(\beta,0)\\ \mu=0}}, \tag{2.32}$$

and for $\theta>\theta_0(0)$

$$\lim_\Lambda \frac{1}{V}\sum_{k\in\Lambda^*,\|k\|>\delta} \omega_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3}\int_{\|k\|>\delta}\left(e^{\beta\varepsilon_k}-1\right)^{-1}d^3k. \tag{2.33}$$

Then, from (1.25), (1.26), (2.21), (2.31)–(2.33) we finally deduce (2.25) by taking the limit $\delta\to 0^+$. ■

Therefore, according to (2.25) and in a close similarity to [11] for $\theta>\theta_0(0)$ and $\rho>\rho_c^B(\theta)$ the WIBG manifests only *one kind* of condensation, namely a *conventional* Bose–Einstein condensation which occurs in the mode $k\neq 0$, whereas for $\theta<\theta_0(0)$ it manifests for $\rho>\rho_c^B(\theta)$ this kind of condensation as a *second* stage after the *non–conventional* Bose condensation $|\hat{c}(\beta,0)|^2$, see (2.21).

Remark 2. *Similar to the model of ref. [11] in domain: $\theta<\theta_0(0)$, $\rho>\rho_c^B(\theta)$, we have the coexistence of two kinds of condensations:*

— the non–conventional one which starts when $\rho>\rho_{sup}^B(\theta)$ ($\rho\leq\rho_c^B(\theta)$), see (1.25)–(1.27),

— and the conventional Bose–Einstein condensation when $\rho>\rho_c^B(\theta)$.

Remark 3. *Before we classify this latter condensation we remind to the readers about the convenience of the nomenclature of conventional (generalized) Bose–Einstein condensations according to [12,13]:*

— a condensation is called of type I when a finite number of levels is macroscopically occupied;

— *it is of type II when an infinite number of levels is macroscopically occupied;*

— *it is called of type III, or the non–extensive condensation, when no levels are macroscopically occupied whereas one has*

$$\lim_{\delta\to 0+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0<\|k\|\leq\delta\}}\langle N_k\rangle=\rho-\rho_c(\theta)\,.$$

Paper [12] demonstrates that these three kinds of *conventional* condensations can be realized for the case of the PBG in an *anisotropic* box $\Lambda\subset\mathbb{R}^3$ with volume the $V=|\Lambda|$ and the Dirichlet boundary conditions, i.e. in a box Λ with $L_x=V^{\alpha_x}$, $L_y=V^{\alpha_y}$ and $L_z=V^{\alpha_z}$ for $\alpha_x+\alpha_y+\alpha_z=1$ and $\alpha_x\leq\alpha_y\leq\alpha_z$. At fixed temperature and for sufficiently large density ρ, we have a condensation of the type I in the fundamental mode $k=\left(\frac{2\pi}{L_x},\frac{2\pi}{L_y},\frac{2\pi}{L_z}\right)$ if $\alpha_z<1/2$ whereas for $\alpha_z=1/2$ one gets a condensation of the type II characterized by a macroscopic occupation of all modes $k=\left(\frac{2\pi}{L_x},\frac{2\pi}{L_y},\frac{2\pi n}{L_Z}\right)$, $n\in\mathbb{N}$ and for $\alpha_z>1/2$ one obtains a condensation of the type III. In [14,15] it was shown that type III condensation can be provoked in the PBG by a weak external potential or (see [13,16]) by a specific choice of boundary conditions and geometry. Another example of the *non–extensive* condensation is given in [10,11] for bosons in an *isotropic* box Λ with *interactions* which spread out the *conventional* condensation of the type I into a *conventional* condensation of the type III.

Therefore, from (2.22) and (2.25) we can deduce only that the *conventional* condensation in the WIBG can be either a condensation of type I in modes $\|k\|=2\pi/L$, or a condensation of the type III if modes $\|k\|=2\pi/L$ are not macroscopically occupied, or finally a combination of the non–extensive condensation with a condensation of the type I in the modes $\|k\|=2\pi/L$.

Corollary 9. *In fact, for $\rho>\rho_c^B(\theta)$ the generalized (conventional) condensation (2.25) is a condensation of the type I in the first 2d modes next to the zero–mode $k=0$, i.e.*

$$\lim_{\Lambda}\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega^B_{\Lambda,\rho}\left(a_k^*a_k\right)=\rho-\rho_c^B(\theta)\,. \tag{2.34}$$

Proof. Since for $\delta>0$

$$\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega^B_{\Lambda,\rho}(N_k)=\rho-\omega^B_{\Lambda,\rho}\left(\frac{a_0^*a_0}{V}\right)-\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\frac{2\pi}{L}<\|k\|<\delta\right\}}\omega^B_{\Lambda,\rho}(N_k)-\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geq\delta\}}\omega^B_{\Lambda,\rho}(N_k)\,,$$

using Lemma 1 we find that

$$\begin{aligned}\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega^B_{\Lambda,\rho}(N_k)\geq{}&\rho-\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\frac{2\pi}{L}<\|k\|<\delta\right\}}\frac{1}{e^{B_k\left(\mu^B_\Lambda(\theta,\rho)\right)}-1}\\&-\omega^B_{\Lambda,\rho}\left(\frac{a_0^*a_0}{V}\right)\left[1+\frac{\beta}{2V}\sum_{\left\{k\in\Lambda^*,\frac{2\pi}{L}<\|k\|<\delta\right\}}\frac{v(k)}{1-e^{-B_k\left(\mu^B_\Lambda(\theta,\rho)\right)}}\right]\\&-\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geq\delta\}}\omega^B_{\Lambda,\rho}(N_k)\,,\end{aligned} \tag{2.35}$$

with $B_k\left(\mu^B_\Lambda(\theta,\rho)\right)$ defined by (4.2). Since by Theorem 1 one gets

$$\mu^B_\Lambda(\theta,\rho)<\varepsilon_{\Lambda,1}<\inf_{k\neq 0}\varepsilon_k=\varepsilon_{\|k\|=\frac{2\pi}{L}}\,,$$

from (1.25), (1.26), (2.32) we deduce

$$\lim_{\Lambda}\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega^B_{\Lambda,\rho}\left(a_k^*a_k\right)\geq\rho-\rho_c^B(\theta) \tag{2.36}$$

by taking the limit $\delta\to 0^+$ in the right–hand side of (2.35) after the thermodynamic limit. Therefore, combining the inequality

$$\lim_{\Lambda}\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega^B_{\Lambda,\rho}\left(N_k\right)$$

$$\leq \lim_{\Lambda}\frac{1}{V}\sum_{\left\{k\in\Lambda^*,0<\|k\|<\delta\right\}}\omega^B_{\Lambda,\rho}\left(N_k\right)$$

with Theorem 5 (cf. (2.25)) and (2.36)), we obtain (2.34). ■

Therefore, for a fixed temperature θ and a fixed total particle density ρ, we obtain *three types* of thermodynamic behaviour of the WIBG for $\theta < \theta_0(0)$:

(i) for $\rho \leq \rho^B_{\inf}(\theta)$, there is no condensation;

(ii) for $\rho^B_{\sup}(\theta) \leq \rho \leq \rho^B_c(\theta)$, there is a *non–conventional* (*dynamical*) condensation (1.21) in the mode $k = 0$ due to non–diagonal interaction in the Bogoliubov Hamiltonian, see Figure 1 and [4,5,17];

(iii) for $\rho^B_c(\theta) \leq \rho$, there is a *second* kind of condensation: the *conventional* type I Bose–Einstein condensation which occurs *after* the *non–conventional* one due to the standard mechanism of the total particle density saturation (Corollary 9).

If $\theta \geq \theta_0(0)$, there are only *two types* of thermodynamic behaviour: they correspond to $\rho \leq \rho^B_c(\theta)$ with no condensation and to $\rho^B_c(\theta) \leq \rho$ with a *conventional* condensation as in (iii). Hence, for $\theta > \theta_0(0)$ the condensation in the WIBG coincides with type I generalized Bose–Einstein condensation in the PBG with *excluded* mode $k = 0$, see Theorem 8 (iii) and [18].

III. CONCLUSION

Papers [4,5] have already discussed the existence of a *non–conventional* condensation of bosons for $k = 0$, for negative μ and $\theta < \theta_0(0)$. The physical reason of this non–conventional (or dynamical) condensation is an *effective attraction* between bosons in the mode $k = 0$ [17]:

$$-\left\{\frac{1}{V^2}\sum_{k\in\Lambda^*,k\neq 0}\frac{\left[v(k)\right]^2}{4\varepsilon_k}\right\}a_0^{*^2}a_0^2 \tag{3.1}$$

which has to dominate the direct repulsion in (1.7):

$$\frac{v(0)}{2V}a_0^{*^2}a_0^2,$$

to ensure this new kind of condensation, see condition (C) (1.17) and discussions in [17]. However, for fixed temperature θ and total particle density ρ the present paper indicates the possibility of a conventional condensation: a *generalized* Bose–Einstein condensation of *the type I* in the first $2d$ modes next to the zero–mode $k = 0$. This second kind of condensation appears only for high densities $\rho \geq \rho^B_c(\theta)$ due to the standard mechanism of the total particle density saturation, see Corollary 9.

Therefore, combining [4,5] with Section II for $\theta < \theta_0(0)$ we obtain for the WIBG three types of thermodynamic behaviour:

(i) for $\rho \leq \rho^B_{\inf}(\theta)$, there is no condensation;

(ii) for $\rho^B_{\sup}(\theta) \leq \rho \leq \rho^B_c(\theta)$, a *non–conventional* (*dynamical*) condensation (1.21) appears in the mode $k = 0$;

(iii) for $\rho^B_c(\theta) \leq \rho$, the WIBG manifests a *conventional* Bose–Einstein condensation of the type I (Corollary 6). Therefore, two kinds of condensation coexist.

For $\theta < \theta_0(0)$, the thermodynamic behaviour of the WIBG is related to the two recent models [11] defined respectively by Hamiltonians

$$H^0_\Lambda \equiv T_\Lambda + U^0_\Lambda, \tag{3.2}$$

and

$$H_\Lambda = H^0_\Lambda + U_\Lambda, \tag{3.3}$$

where

$$T_\Lambda = \sum_{k\in\Lambda^*\backslash\{0\}}\varepsilon_k a_k^* a_k,\ \varepsilon_{k\neq 0} = \hbar^2k^2/2m,$$

$$U^0_\Lambda = \varepsilon_0 a_0^* a_0 + \frac{g_0}{V}a_0^*a_0^*a_0a_0,\ \varepsilon_0\in\mathbb{R}^1,\ g_0>0, \tag{3.4}$$

$$U_\Lambda = \frac{1}{V}\sum_{k\in\Lambda^*,k\neq 0} g_k(V)\, a_k^*a_k^*a_ka_k,$$

with $0 < g_k(V) \leq \gamma_k V^{\alpha_k}$ for $k \in \Lambda^*\backslash\{0\}$, $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$. Notice that in these models, $\varepsilon_0 \in \mathbb{R}^1$ is *not* equal to $\varepsilon_{\|k\|=0} = 0$. Paper [11] shows the possibility of coexistence of two kinds of Bose condensations equally for models (3.2) and (3.3). In particular the WIBG is close to model (3.2) for $\theta < \theta_0(0)$ in the sense that the Bose gas (3.2) manifests the same three types of thermodynamic behaviour $(i)-(iii)$ as above but there is no limiting temperature $\theta_0(0)$ and no discontinuity of the condensate and the total particle density. The peculiarity of model (3.3) is that under conditions $g_{k\neq 0}(V) \geq g_- > 0$ or $\inf_{\|k\|<\delta_0,V} g_k(V) > 0$ in a band $\delta_0 > 0$, the *direct* repulsion U_Λ (3.4) spreads out the *conventional* Bose–Einstein condensation, originally of the type I in modes $\|k\| = \frac{2\pi}{L}$, into a *conventional* Bose–Einstein condensation of the type III (cf. [10,11]). Notice that the conventional Bose–Einstein condensation persists in the model (3.3) even if for $k \in \Lambda^*\backslash\{0\}$ $g_k(V) = \gamma_k V^{\alpha_k} \overset{V\to+\infty}{\longrightarrow} +\infty$ ($\alpha_k \leq \alpha_+ < 1$) which is similar to the WIBG where in the effective two–bosons *repulsion* for $k, q \neq 0$

$$g_{\Lambda,kq}a_k^*a_{-k}^*a_{-q}a_q,$$

the "form–factor" $g_{\Lambda,kq} > 0$ diverges with volume as $V^{2/3}$, see [17]. However, an important difference is that this *effective* interaction (which is due to non–diagonal term (1.7)) does not able to spread out the Bose–Einstein condensation into the type III as in the model (3.3) for the WIBG: it rests as a condensation of the type I.

For $\theta \geq \theta_0(0)$, there are only *two types* of thermodynamic behaviour: they correspond to the domain $\rho \leq \rho_c^B(\theta)$ where there is no condensation and to $\rho_c^B(\theta) \leq \rho$ where we have a *conventional* condensation as in (*iii*). Hence, for $\theta > \theta_0(0)$ the condensation in the WIBG coincides with the type I generalized Bose–Einstein condensation in the PBG with *excluded* mode $k = 0$, see (*iii*) in Theorem 8 and [18].

Notice that one of the possibility to correct the instabilities of the WIBG for $\mu > 0$ (originally discovered in [21]) would be to add to H_Λ^B (1.4) the "forward–scattering" repulsive interaction between particles next to the zero–mode $k = 0$:

$$H_\Lambda = H_\Lambda^B + \frac{v(0)}{2V} \sum_{k,q \in \Lambda^* \setminus \{0\}} a_k^* a_q^* a_q a_k. \qquad (3.5)$$

Paper [21] proposes to use the superstable Hamiltonian (3.5) to extract the gapless spectrum by doing the Bogoliubov approximation (see Definition 1) only in the operator $H_\Lambda^B - v(0) a_0^{*^2} a_0^2 / 2V$ (see also [22,23]). In fact the problem of the thermodynamics and the gapless spectrum for stabilized WIBG models is rather delicate, see discussions in [21–23]. The reason is that the interaction in the WIBG is in fact a long–range one, which implies the appearance of the gap when one has the non–conventional condensation in the zero–mode, see [5].

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IV. APPENDIX A

Lemma 10.

Let $\|k\| > 2\pi/L$. Then for the Gibbs state $\omega_{\Lambda,\rho}^B(-)$ we have:

$$\omega_{\Lambda,\rho}^B(N_k) \leq \frac{1}{e^{B_k(\mu_\Lambda^B(\theta,\rho))} - 1} + \beta \frac{v(k)}{2V} \frac{\omega_{\Lambda,\rho}^B(a_0^* a_0)}{1 - e^{-B_k(\mu_\Lambda^B(\theta,\rho))}}, \qquad (4.1)$$

with

$$B_k(\mu = \mu_\Lambda^B(\theta,\rho)) \equiv \beta \left[\varepsilon_k - \mu_\Lambda^B(\theta,\rho) - \frac{v(k)}{2V} \right]. \qquad (4.2)$$

Proof. By the correlation inequalities for the Gibbs state $\omega_\Lambda^B(-) \equiv \langle - \rangle_{H_\Lambda^B}(\beta,\mu)$ (see [19,20]):

$$\beta \omega_\Lambda^B\left(X^* \left[H_\Lambda^B(\mu), X\right]\right) \geq \omega_\Lambda^B(X^* X) \ln \frac{\omega_\Lambda^B(X^* X)}{\omega_\Lambda^B(X X^*)}, \qquad (4.3)$$

where X is an observable from the domain of the commutator $\left[H_\Lambda^B(\mu), .\right]$, we deduce

$$\beta \omega_\Lambda^B\left(a_k^* \left[H_\Lambda^B(\mu), a_k\right]\right) \geq \omega_\Lambda^B(N_k) \ln \frac{\omega_\Lambda^B(N_k)}{\omega_\Lambda^B(N_k) + 1}, \qquad (4.4)$$

for $X = a_k$. Since for $\|k\| > 2\pi/L$

$$\left[H_\Lambda^B(\mu), a_k\right] = -\left(\varepsilon_k - \mu - [v(0) + v(k)] \frac{a_0^* a_0}{V}\right) a_k - \frac{v(k)}{V} a_0^2 a_{-k}^*,$$

one gets for $\mu = \mu_\Lambda^B(\theta,\rho)$ that

$$\begin{aligned} \omega_{\Lambda,\rho}^B\left(a_k^* \left[H_\Lambda^B\left(\mu_\Lambda^B(\theta,\rho)\right), a_k\right]\right) &= -\left[\varepsilon_k - \mu_\Lambda^B(\theta,\rho)\right] \omega_{\Lambda,\rho}^B(N_k) - [v(0) + v(k)] \frac{\omega_{\Lambda,\rho}^B(a_0^* a_0 N_k)}{V} \\ &- v(k) \frac{\omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right)}{V} \end{aligned} \qquad (4.5)$$

Notice that $\omega_{\Lambda,\rho}^B\left(a_k^* \left[H_\Lambda^B\left(\mu_\Lambda^B(\theta,\rho)\right), a_k\right]\right) \in \mathbb{R}$, then by (4.5) $\omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right) \in \mathbb{R}$. Therefore,

$$2\omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right) = \omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right) + \omega_{\Lambda,\rho}^B\left(a_k a_{-k} a_0^{*2}\right). \qquad (4.6)$$

Moreover, since the functions ε_k and $v(k)$ are even, we have

$$\omega^B_{\Lambda,\rho}\left(a^*_0 a_0 N_k\right) = \omega^B_{\Lambda,\rho}\left(a^*_0 a_0 N_{-k}\right). \tag{4.7}$$

Thus (4.5)–(4.7) imply

$$\begin{aligned}\omega^B_{\Lambda,\rho}\left(a^*_k\left[H^B_\Lambda\left(\mu^B_\Lambda(\theta,\rho)\right),a_k\right]\right) &= -\left[\varepsilon_k-\mu^B_\Lambda(\theta,\rho)\right]\omega^B_{\Lambda,\rho}\left(a^*_k a_k\right)-\frac{v(k)}{2V}\omega^B_{\Lambda,\rho}\left(a^2_0 a^*_k a^*_{-k}+a^{*2}_0 a_k a_{-k}\right)\\ &-\frac{[v(0)+v(k)]}{2V}\omega^B_{\Lambda,\rho}\left(a^*_0 a_0 N_k + a^*_0 a_0 N_{-k}\right).\end{aligned} \tag{4.8}$$

Now applying the identity

$$a^2_0 a^*_k a^*_{-k} + a^{*2}_0 a_k a_{-k} + a^*_0 a_0 a^*_k a_k + a^*_0 a_0 a^*_{-k} a_{-k} = \left(a^*_0 a_k + a^*_{-k} a_0\right)^*\left(a^*_0 a_k + a^*_{-k} a_0\right) - a^*_k a_k - a^*_0 a_0, \tag{4.9}$$

we deduce from (4.8) the estimate:

$$\omega^B_{\Lambda,\rho}\left(a^*_k\left[H^B_\Lambda\left(\mu^B_\Lambda(\theta,\rho)\right),a_k\right]\right) \le -\left[\varepsilon_k-\mu^B_\Lambda(\theta,\rho)-\frac{v(k)}{2V}\right]\omega^B_{\Lambda,\rho}(N_k)+\frac{v(k)}{2V}\omega^B_{\Lambda,\rho}\left(a^*_0 a_0\right). \tag{4.10}$$

Therefore, combining (4.4) with (4.10) we find that:

$$B_k\left(\mu^B_\Lambda(\theta,\rho)\right)\omega^B_{\Lambda,\rho}(N_k)-\beta\frac{v(k)}{2V}\omega^B_{\Lambda,\rho}\left(a^*_0 a_0\right)\le\omega^B_{\Lambda,\rho}(N_k)\ln\frac{\omega^B_{\Lambda,\rho}(N_k)+1}{\omega^B_{\Lambda,\rho}(N_k)}, \tag{4.11}$$

with $B_k\left(\mu^B_\Lambda(\theta,\rho)\right)$ defined by (4.2). Notice that, since

$$\mu^B_\Lambda(\theta,\rho) < \varepsilon_{\Lambda,1} < \widehat{\varepsilon}_{\Lambda,1} = \inf_{k\neq 0}\varepsilon_k$$

and $\|k\| > 2\pi/L$, one has $B_k\left(\mu^B_\Lambda(\theta,\rho)\right) > 0$. Hence we have to solve the inequality

$$B_k\left(\mu^B_\Lambda(\theta,\rho)\right)x-\beta\frac{v(k)}{2V}\omega^B_{\Lambda,\rho}\left(a^*_0 a_0\right)\le x\ln\frac{x+1}{x}, \tag{4.12}$$

for $x=\omega^B_{\Lambda,\rho}(N_k)\ge 0$. Notice that the solution of (4.12) is the set $\{0\le x\le x_2\}$ where x_2 is a solution of the equation

$$B_k\left(\mu^B_\Lambda(\theta,\rho)\right)x_2-\beta\frac{v(k)}{2V}\omega^B_{\Lambda,\rho}\left(a^*_0 a_0\right)=x_2\ln\frac{x_2+1}{x_2}.$$

Let

$$x_1=\frac{1}{e^{B_k\left(\mu^B_\Lambda(\theta,\rho)\right)}-1} \tag{4.13}$$

be a nontrivial solution of the equation

$$B_k\left(\mu^B_\Lambda(\theta,\rho)\right)x = x\ln\frac{x+1}{x}.$$

Then the inequality $x\le x_2$ can be rewritten as

$$x\le x_1+(x_2-x_1). \tag{4.14}$$

Since the function $f(x)\equiv x\ln\frac{x+1}{x}$ defined for $x\ge 0$ is concave, we get

$$\frac{f(x_2)-f(x_1)}{f'(x_1)}\le x_2-x_1,$$

from which by (4.13), (4.14) we get (4.1) for $\|k\|>2\pi/L$. ■

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СЛАБОНЕІДЕАЛЬНИЙ БОЗЕ–ГАЗ БОГОЛЮБОВА

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Показано, що конденсація слабонеідеального бозе–газу Боголюбова може мати дві стадії. Якщо взаємодія є такою, що тиск слабонеідеального бозе–газу не збігається з тиском ідеального бозе–газу, то слабонеідеальний бозе–газ може виявляти два типи коденсацій: *незвичну* конденсацію в нульовій моді завдяки взаємодії (у першій стадії) і *звичну* (у загальному типу I) бозе–айнштайнівську конденсацію в модах, суміжних із нульовою, завдяки насиченості густини частинок (у другій стадії).