JORDAN–WIGNER FERMIONIZATION FOR SPIN–$\frac{1}{2}$ SYSTEMS IN TWO DIMENSIONS: A BRIEF REVIEW

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We review the papers on the Jordan–Wigner transformation in two dimensions to comment on a possibility of examining the statistical mechanics properties of two–dimensional spin–$\frac{1}{2}$ models. We discuss in some detail the two–dimensional spin–$\frac{1}{2}$ isotropic XY and Heisenberg models.
Key words: 2D XY model, 2D Heisenberg model, Jordan–Wigner fermionization

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I. INTRODUCTORY REMARKS

A mapping of the spin–$\frac{1}{2}$ operators onto Fermi operators by means of the Jordan–Wigner transformation was used by Lieb, Schultz and Mattis [1] to introduce the exactly solvable one–dimensional spin–$\frac{1}{2}$ XY model. Later the famous Onsager's solution for the two–dimensional Ising model was reproduced by using the Jordan–Wigner transformation for the transfer matrix of that model [2]. The Jordan–Wigner transformation is the essential constituent of the studies of the statistical mechanics properties of quantum spin chains [3]. Much effort has been devoted to generalize the fermionization procedure for two [4–14] and three [15,16] dimensions. In the present paper we review the Jordan–Wigner transformation in two dimensions as well as some existing applications of this mapping for the spin system theory.

We shall consider a spin model consisting of $N = N_xN_y$ ($N_x \to \infty$, $N_y \to \infty$) spins $\frac{1}{2}$ on a square lattice of the size L_xL_y ($L_x \to \infty$, $L_y \to \infty$) governed by the Heisenberg Hamiltonian

$$H = \sum_{\langle\mathbf{i},\mathbf{j}\rangle} J\mathbf{s}_{\mathbf{i}}\cdot\mathbf{s}_{\mathbf{j}} + \sum_{\mathbf{i}} hs^z_{\mathbf{i}}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} J\left(\mathbf{s}_{i,j}\cdot\mathbf{s}_{i+1,j} + \mathbf{s}_{i,j}\cdot\mathbf{s}_{i,j+1}\right)$$
$$+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} hs^z_{i,j}. \tag{1}$$

Here $\langle\mathbf{i},\mathbf{j}\rangle$ denote all different nearest neighbouring sites at the square lattice, J is the exchange interaction between the neighbouring sites, h is the external field. The isotropic Heisenberg interaction consists of the isotropic XY part and the Ising part

$$\mathbf{s}_{\mathbf{i}}\cdot\mathbf{s}_{\mathbf{j}} = \left(s^x_{\mathbf{i}}s^x_{\mathbf{j}} + s^y_{\mathbf{i}}s^y_{\mathbf{j}}\right) + s^z_{\mathbf{i}}s^z_{\mathbf{j}}$$
$$= \frac{1}{2}\left(s^+_{\mathbf{i}}s^-_{\mathbf{j}} + s^-_{\mathbf{i}}s^+_{\mathbf{j}}\right)$$
$$+ \left(s^+_{\mathbf{i}}s^-_{\mathbf{i}} - \frac{1}{2}\right)\left(s^+_{\mathbf{j}}s^-_{\mathbf{j}} - \frac{1}{2}\right), \tag{2}$$

where we have introduced the spin raising and lowering operators $s^{\pm} = s^x \pm \mathrm{i}s^y$ and $s^z = s^+s^- - \frac{1}{2}$. It is important to note that the operators s^+, s^- obey the Fermi commutation rules at the same site

$$\left\{s^-_{\mathbf{i}}, s^+_{\mathbf{i}}\right\} = 1, \quad \left\{s^+_{\mathbf{i}}, s^+_{\mathbf{i}}\right\} = \left\{s^-_{\mathbf{i}}, s^-_{\mathbf{i}}\right\} = 0 \tag{3}$$

and the Bose commutation rules at different sites $\mathbf{i} \neq \mathbf{j}$

$$\left[s^-_{\mathbf{i}}, s^+_{\mathbf{j}}\right] = \left[s^+_{\mathbf{i}}, s^+_{\mathbf{j}}\right] = \left[s^-_{\mathbf{i}}, s^-_{\mathbf{j}}\right] = 0. \tag{4}$$

The aim of the Jordan–Wigner trick is to transform the spin variables into pure fermion variables.

We start the paper by giving a short reminder of the Jordan–Wigner transformation in one dimension (Section II). Then we discuss the extensions for two dimensions suggested by M. Azzouz [8] (Section III), Y. R. Wang [5] (Section IV), and E. Fradkin [4] (Section V). The mean–field–like treatment of the Jordan–Wigner fermions for the isotropic XY model is discussed in detail in Section VI. The consideration of a model with Ising term in fermionic language is given separately in Section VII. Finally, we summarize some of the results obtained using this approximate approach and comment on a comparison with the results derived using other methods (Section VIII).

II. THE JORDAN–WIGNER TRANSFORMATION IN ONE DIMENSION

With the help of the Jordan–Wigner transformation we introduce instead of the operators s^+, s^- satisfying (3), (4) the operators c^+, c satisfying the Fermi commutation rules (both at the same and different sites) in terms of which the Hamiltonian of the isotropic XY chain is a bilinear form and the Ising interaction yields the products of four Fermi operators. Explicitly the Jordan–Wigner transformation reads

$$c_n^+ = \mathrm{e}^{\mathrm{i}\alpha_n} s_n^+, \quad c_n = \mathrm{e}^{-\mathrm{i}\alpha_n} s_n^-,$$
$$s_n^+ = \mathrm{e}^{-\mathrm{i}\alpha_n} c_n^+, \quad s_n^- = \mathrm{e}^{\mathrm{i}\alpha_n} c_n,$$
$$\alpha_n = \pi \sum_{j=0}^{n-1} n_j, \quad n_j = c_j^+ c_j. \tag{5}$$

The signs in the exponents and the order of the multipliers (first and second lines in (5)) are not important. To get the operator c_n (c_n^+) we must multiply s_n^- (s_n^+) by the exponent containing a sum of n_j at all the previous sites $0 \le j \le n-1$ as can be seen from (5) and is shown symbolically in Fig. 1.

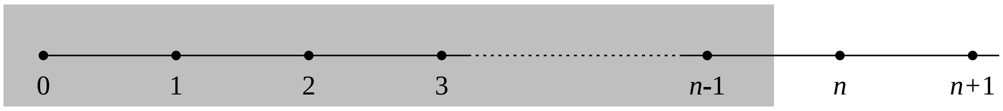

Fig. 1. Towards the Jordan–Wigner transformation in one dimension.

To demonstrate how transformation (5) works we need the following equations for Fermi operators

$$c_j^+ \mathrm{e}^{\mathrm{i}\pi c_j^+ c_j} = c_j^+, \quad \mathrm{e}^{\mathrm{i}\pi c_j^+ c_j} c_j^+ = -c_j^+,$$
$$c_j \mathrm{e}^{\mathrm{i}\pi c_j^+ c_j} = -c_j, \quad \mathrm{e}^{\mathrm{i}\pi c_j^+ c_j} c_j = c_j, \tag{6}$$

which can be easily obtained since

$$\mathrm{e}^{\mathrm{i}\pi c_j^+ c_j} = 1 + \mathrm{i}\pi c_j^+ c_j + \frac{1}{2!}(\mathrm{i}\pi)^2 c_j^+ c_j + \ldots$$
$$= 1 - c_j^+ c_j + \mathrm{e}^{\mathrm{i}\pi} c_j^+ c_j = 1 - 2c_j^+ c_j.$$

From (6) one finds that

$$\mathrm{e}^{\mathrm{i}\pi c_n^+ c_n} c_l = c_l \mathrm{e}^{\mathrm{i}\pi c_n^+ c_n},$$
$$\mathrm{e}^{\mathrm{i}\pi c_n^+ c_n} c_l^+ = c_l^+ \mathrm{e}^{\mathrm{i}\pi c_n^+ c_n}, \tag{7}$$

if $n \neq l$, but

$$\mathrm{e}^{\mathrm{i}\pi c_l^+ c_l} c_l = -c_l \mathrm{e}^{\mathrm{i}\pi c_l^+ c_l},$$
$$\mathrm{e}^{\mathrm{i}\pi c_l^+ c_l} c_l^+ = -c_l^+ \mathrm{e}^{\mathrm{i}\pi c_l^+ c_l}. \tag{8}$$

Besides,

$$\mathrm{e}^{2\mathrm{i}\pi c_l^+ c_l} = 1. \tag{9}$$

Let us show that the commutation rules for the spin raising and lowering operators s^+, s^- (5) are given by (3), (4) if c^+, c are Fermi operators. At the same site we find

$$s_n^- s_n^+ = c_n \mathrm{e}^{\mathrm{i}\alpha_n} \mathrm{e}^{-\mathrm{i}\alpha_n} c_n^+ = c_n c_n^+, \quad s_n^+ s_n^- = c_n^+ c_n,$$
$$s_n^+ s_n^+ = c_n^+ c_n^+ = 0, \quad s_n^- s_n^- = c_n c_n = 0, \tag{10}$$

and as a result Eq. (3) becomes evident. At different sites n and $n+m$ (without any loss of generality $m > 0$) we have

$$s_n^- s_{n+m}^+ = c_n \mathrm{e}^{\mathrm{i}\pi \sum_{l=n}^{n+m-1} c_l^+ c_l} c_{n+m}^+$$
$$= c_n c_{n+m}^+ \mathrm{e}^{\mathrm{i}\pi \sum_{l=n}^{n+m-1} c_l^+ c_l},$$
$$s_{n+m}^+ s_n^- = c_{n+m}^+ \mathrm{e}^{\mathrm{i}\pi \sum_{l=n}^{n+m-1} c_l^+ c_l} c_n$$
$$= -c_{n+m}^+ c_n \mathrm{e}^{\mathrm{i}\pi \sum_{l=n}^{n+m-1} c_l^+ c_l}, \tag{11}$$

etc. which immediately yields Eq. (4).

Let us write down the transformed Hamiltonian. Besides $s_j^+ s_j^- = c_j^+ c_j$ (10) we have

$$s_j^+ s_{j+1}^- = c_j^+ \mathrm{e}^{\mathrm{i}\pi c_j^+ c_j} c_{j+1} = c_j^+ c_{j+1},$$

$$s_j^- s_{j+1}^+ = c_j \mathrm{e}^{\mathrm{i}\pi c_j^+ c_j} c_{j+1}^+ = -c_j c_{j+1}^+, \tag{12}$$

and as a result the one–dimensional spin–$\frac{1}{2}$ Heisenberg Hamiltonian (1), (2) becomes

$$H = \sum_j \left(\frac{1}{2} J \left(c_j^+ c_{j+1} - c_j c_{j+1}^+ \right) \right.$$
$$\left. + J \left(c_j^+ c_j - \frac{1}{2} \right) \left(c_{j+1}^+ c_{j+1} - \frac{1}{2} \right) \right)$$
$$+ \sum_j h \left(c_j^+ c_j - \frac{1}{2} \right). \tag{13}$$

It is clear now to what extent the Jordan–Wigner transformation simplifies further statistical mechanics calculations. A nontrivial in spin language isotropic XY chain transforms into tight–binding spinless fermions and further rigorous treatment becomes possible. For the anisotropic XY chain the operators $c_j^+ c_{j+1}^+$, $c_j c_{j+1}$ enter Eq. (13) besides, and therefore the Bogolyubov transformation is required in addition. The Ising term leads to an interaction between the Jordan–Wigner fermions, however, the low–energy properties may be analysed using the bosonization techniques [3]. While interesting only in the low–energy physics of $s > \frac{1}{2}$ spin chains one may represent the spin–s operators as a sum of $2s$ spin–$\frac{1}{2}$ operators and then, making use of the Jordan–Wigner representation for the latter operators, proceed in fermionic language.

III. THE JORDAN–WIGNER TRANSFORMATION IN TWO DIMENSIONS (M. AZZOUZ, 1993)

Consider the spin model (1) on a square lattice (Fig. 2). Two coordinates i and j are taken at the x and y axes, respectively, to specify a given site.

M. Azzouz defined [8] the extended Jordan–Wigner transformation as

$$s_{i,j}^- = c_{i,j} \mathrm{e}^{\mathrm{i}\alpha_{i,j}} = \mathrm{e}^{\mathrm{i}\alpha_{i,j}} c_{i,j},$$
$$s_{i,j}^+ = c_{i,j}^+ \mathrm{e}^{-\mathrm{i}\alpha_{i,j}} = \mathrm{e}^{-\mathrm{i}\alpha_{i,j}} c_{i,j}^+,$$
$$\alpha_{i,j} = \pi \left(\sum_{d=0}^{i-1} \sum_{f=0}^{\infty} n_{d,f} + \sum_{f=0}^{j-1} n_{i,f} \right),$$
$$n_{d,f} = c_{d,f}^+ c_{d,f} \tag{14}$$

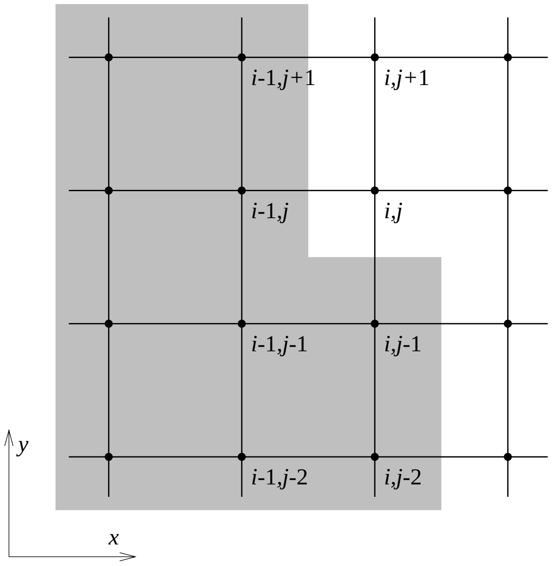

Fig. 2. Towards the Jordan–Wigner transformation in two dimensions [8].

(compare with Eqs. (5)). The signs in the exponents in (14) (and the order of the multipliers in the first and second lines in (14)) are not important. Let us show that the introduced transformation (14) enables one to construct a fermion representation for two–dimensional spin–$\frac{1}{2}$ models.

We start with the commutation rules. At the same site one has

$$s_{q,p}^- s_{q,p}^+ = c_{q,p} \mathrm{e}^{\mathrm{i}\alpha_{q,p}} \mathrm{e}^{-\mathrm{i}\alpha_{q,p}} c_{q,p}^+ = c_{q,p} c_{q,p}^+,$$
$$s_{q,p}^+ s_{q,p}^- = c_{q,p}^+ c_{q,p},$$
$$s_{q,p}^+ s_{q,p}^+ = c_{q,p}^+ c_{q,p}^+, \quad s_{q,p}^- s_{q,p}^- = c_{q,p} c_{q,p}, \tag{15}$$

and the Fermi type commutation rules remain unchanged. To illustrate how transformation (14) works in the case of different sites let us consider, for example, two sites q, p and $q, p+m$, $m > 0$. Then, similarly to Eq. (11) one finds

$$s_{q,p}^- s_{q,p+m}^+ = c_{q,p} \mathrm{e}^{\mathrm{i}\pi \sum_{f=p}^{p+m-1} n_{q,f}} c_{q,p+m}^+$$
$$= c_{q,p} c_{q,p+m}^+ \mathrm{e}^{\mathrm{i}\pi \sum_{f=p}^{p+m-1} n_{q,f}},$$
$$s_{q,p+m}^+ s_{q,p}^- = c_{q,p+m}^+ \mathrm{e}^{\mathrm{i}\pi \sum_{f=p}^{p+m-1} n_{q,f}} c_{q,p}$$
$$= -c_{q,p+m}^+ c_{q,p} \mathrm{e}^{\mathrm{i}\pi \sum_{f=p}^{p+m-1} n_{q,f}} \tag{16}$$

(we have used an analogue of Eqs. (7), (8) in two dimensions) and hence the operators $s_{q,p}^-$ and $s_{q,p+m}^+$ commute if the operators $c_{q,p}$ and $c_{q,p+m}^+$ anticommute. Following the same reasoning one can check the rest of commutation rules.

Consider further the transformed spin Hamiltonian. Let us treat somewhat more general nearest neighbour interactions than those in Eq. (1). Namely, we assume different values of interaction in different directions on a square lattice as shown in Fig. 3. (Evidently, we can perform the fermionization presented below for a completely nonuniform model characterized by a set of intersite interactions $\{\ldots, J_{i,j;i+1,j}, \ldots; \ldots, J_{i,j;i,j+1}, \ldots\}$.) If $J = J' = J_\perp = J'_\perp$ one faces the uniform square lattice. To examine the effects of interchain interactions in quasi–one–dimensional systems one may consider the case $J_\perp$, $J'_\perp \ll J$, J' (with J not equal to J' if the dimerised chain is considered). If $J'_\perp \ll J, J', J_\perp$ one has a model of interacting two–leg ladders. In the limiting case of $J'_\perp = 0$ (the noninteracting two–leg ladders) the model may be reduced to a one–dimensional system with interactions extending over the nearest sites.

We begin with the isotropic XY interaction (see Fig. 4).

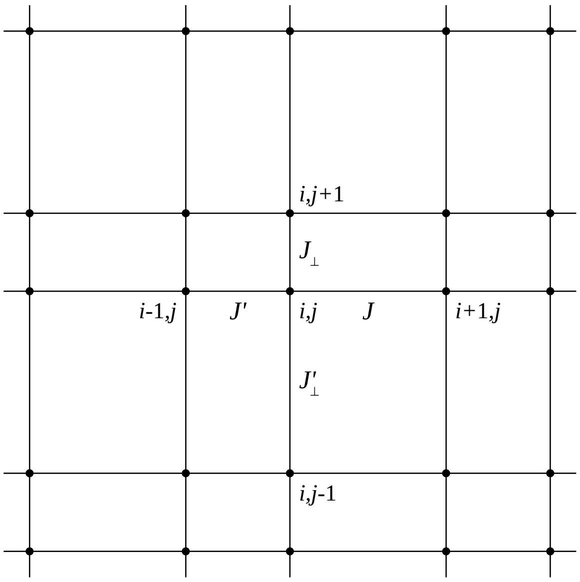

Fig. 3. Nearest neighbour interactions on a square lattice.

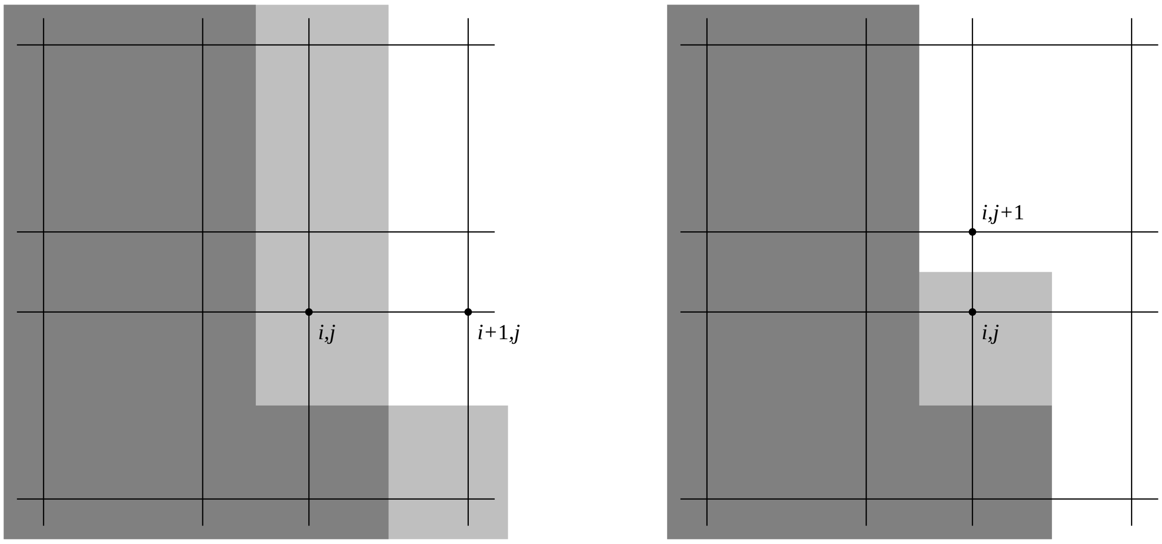

Fig. 4. Towards the fermionization of H_{XY}.

Inserting (14) into (1), (2) one finds

$$
\begin{aligned}
H_{XY} &= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(J_{i,j;i+1,j}\left(s_{i,j}^{-}s_{i+1,j}^{+}+s_{i,j}^{+}s_{i+1,j}^{-}\right)+J_{i,j;i,j+1}\left(s_{i,j}^{-}s_{i,j+1}^{+}+s_{i,j}^{+}s_{i,j+1}^{-}\right)\right)\\
&= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big(J_{i,j;i+1,j}\Big(c_{i,j}\mathrm{e}^{-\mathrm{i}\pi\left(\sum_{f=j}^{\infty}n_{i,f}+\sum_{f=0}^{j-1}n_{i+1,f}\right)}c_{i+1,j}^{+}+c_{i,j}^{+}\mathrm{e}^{\mathrm{i}\pi\left(\sum_{f=j}^{\infty}n_{i,f}+\sum_{f=0}^{j-1}n_{i+1,f}\right)}c_{i+1,j}\Big)\\
&+ J_{i,j;i,j+1}\Big(c_{i,j}\mathrm{e}^{-\mathrm{i}\pi n_{i,j}}c_{i,j+1}^{+}+c_{i,j}^{+}\mathrm{e}^{\mathrm{i}\pi n_{i,j}}c_{i,j+1}\Big)\Big)\\
&= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big(J_{i,j;i+1,j}\Big(-c_{i,j}\mathrm{e}^{-\mathrm{i}\pi\left(\sum_{f=j+1}^{\infty}n_{i,f}+\sum_{f=0}^{j-1}n_{i+1,f}\right)}c_{i+1,j}^{+}+c_{i,j}^{+}\mathrm{e}^{\mathrm{i}\pi\left(\sum_{f=j+1}^{\infty}n_{i,f}+\sum_{f=0}^{j-1}n_{i+1,f}\right)}c_{i+1,j}\Big)\\
&+ J_{i,j;i,j+1}\left(-c_{i,j}c_{i,j+1}^{+}+c_{i,j}^{+}c_{i,j+1}\right)\Big)
\end{aligned}
\tag{17}
$$

(to get the last equality we have used an analogue of Eq. (6) in two dimensions). After introducing the notations

$$
\begin{aligned}
\phi_{i,i+1}(j) &= \pi\left(\sum_{f=j}^{\infty}n_{i,f}+\sum_{f=0}^{j-1}n_{i+1,f}\right),\\
\tilde{\phi}_{i,i+1}(j) &= \pi\left(\sum_{f=j+1}^{\infty}n_{i,f}+\sum_{f=0}^{j-1}n_{i+1,f}\right),\\
\varphi_{j,j+1}(i) &= \pi n_{i,j}
\end{aligned}
\tag{18}
$$

the Hamiltonian (17) becomes as follows

$$
\begin{aligned}
H_{XY} &= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big(J_{i,j;i+1,j}\left(c_{i,j}\mathrm{e}^{-\mathrm{i}\phi_{i,i+1}(j)}c_{i+1,j}^{+}+c_{i,j}^{+}\mathrm{e}^{\mathrm{i}\phi_{i,i+1}(j)}c_{i+1,j}\right)\\
&+ J_{i,j;i,j+1}\left(c_{i,j}\mathrm{e}^{-\mathrm{i}\varphi_{j,j+1}(i)}c_{i,j+1}^{+}+c_{i,j}^{+}\mathrm{e}^{\mathrm{i}\varphi_{j,j+1}(i)}c_{i,j+1}\right)\Big)\\
&= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big(J_{i,j;i+1,j}\left(-c_{i,j}\mathrm{e}^{-\mathrm{i}\tilde{\phi}_{i,i+1}(j)}c_{i+1,j}^{+}+c_{i,j}^{+}\mathrm{e}^{\mathrm{i}\tilde{\phi}_{i,i+1}(j)}c_{i+1,j}\right)\\
&+ J_{i,j;i,j+1}\left(-c_{i,j}c_{i,j+1}^{+}+c_{i,j}^{+}c_{i,j+1}\right)\Big).
\end{aligned}
\tag{19}
$$

Eq. (19) can be viewed as the Hamiltonian of a two–dimensional tight–binding–like spinless fermions with the hopping amplitudes

$$
\mp\frac{1}{2}J_{i,j;i+1,j}\mathrm{e}^{\mp\mathrm{i}\tilde{\phi}_{i,i+1}(j)}
\tag{20}
$$

in the x direction and

$$
\mp\frac{1}{2}J_{i,j;i,j+1}
\tag{21}
$$

in the y direction. Those hoppings depend in a complicated way on a configuration of the 'intermediate' sites. Their complexity explains how the isotropic XY model becomes difficult to examine in two dimensions in comparison with an obvious analysis in one dimension.

There are no difficulties in rewriting the Ising interaction in fermionic language

$$H_Z=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(J_{i,j;i+1,j}\left(s_{i,j}^{+}s_{i,j}^{-}-\frac{1}{2}\right)\left(s_{i+1,j}^{+}s_{i+1,j}^{-}-\frac{1}{2}\right)+J_{i,j;i,j+1}\left(s_{i,j}^{+}s_{i,j}^{-}-\frac{1}{2}\right)\left(s_{i,j+1}^{+}s_{i,j+1}^{-}-\frac{1}{2}\right)\right)$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(J_{i,j;i+1,j}\left(c_{i,j}^{+}c_{i,j}-\frac{1}{2}\right)\left(c_{i+1,j}^{+}c_{i+1,j}-\frac{1}{2}\right)+J_{i,j;i,j+1}\left(c_{i,j}^{+}c_{i,j}-\frac{1}{2}\right)\left(c_{i,j+1}^{+}c_{i,j+1}-\frac{1}{2}\right)\right) \tag{22}$$

and the interaction with an external field

$$H_f=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}h\left(s_{i,j}^{+}s_{i,j}^{-}-\frac{1}{2}\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}h\left(c_{i,j}^{+}c_{i,j}-\frac{1}{2}\right). \tag{23}$$

Formulas (19), (22), (23) realize the fermionic representation of the spin–$\frac{1}{2}$ isotropic Heisenberg model on a square lattice (1), (2).

IV. THE JORDAN–WIGNER TRANSFORMATION IN TWO DIMENSIONS (Y. R. WANG, 1991)

Let us turn back to the Jordan–Wigner transformation in one dimension (5) using this case as a guideline and define a particle–annihilation operator as

$$d_{\mathbf{i}}=\mathrm{e}^{-\mathrm{i}\alpha_{\mathbf{i}}}s_{\mathbf{i}}^{-},$$

$$\alpha_{\mathbf{i}}=\sum_{\mathbf{j}(\neq\mathbf{i})}B_{\mathbf{ij}}n_{\mathbf{j}}, \tag{24}$$

where $B_{\mathbf{ij}}$ is the c–number matrix element, $n_{\mathbf{j}}=d_{\mathbf{j}}^{+}d_{\mathbf{j}}$. A particle–creation operator is given by

$$d_{\mathbf{i}}^{+}=s_{\mathbf{i}}^{+}\mathrm{e}^{\mathrm{i}\alpha_{\mathbf{i}}}=\mathrm{e}^{\mathrm{i}\alpha_{\mathbf{i}}}s_{\mathbf{i}}^{+}, \tag{25}$$

whereas the inverse to Eqs. (24), (25) formulas read

$$s_{\mathbf{i}}^{-}=\mathrm{e}^{\mathrm{i}\alpha_{\mathbf{i}}}d_{\mathbf{i}}=d_{\mathbf{i}}\mathrm{e}^{\mathrm{i}\alpha_{\mathbf{i}}}, \tag{26}$$

$$s_{\mathbf{i}}^{+}=\mathrm{e}^{-\mathrm{i}\alpha_{\mathbf{i}}}d_{\mathbf{i}}^{+}=d_{\mathbf{i}}^{+}\mathrm{e}^{-\mathrm{i}\alpha_{\mathbf{i}}}. \tag{27}$$

We want the introduced operators d^{+}, d to obey the Fermi type commutation rules. They are indeed the Fermi operators at the same site due to Eq. (3). Consider further two different sites $\mathbf{i}\neq\mathbf{j}$. Assuming that d^{+}, d are Fermi operators one finds

$$\left[s_{\mathbf{i}}^{+},s_{\mathbf{j}}^{-}\right]=\mathrm{e}^{-\mathrm{i}\alpha_{\mathbf{i}}}d_{\mathbf{i}}^{+}\mathrm{e}^{\mathrm{i}\alpha_{\mathbf{j}}}d_{\mathbf{j}}-\mathrm{e}^{\mathrm{i}\alpha_{\mathbf{j}}}d_{\mathbf{j}}\mathrm{e}^{-\mathrm{i}\alpha_{\mathbf{i}}}d_{\mathbf{i}}^{+} \tag{28}$$

$$=\mathrm{e}^{-\mathrm{i}n_{\mathbf{j}}B_{\mathbf{ij}}}d_{\mathbf{i}}^{+}\mathrm{e}^{\mathrm{i}n_{\mathbf{i}}B_{\mathbf{ji}}}d_{\mathbf{j}}-\mathrm{e}^{\mathrm{i}n_{\mathbf{i}}B_{\mathbf{ji}}}d_{\mathbf{j}}\mathrm{e}^{-\mathrm{i}n_{\mathbf{j}}B_{\mathbf{ij}}}d_{\mathbf{i}}^{+}.$$

Since

$$\mathrm{e}^{-\mathrm{i}n_{\mathbf{j}}B_{\mathbf{ij}}}=1+\left(\mathrm{e}^{-\mathrm{i}B_{\mathbf{ij}}}-1\right)n_{\mathbf{j}},$$

$$\mathrm{e}^{\mathrm{i}n_{\mathbf{i}}B_{\mathbf{ji}}}=1+\left(\mathrm{e}^{\mathrm{i}B_{\mathbf{ji}}}-1\right)n_{\mathbf{i}}$$

one can easily proceed in the calculation of the r.h.s. of Eq. (28) finally arriving at

$$\left[s_{\mathbf{i}}^{+},s_{\mathbf{j}}^{-}\right]=\left(1+\mathrm{e}^{-\mathrm{i}B_{\mathbf{ij}}}\mathrm{e}^{\mathrm{i}B_{\mathbf{ji}}}\right)d_{\mathbf{i}}^{+}d_{\mathbf{j}}. \tag{29}$$

The result equals 0 (as is the case for the commutator of the operators s^{+} and s^{-} attached to different sites) if

$$\mathrm{e}^{\mathrm{i}B_{\mathbf{ij}}}=-\mathrm{e}^{\mathrm{i}B_{\mathbf{ji}}}. \tag{30}$$

Thus we assume that $B_{\mathbf{ij}}$'s in (24), (25), (26), (27) satisfy relation (30) that yields the spin commutation relations for s^{+}, s (3), (4) if d^{+}, d are the Fermi operators.

Y. R. Wang suggested [5] the following choice for $B_{\mathbf{ij}}$. Consider two complex numbers

$$\tau_{\mathbf{i}}=i_x+\mathrm{i}i_y \tag{31}$$

and

$$\tau_{\mathbf{j}}=j_x+\mathrm{i}j_y, \tag{32}$$

which correspond to the sites $\mathbf{i}=i_x\mathbf{n}_x+i_y\mathbf{n}_y$ and $\mathbf{j}=j_x\mathbf{n}_x+j_y\mathbf{n}_y$, respectively. Here $\mathbf{n}_x$ and $\mathbf{n}_y$ are the unit vectors directed along x and y axes, respectively. Assume that

$$B_{\mathbf{ij}}=\arg\left(\tau_{\mathbf{j}}-\tau_{\mathbf{i}}\right). \tag{33}$$

Evidently

$$\mathrm{e}^{\mathrm{i}B_{\mathbf{ji}}} = \mathrm{e}^{\mathrm{i}\arg(\tau_{\mathbf{i}}-\tau_{\mathbf{j}})} = \mathrm{e}^{\mathrm{i}(\arg(\tau_{\mathbf{j}}-\tau_{\mathbf{i}})\pm\pi)} = -\mathrm{e}^{\mathrm{i}B_{\mathbf{ij}}}, \tag{34}$$

that is the required condition (30). Since $\tau_{\mathbf{i}} - \tau_{\mathbf{j}} = |\tau_{\mathbf{i}} - \tau_{\mathbf{j}}|\mathrm{e}^{\mathrm{i}\arg(\tau_{\mathbf{i}}-\tau_{\mathbf{j}})}$ Eq. (33) can be rewritten in the form

$$B_{\mathbf{ij}} = \mathrm{Im}\ln(\tau_{\mathbf{j}} - \tau_{\mathbf{i}}), \tag{35}$$

and hence the introduced transformations (24), (25), (26), (27) contain

$$\alpha_{\mathbf{i}} = \sum_{\mathbf{j}(\neq\mathbf{i})} \mathrm{Im}\ln(\tau_{\mathbf{j}} - \tau_{\mathbf{i}})\, n_{\mathbf{j}}. \tag{36}$$

It is worth noting that the transformation of M. Azzouz (14) can also be written as Eqs. (26), (27) with

$$B_{\mathbf{ij}} = \pi\left(\Theta(i_x - j_x)(1 - \delta_{i_x,j_x}) + \delta_{i_x,j_x}\Theta(i_y - j_y)(1 - \delta_{i_y,j_y})\right) \tag{37}$$

where $\Theta(x)$ is the step function, and a necessary condition for having spin to fermion mapping (30) fulfilled.

The advantage of the choice of Y. R. Wang (36) becomes clear when one tries to introduce an approximate treatment of the transformed Hamiltonian. After inserting (26), (27) into (1), (2) one finds

$$H = \sum_{\langle\mathbf{i},\mathbf{j}\rangle}\left(\frac{1}{2}J_{\mathbf{i},\mathbf{j}}\left(d_{\mathbf{i}}^{+}\mathrm{e}^{\mathrm{i}(\alpha_{\mathbf{j}}-\alpha_{\mathbf{i}})}d_{\mathbf{j}} + d_{\mathbf{i}}\mathrm{e}^{\mathrm{i}(\alpha_{\mathbf{i}}-\alpha_{\mathbf{j}})}d_{\mathbf{j}}^{+}\right) + J_{\mathbf{i},\mathbf{j}}\left(d_{\mathbf{i}}^{+}d_{\mathbf{i}} - \frac{1}{2}\right)\left(d_{\mathbf{j}}^{+}d_{\mathbf{j}} - \frac{1}{2}\right)\right) + \sum_{\mathbf{i}} h\left(d_{\mathbf{i}}^{+}d_{\mathbf{i}} - \frac{1}{2}\right). \tag{38}$$

Note that

$$\alpha_{\mathbf{j}} - \alpha_{\mathbf{i}} = \int_{\mathbf{i}}^{\mathbf{j}} \mathrm{d}\mathbf{r}\cdot\mathbf{A}(\mathbf{r}) \tag{39}$$

where $\mathbf{A}(\mathbf{r}) = \nabla_{\mathbf{r}}\alpha_{\mathbf{r}}$ and hence

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\partial}{\partial r_x}\left(\sum_{\mathbf{r}'(\neq\mathbf{r})} n_{\mathbf{r}'}\mathrm{Im}\ln\left(r'_x - r_x + \mathrm{i}\left(r'_y - r_y\right)\right)\right)\mathbf{n}_x + \frac{\partial}{\partial r_y}\left(\sum_{\mathbf{r}'(\neq\mathbf{r})} n_{\mathbf{r}'}\mathrm{Im}\ln\left(r'_x - r_x + \mathrm{i}\left(r'_y - r_y\right)\right)\right)\mathbf{n}_y \\ &= \sum_{\mathbf{r}'(\neq\mathbf{r})} n_{\mathbf{r}'}\frac{\left(r'_y - r_y\right)\mathbf{n}_x - \left(r'_x - r_x\right)\mathbf{n}_y}{\left(\mathbf{r}' - \mathbf{r}\right)^2} \\ &= -\sum_{\mathbf{r}'(\neq\mathbf{r})} n_{\mathbf{r}'}\frac{\mathbf{n}_z\times\left(\mathbf{r}' - \mathbf{r}\right)}{\left(\mathbf{r}' - \mathbf{r}\right)^2}.\end{aligned} \tag{40}$$

The crucial approximation to proceed is to make a change in (40)

$$n_{\mathbf{r}} \to \langle n_{\mathbf{r}}\rangle = \langle s_{\mathbf{r}}^z\rangle + \frac{1}{2} \to \frac{1}{2}. \tag{41}$$

Here the angular brackets denote the thermodynamical canonical average with the Hamiltonian (1). Thus, we have postulated the mean–field description of the phase factors in (38). A similar treatment was adopted in Refs. [17,18]. Apparently, assuming further in (41) that $\langle s_{\mathbf{r}}^z\rangle = 0$ one should suppose that $h = 0$ (although in Refs. [10,11] the uniform magnetic field was included into the Hamiltonian). In principle Eq. (41) simplifies the problem drastically since one faces a tight–binding spinless fermions on a square lattice. However, in practice it is hard to proceed because of nonuniformity of that model.

The Hamiltonian (38), (39), (40), (41) describes the (charged) spinless fermions moving in a plane in an external uniform (classical) magnetic field which is perpendicular to the plane. Due to (41) $\langle n_{\mathbf{r}'}\rangle$ can be taken out from the summation and in the continuum limit the vector potential of the field $\mathbf{A}(\mathbf{r})$ can be written [7] in the following form

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= -\langle n_{\mathbf{r}'}\rangle\sum_{\mathbf{r}'(\neq\mathbf{r})}\frac{\mathbf{n}_z\times\left(\mathbf{r}' - \mathbf{r}\right)}{\left(\mathbf{r}' - \mathbf{r}\right)^2} \\ &= -\langle n_{\mathbf{r}'}\rangle\frac{1}{S_0}\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}}\mathrm{d}r'_x\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}}\mathrm{d}r'_y\,\frac{\mathbf{n}_z\times\left(\mathbf{r}' - \mathbf{r}\right)}{\left(\mathbf{r}' - \mathbf{r}\right)^2} \\ &= \ldots = \langle n_{\mathbf{r}'}\rangle\frac{\pi}{S_0}\mathbf{n}_z\times\mathbf{r},\end{aligned} \tag{42}$$

where S_0 is the area of the elementary plaquette in the plane (see Fig. 5) and $L_x = L_y = L \to \infty$. The corresponding magnetic field $\mathbf{H}(\mathbf{r})$ immediately follows from Eq. (42)

$$\begin{aligned}\mathbf{H}(\mathbf{r}) &= \mathrm{rot}\mathbf{A}(\mathbf{r}) = \langle n_{\mathbf{r}'}\rangle\frac{\pi}{S_0}\begin{vmatrix}\mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -r_y & r_x & 0\end{vmatrix} \\ &= \langle n_{\mathbf{r}'}\rangle\frac{2\pi}{S_0}\mathbf{n}_z,\end{aligned} \tag{43}$$

so that $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{H}(\mathbf{r})\times\mathbf{r}$. The flux per elementary plaquette equals

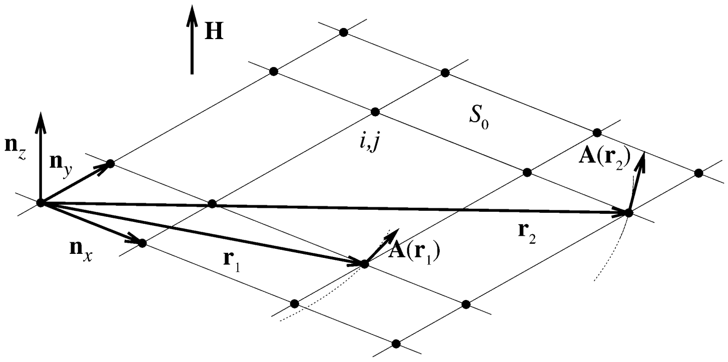

Fig. 5. Fermions in the magnetic field, which appears within the mean–field treatment of the phase factors in the Hamiltonian (38).

$$\Phi_0 = \mathbf{H}(\mathbf{r}) \cdot S_0 \mathbf{n}_z = 2\pi \langle n_{\mathbf{r}'} \rangle = \pi. \tag{44}$$

The vector potential $\mathbf{A}(\mathbf{r})$ (42) shown schematically in Fig. 5 is not convenient to be concerned with. Because of the gauge invariance one may perform a gauge transformation introducing a new vector potential $\tilde{\mathbf{A}}(\mathbf{r})$ which yields the same flux per elementary plaquette $\Phi_0 = \pi$. Namely, assume that $\tilde{\mathbf{A}}(\mathbf{r})$ is such that

$$\alpha_{i+1,j} - \alpha_{i,j} = \int_{i,j}^{i+1,j} \mathrm{d}\mathbf{r} \cdot \tilde{\mathbf{A}}(\mathbf{r}) = \pi, \tag{45}$$

whereas

$$\begin{aligned}\alpha_{i+1,j+1} - \alpha_{i+1,j} &= \alpha_{i,j+1} - \alpha_{i+1,j+1} \\ &= \alpha_{i,j} - \alpha_{i,j+1} = 0\end{aligned} \tag{46}$$

(see Fig. 6). From Eqs. (45), (46) one finds that

$$\oint \mathrm{d}\mathbf{r} \cdot \tilde{\mathbf{A}}(\mathbf{r}) = \pi, \tag{47}$$

and on the other hand

$$\begin{aligned}\oint \mathrm{d}\mathbf{r} \cdot \tilde{\mathbf{A}}(\mathbf{r}) &= \int \mathrm{d}\mathbf{S} \cdot \mathrm{rot} \tilde{\mathbf{A}}(\mathbf{r}) \\ &= \int \mathrm{d}\mathbf{S} \cdot \mathbf{H}(\mathbf{r}) = \Phi_0,\end{aligned} \tag{48}$$

and hence the flux per elementary plaquette Φ_0 remains equal to π.

Let us turn back to the Hamiltonian (38). Only now we are in the position to proceed with the statistical mechanics analysis. Within the frames of the introduced mean–field treatment of the phase factors (45), (46) the Hamiltonian (38) can be rewritten as

$$H = \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \frac{1}{2} J_{\mathbf{i},\mathbf{j}} \left(d_{\mathbf{i}}^{+} d_{\mathbf{j}} - d_{\mathbf{i}} d_{\mathbf{j}}^{+} \right) + H_Z + H_f, \tag{49}$$

where even for initially uniform lattice $J = J' = J_\perp = J'_\perp$ in the Hamiltonian H_{XY} one has

$$\begin{aligned}&J_{i,j;i+1,j} = -J, \quad J_{i,j;i,j+1} = J, \\ &J_{i+1,j;i+2,j} = J, \quad J_{i+1,j;i+1,j+1} = J\end{aligned} \tag{50}$$

etc. (see Fig. 6).

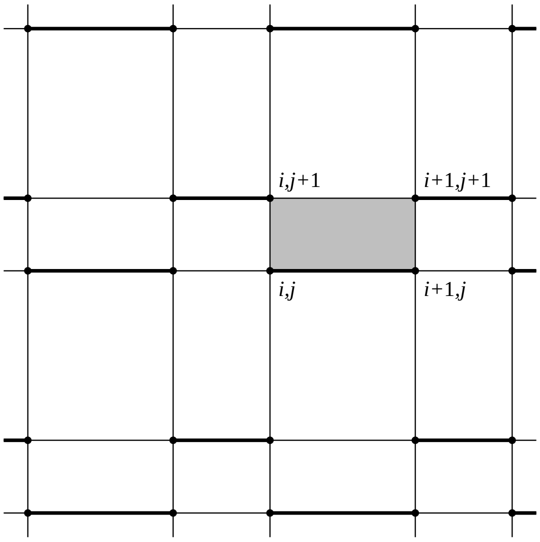

Fig. 6. Towards Eqs. (45), (46) and Eqs. (49), (50).

As can be seen from Eqs. (49), (50) the isotropic XY model can be examined now without making any other additional approximations since it corresponds to a model of tight–binding spinless fermions on a bipartite square lattice (see Section VI). Conversely, the Heisenberg model requires further approximations to proceed because of the interaction between spinless fermions (see Section VII).

V. THE JORDAN–WIGNER TRANSFORMATION IN TWO DIMENSIONS (E. FRADKIN, 1989)

The Fermi–Bose correspondence in two dimensions, i.e., the Jordan–Wigner transformation for two–dimensional spin–$\frac{1}{2}$ systems on a lattice, was discussed even earlier [4], however, without applications to the theory of concrete spin models. Consider a system of spinless fermions, i.e., a matter field, $a(\mathbf{r})$ on the sites of a square lattice (Fig. 7)

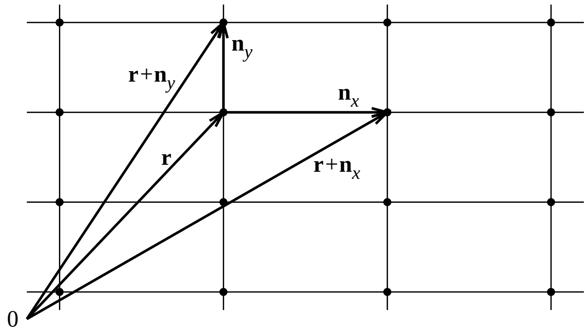

Fig. 7. Towards the Hamiltonian (51).

and gauge field $A_x(\mathbf{r})$, $A_y(\mathbf{r})$ on the links of the lattice (Fig. 7). The Hamiltonian of the system is

$$H = t\sum_{\mathbf{r}} \Big(a^+(\mathbf{r}) \mathrm{e}^{\mathrm{i}A_x(\mathbf{r})} a(\mathbf{r}+\mathbf{n}_x) + a^+(\mathbf{r}) \mathrm{e}^{\mathrm{i}A_y(\mathbf{r})} a(\mathbf{r}+\mathbf{n}_y) + \mathrm{h.c.}\Big) \tag{51}$$

with the constraint

$$a^+(\mathbf{r})a(\mathbf{r}) = \theta\left(A_y(\mathbf{r}+\mathbf{n}_x) - A_y(\mathbf{r}) - A_x(\mathbf{r}+\mathbf{n}_y) + A_x(\mathbf{r})\right), \tag{52}$$

where θ is the parameter which will be defined later. E. Fradkin showed [4] that the gauge field can be eliminated at the expense of a change in the commutation relations of the matter field. Namely, introducing [4] the Jordan–Wigner operators $\tilde{a}(\mathbf{r})$, $\tilde{a}^+(\mathbf{r})$ which obey

$$\tilde{a}(\mathbf{r}')\tilde{a}^+(\mathbf{r}) = \delta_{\mathbf{r}',\mathbf{r}} - \mathrm{e}^{\frac{\mathrm{i}}{2\theta}}\tilde{a}^+(\mathbf{r})\tilde{a}(\mathbf{r}') \tag{53}$$

with $\frac{1}{2\theta} = \pi$ the Hamiltonian (51) is

$$H = t\sum_{\mathbf{r}} \big(\tilde{a}^+(\mathbf{r})\tilde{a}(\mathbf{r}+\mathbf{n}_x) + \tilde{a}^+(\mathbf{r})\tilde{a}(\mathbf{r}+\mathbf{n}_y) + \mathrm{h.c.}\big). \tag{54}$$

Eq. (54) is easily recognized as the Hamiltonian of the spin–$\frac{1}{2}$ isotropic XY model on a square lattice with exchange interaction between the nearest sites $J = 2t$ and the correspondences

$$s_{\mathbf{r}}^+ = \tilde{a}^+(\mathbf{r}), \quad s_{\mathbf{r}}^- = \tilde{a}(\mathbf{r}), \quad s_{\mathbf{r}}^z = \tilde{a}^+(\mathbf{r})\tilde{a}(\mathbf{r}) - \frac{1}{2}.$$

Further discussions on the extension of the Jordan–Wigner transformation to three– or more–dimensional cases can be found in Ref. [15].

VI. 2D $S = \frac{1}{2}$ ISOTROPIC XY MODEL

Let us show how the two–dimensional Jordan–Wigner transformation with the mean–field treatment of the phase factors can be used in the theory of spin models. As a result we come to an approximate approach for a study of 2D quantum spin models.

We begin with the 2D isotropic XY model ($H_Z = H_f = 0$) considering for concreteness the case of $J = J'$, $J_\perp = J'_\perp$. In accordance with (49), (50) we start from

$$\begin{aligned} H_{XY} &= \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big(J\,(-1)^{i+j}\left(d_{i,j}^+ d_{i+1,j} - d_{i,j}d_{i+1,j}^+\right) + J_\perp\left(d_{i,j}^+ d_{i,j+1} - d_{i,j}d_{i,j+1}^+\right)\Big) \\ &= \frac{1}{2}J\left(\ldots - a_{i,j}^+ b_{i+1,j} + a_{i,j}b_{i+1,j}^+ + b_{i+1,j}^+ a_{i+2,j} - b_{i+1,j}a_{i+2,j}^+ + \ldots\right) \\ &+ \frac{1}{2}J_\perp\left(\ldots + a_{i,j}^+ b_{i,j+1} - a_{i,j}b_{i,j+1}^+ + b_{i+1,j}^+ a_{i+1,j+1} - b_{i+1,j}a_{i+1,j+1}^+ + \ldots\right), \end{aligned} \tag{55}$$

where, to emphasize a bipartite character of the square lattice that appeared, we have introduced notations $a_{i,j} = d_{i,j}$,

$b_{i+1,j}=d_{i+1,j}$ etc..

Notwithstanding the fact that we have an approximate theory, it contains the exact result in the 1D limit if one puts in Eq. (55) either $J_{\perp}=0$ or $J=0$ coming to a system of noninteracting chains extended in either horizontal or vertical direction, respectively. In the latter case Eq. (55) corresponds to a system of noninteracting chains each with the XY part of the Hamiltonian (13)

$$H_{XY}(i)=\sum_{j=0}^{\infty}\frac{1}{2}J_{\perp}\left(d_{i,j}^{+}d_{i,j+1}-d_{i,j}d_{i,j+1}^{+}\right). \tag{56}$$

In the former case one gets a system of noninteracting chains each with the Hamiltonian

$$H_{XY}(j)=(-1)^{j}\sum_{i=0}^{\infty}\frac{1}{2}J\,(-1)^{i}\left(d_{i,j}^{+}d_{i+1,j}-d_{i,j}d_{i+1,j}^{+}\right) \tag{57}$$

and to recover the 1D limit explicitly the transformation

$$d_{i,j}^{+}=\mathrm{e}^{\mathrm{i}\pi\psi_i}f_i^{+},\,d_{i+1,j}=\mathrm{e}^{-\mathrm{i}\pi\psi_{i+1}}f_{i+1},\ldots,\quad \psi_0=0,\psi_{i+1}=\psi_i+i \tag{58}$$

(e.g., $d_{0,j}^{+}=f_0^{+}$, $d_{1,j}^{+}=-f_1^{+}$, $d_{2,j}^{+}=-f_2^{+}$, $d_{3,j}^{+}=f_3^{+}$ etc.) should be performed.

After the Fourier transformation

$$d_{i,j}=\frac{1}{\sqrt{N_xN_y}}\sum_{k_x,k_y}\mathrm{e}^{\mathrm{i}(k_xi+k_yj)}d_{k_x,k_y} \tag{59}$$

etc. or in short

$$\begin{aligned}
&d_{\mathbf{i}}=\frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{i}}d_{\mathbf{k}},\quad d_{\mathbf{i}}^{+}=\frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{i}}d_{\mathbf{k}}^{+},\quad d_{\mathbf{k}}=\frac{1}{\sqrt{N}}\sum_{\mathbf{i}}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{i}}d_{\mathbf{i}},\quad d_{\mathbf{k}}^{+}=\frac{1}{\sqrt{N}}\sum_{\mathbf{i}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{i}}d_{\mathbf{i}}^{+},\\
&k_x=\frac{2\pi}{N_x}n_x,\quad n_x=-\frac{N_x}{2},-\frac{N_x}{2}+1,\ldots,\frac{N_x}{2}-1,\quad k_y=\frac{2\pi}{N_y}n_y,\quad n_y=-\frac{N_y}{2},-\frac{N_y}{2}+1,\ldots,\frac{N_y}{2}-1,\\
&\left\{d_{\mathbf{k}_1},d_{\mathbf{k}_2}^{+}\right\}=\delta_{\mathbf{k}_1,\mathbf{k}_2},\quad \left\{d_{\mathbf{k}_1},d_{\mathbf{k}_2}\right\}=\left\{d_{\mathbf{k}_1}^{+},d_{\mathbf{k}_2}^{+}\right\}=0
\end{aligned} \tag{60}$$

(N_x, N_y are even), the Hamiltonian (55) becomes

$$\begin{aligned}
H_{XY}&=\frac{1}{2}\sum_{\mathbf{k}}\left(\mathrm{i}J\sin k_x\left(b_{\mathbf{k}}^{+}a_{\mathbf{k}}-a_{\mathbf{k}}^{+}b_{\mathbf{k}}\right)+J_{\perp}\cos k_y\left(b_{\mathbf{k}}^{+}b_{\mathbf{k}}-a_{\mathbf{k}}^{+}a_{\mathbf{k}}\right)\right)\\
&=\frac{1}{2}\sum_{\mathbf{k}}|E_{\mathbf{k}}|\left(\cos\gamma_{\mathbf{k}}\left(b_{\mathbf{k}}^{+}b_{\mathbf{k}}-a_{\mathbf{k}}^{+}a_{\mathbf{k}}\right)+\mathrm{i}\sin\gamma_{\mathbf{k}}\left(b_{\mathbf{k}}^{+}a_{\mathbf{k}}-a_{\mathbf{k}}^{+}b_{\mathbf{k}}\right)\right)
\end{aligned} \tag{61}$$

where we have used the notations

$$\begin{aligned}
&b_{\mathbf{k}}^{+}=d_{k_x,k_y}^{+},\quad a_{\mathbf{k}}=d_{k_x\pm\pi,k_y\pm\pi},\quad \ldots,\quad E_{\mathbf{k}}=J_{\perp}\cos k_y+\mathrm{i}J\sin k_x=|E_{\mathbf{k}}|\mathrm{e}^{\mathrm{i}\gamma_{\mathbf{k}}},\\
&|E_{\mathbf{k}}|=\sqrt{J_{\perp}^2\cos^2 k_y+J^2\sin^2 k_x},\quad \cos\gamma_{\mathbf{k}}=\frac{J_{\perp}\cos k_y}{|E_{\mathbf{k}}|}\quad \sin\gamma_{\mathbf{k}}=\frac{J\sin k_x}{|E_{\mathbf{k}}|}.
\end{aligned} \tag{62}$$

The Hamiltonian (61) can be rewritten as

$$H_{XY}={\sum_{\mathbf{k}}}'|E_{\mathbf{k}}|\left(\cos\gamma_{\mathbf{k}}\left(b_{\mathbf{k}}^{+}b_{\mathbf{k}}-a_{\mathbf{k}}^{+}a_{\mathbf{k}}\right)+\mathrm{i}\sin\gamma_{\mathbf{k}}\left(b_{\mathbf{k}}^{+}a_{\mathbf{k}}-a_{\mathbf{k}}^{+}b_{\mathbf{k}}\right)\right) \tag{63}$$

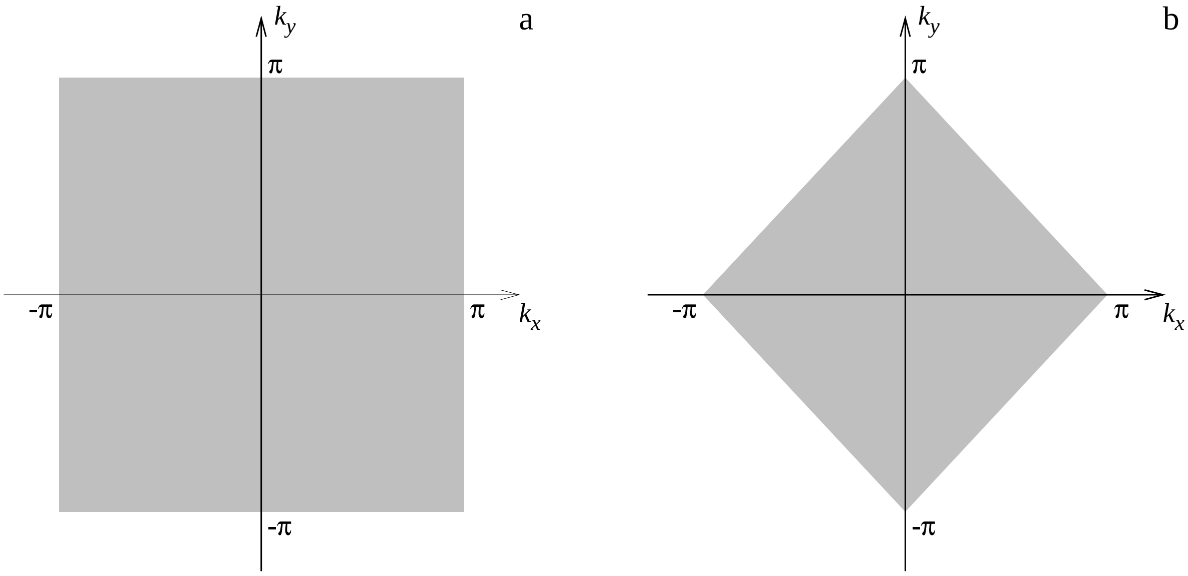

Fig. 8. The region in which $\mathbf{k}$ varies in the sum in Eq. (61) (see Eq. (60)) (a) and in Eq. (63) (b).

where the prime denotes that $\mathbf{k}$ varies in the region shown in Fig. 8b.

At last one introduces the operators

$$\alpha_{\mathbf{k}} = \cos\frac{\gamma_{\mathbf{k}}}{2} b_{\mathbf{k}} + \mathrm{i}\sin\frac{\gamma_{\mathbf{k}}}{2} a_{\mathbf{k}}, \quad \beta_{\mathbf{k}} = \sin\frac{\gamma_{\mathbf{k}}}{2} b_{\mathbf{k}} - \mathrm{i}\cos\frac{\gamma_{\mathbf{k}}}{2} a_{\mathbf{k}},$$
$$\left\{\alpha_{\mathbf{k}_1}, \alpha^+_{\mathbf{k}_2}\right\} = \delta_{\mathbf{k}_1,\mathbf{k}_2}, \quad \left\{\beta_{\mathbf{k}_1}, \beta^+_{\mathbf{k}_2}\right\} = \delta_{\mathbf{k}_1,\mathbf{k}_2}, \quad \left\{\alpha_{\mathbf{k}_1}, \beta^+_{\mathbf{k}_2}\right\} = \left\{\beta_{\mathbf{k}_1}, \alpha^+_{\mathbf{k}_2}\right\} = 0, \tag{64}$$

etc. to get from Eq. (63) the final form of the 2D spin–$\frac{1}{2}$ isotropic XY model Hamiltonian in fermionic language within approximation (41)

$$H_{XY} = \sum_{\mathbf{k}}{}' \Lambda_{\mathbf{k}} \left(\alpha^+_{\mathbf{k}}\alpha_{\mathbf{k}} - \beta^+_{\mathbf{k}}\beta_{\mathbf{k}}\right),$$
$$\Lambda_{\mathbf{k}} = |E_{\mathbf{k}}| = \sqrt{J^2\sin^2 k_x + J_\perp^2\cos^2 k_y} \geq 0. \tag{65}$$

It is easy now to calculate the thermodynamic functions of the spin model which correspond to Eq. (65). For example, the ground state energy per site is

$$\frac{E_0}{N} = -\int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}\int_{-\pi+|k_x|}^{\pi-|k_x|}\frac{\mathrm{d}k_y}{2\pi}\sqrt{J^2\sin^2 k_x + J_\perp^2\cos^2 k_y} = -\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_y}{2\pi}\sqrt{J^2\sin^2 k_x + J_\perp^2\cos^2 k_y}. \tag{66}$$

In 1D limit ($J_\perp = 0$ or $J = 0$) Eq. (66) becomes

$$\frac{E_0}{N} = -\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}|J\sin k_x| = -\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_y}{2\pi}|J_\perp\cos k_y| = -\frac{|J|}{\pi} \tag{67}$$

that is a well–known exact result.

It is remarkable to note that the Hamiltonian (65) (and hence the ground state energy per site (66)) arises also for the 2D spin–$\frac{1}{2}$ Heisenberg model and is known as the *uniform flux* ($\langle n_{\mathbf{r}}\rangle = \frac{1}{2}$) solution [5].

VII. 2D $S=\frac{1}{2}$ HEISENBERG MODEL

In the remainder of the paper we consider the 2D spin–$\frac{1}{2}$ isotropic Heisenberg model (1), (2) treating the phase factors which appear after making use of the 2D Jordan–Wigner transformation within the frames of the mean–field approximation (41). The Heisenberg model besides the H_{XY} term (55) includes the H_Z term which contains

$$d_{\mathbf{i}}^{+}d_{\mathbf{i}}d_{\mathbf{j}}^{+}d_{\mathbf{j}}-\frac{1}{2}d_{\mathbf{i}}^{+}d_{\mathbf{i}}-\frac{1}{2}d_{\mathbf{j}}^{+}d_{\mathbf{j}}+\frac{1}{4}. \tag{68}$$

Thus, the Jordan–Wigner spinless fermions interact and further approximations are required. The first term in (68) can be changed by

$$\begin{aligned} d_{\mathbf{i}}^{+}d_{\mathbf{i}}d_{\mathbf{j}}^{+}d_{\mathbf{j}} &\to d_{\mathbf{i}}^{+}d_{\mathbf{i}}\langle d_{\mathbf{j}}^{+}d_{\mathbf{j}}\rangle+\langle d_{\mathbf{i}}^{+}d_{\mathbf{i}}\rangle d_{\mathbf{j}}^{+}d_{\mathbf{j}}\\ &-\langle d_{\mathbf{i}}^{+}d_{\mathbf{i}}\rangle\langle d_{\mathbf{j}}^{+}d_{\mathbf{j}}\rangle+d_{\mathbf{i}}^{+}d_{\mathbf{j}}\langle d_{\mathbf{i}}d_{\mathbf{j}}^{+}\rangle\\ &+\langle d_{\mathbf{i}}^{+}d_{\mathbf{j}}\rangle d_{\mathbf{i}}d_{\mathbf{j}}^{+}-\langle d_{\mathbf{i}}^{+}d_{\mathbf{j}}\rangle\langle d_{\mathbf{i}}d_{\mathbf{j}}^{+}\rangle\\ &=d_{\mathbf{i}}^{+}d_{\mathbf{i}}\langle n_{\mathbf{j}}\rangle+d_{\mathbf{j}}^{+}d_{\mathbf{j}}\langle n_{\mathbf{i}}\rangle\\ &-\langle n_{\mathbf{i}}\rangle\langle n_{\mathbf{j}}\rangle+d_{\mathbf{i}}^{+}d_{\mathbf{j}}\Delta_{\mathbf{i},\mathbf{j}}\mathrm{e}^{\mathrm{i}\theta_{\mathbf{i},\mathbf{j}}}\\ &-d_{\mathbf{i}}d_{\mathbf{j}}^{+}\Delta_{\mathbf{j},\mathbf{i}}\mathrm{e}^{\mathrm{i}\theta_{\mathbf{j},\mathbf{i}}}+\Delta_{\mathbf{i},\mathbf{j}}\Delta_{\mathbf{j},\mathbf{i}}\mathrm{e}^{\mathrm{i}(\theta_{\mathbf{i},\mathbf{j}}+\theta_{\mathbf{j},\mathbf{i}})}, \end{aligned} \tag{69}$$

where we have introduced the notation

$$\langle d_{\mathbf{i}}d_{\mathbf{j}}^{+}\rangle=\Delta_{\mathbf{i},\mathbf{j}}\mathrm{e}^{\mathrm{i}\theta_{\mathbf{i},\mathbf{j}}}=-\langle d_{\mathbf{j}}^{+}d_{\mathbf{i}}\rangle. \tag{70}$$

In accordance with (69) there may be four ways of treating the Ising interaction, i.e. assuming either

$$\Delta_{\mathbf{i},\mathbf{j}}=0,\quad m=\langle s_{\mathbf{j}}^{z}\rangle=\langle n_{\mathbf{j}}\rangle-\frac{1}{2}=0 \tag{71}$$

or

$$\Delta_{\mathbf{i},\mathbf{j}}\neq 0,\quad m=0 \tag{72}$$

or

$$\Delta_{\mathbf{i},\mathbf{j}}=0,\quad m\neq 0 \tag{73}$$

or

$$\Delta_{\mathbf{i},\mathbf{j}}\neq 0,\quad m\neq 0. \tag{74}$$

The first possibility (71) yielding the *uniform flux* solution was considered in the previous Section (Eqs. (65), (66)).

Let us consider the second possibility (72). In such a case the Ising term becomes

$$\begin{aligned} H_Z=\sum_{\langle\mathbf{i},\mathbf{j}\rangle}J_{\mathbf{i},\mathbf{j}}\Big(&\Delta_{\mathbf{i},\mathbf{j}}\mathrm{e}^{\mathrm{i}\theta_{\mathbf{i},\mathbf{j}}}d_{\mathbf{i}}^{+}d_{\mathbf{j}}\\ &-\Delta_{\mathbf{j},\mathbf{i}}\mathrm{e}^{\mathrm{i}\theta_{\mathbf{j},\mathbf{i}}}d_{\mathbf{i}}d_{\mathbf{j}}^{+}+\Delta_{\mathbf{i},\mathbf{j}}\Delta_{\mathbf{j},\mathbf{i}}\mathrm{e}^{\mathrm{i}(\theta_{\mathbf{i},\mathbf{j}}+\theta_{\mathbf{j},\mathbf{i}})}\Big), \end{aligned} \tag{75}$$

where to get the *in–phase flux* solution one puts

$$\begin{aligned} \Delta_{i,j;i+1,j}&=\Delta_{i+1,j;i,j}=\Delta_{i+1,j;i+2,j}\\ &=\Delta_{i+2,j;i+1,j}=Q,\\ \mathrm{e}^{\mathrm{i}\theta_{i,j;i+1,j}}&=\mathrm{e}^{\mathrm{i}\theta_{i+1,j;i,j}}=-1,\\ \mathrm{e}^{\mathrm{i}\theta_{i+1,j;i+2,j}}&=\mathrm{e}^{\mathrm{i}\theta_{i+2,j;i+1,j}}=1,\\ \Delta_{i,j;i,j+1}&=\Delta_{i,j+1;i,j}=\Delta_{i+1,j;i+1,j+1}\\ &=\Delta_{i+1,j+1;i+1,j}=P,\\ \mathrm{e}^{\mathrm{i}\theta_{i,j;i,j+1}}&=\mathrm{e}^{\mathrm{i}\theta_{i,j+1;i,j}}=\mathrm{e}^{\mathrm{i}\theta_{i+1,j;i+1,j+1}}\\ &=\mathrm{e}^{\mathrm{i}\theta_{i+1,j+1;i+1,j}}=1 \end{aligned} \tag{76}$$

(see Fig. 9) and the parameters Q and P are calculated self–consistently (see Eqs. (80), (81) below). Now the Heisenberg model Hamiltonian only slightly differs from that of the XY model (55) becoming

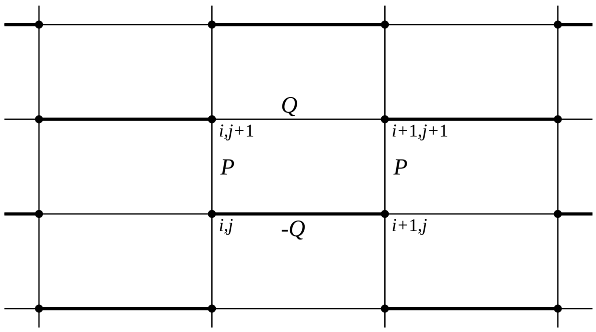

Fig. 9. Towards the in–phase flux solution for the Heisenberg model; the values of $\Delta_{\mathbf{i},\mathbf{j}}\mathrm{e}^{\mathrm{i}\theta_{\mathbf{i},\mathbf{j}}}$ are attached to the bonds.

$$H=\frac{1}{2}J\left(1+2Q\right)\left(\ldots-a_{i,j}^{+}b_{i+1,j}+a_{i,j}b_{i+1,j}^{+}+b_{i+1,j}^{+}a_{i+2,j}-b_{i+1,j}a_{i+2,j}^{+}+\ldots\right)$$

$$+\frac{1}{2}J_{\perp}\left(1+2P\right)\left(\ldots+a^{+}_{i,j}b_{i,j+1}-a_{i,j}b^{+}_{i,j+1}+b^{+}_{i+1,j}a_{i+1,j+1}-b_{i+1,j}a^{+}_{i+1,j+1}+\ldots\right)+NJQ^{2}+NJ_{\perp}P^{2}. \tag{77}$$

Acting along the line described in Section VI one finds that

$$H=\sum_{\mathbf{k}}{}'\Lambda_{\mathbf{k}}\left(\alpha^{+}_{\mathbf{k}}\alpha_{\mathbf{k}}-\beta^{+}_{\mathbf{k}}\beta_{\mathbf{k}}\right)+NJQ^{2}+NJ_{\perp}P^{2},$$
$$\Lambda_{\mathbf{k}}=\sqrt{J^{2}\left(1+2Q\right)^{2}\sin^{2}k_{x}+J^{2}_{\perp}\left(1+2P\right)^{2}\cos^{2}k_{y}}\geq 0. \tag{78}$$

The ground state energy per site is given by

$$\frac{E_{0}}{N}=-\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_{x}}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_{y}}{2\pi}\sqrt{J^{2}\left(1+2Q\right)^{2}\sin^{2}k_{x}+J^{2}_{\perp}\left(1+2P\right)^{2}\cos^{2}k_{y}}+JQ^{2}+J_{\perp}P^{2}, \tag{79}$$

where parameters Q and P are determined from the conditions $\frac{\partial}{\partial Q}\frac{E_0}{N}=0$ and $\frac{\partial}{\partial P}\frac{E_0}{N}=0$, i.e.,

$$Q=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_{x}}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_{y}}{2\pi}\frac{J\sin^{2}k_{x}\left(1+2Q\right)}{\sqrt{J^{2}\left(1+2Q\right)^{2}\sin^{2}k_{x}+J^{2}_{\perp}\left(1+2P\right)^{2}\cos^{2}k_{y}}} \tag{80}$$

and

$$P=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_{x}}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_{y}}{2\pi}\frac{J_{\perp}\cos^{2}k_{y}\left(1+2P\right)}{\sqrt{J^{2}\left(1+2Q\right)^{2}\sin^{2}k_{x}+J^{2}_{\perp}\left(1+2P\right)^{2}\cos^{2}k_{y}}}. \tag{81}$$

Treating the Ising term (69) within assumptions (73) or (74) one assumes the Néel order with the sublattice magnetizations m and $-m$

$$\langle n_{i,j}\rangle=\langle n_{i+1,j+1}\rangle=\ldots=m+\frac{1}{2},\quad \langle n_{i,j+1}\rangle=\langle n_{i+1,j}\rangle=\ldots=-m+\frac{1}{2} \tag{82}$$

(see Fig. 10). (The given assumption, by the way, apparently contradicts the mean–field approximation (41) for the phase factors; this inconsistency, however, to our best knowledge has not yet been discussed.) Therefore, case (73), i.e., the uniform flux with the Néel order, is called the *Néel flux* solution and case (74), i.e., the in–flux with the Néel order, is called the *in–phase Néel flux* solution. In the latter case the Hamiltonian reads

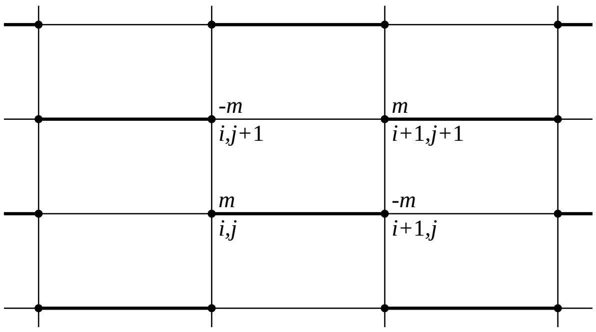

Fig. 10. The Néel order for the Heisenberg model; the values of magnetization $\langle s^{z}_{\mathrm{i}}\rangle$ are attached to the sites.

$$H = \frac{1}{2}J\left(1+2Q\right)\left(\ldots - a^+_{i,j}b_{i+1,j} + a_{i,j}b^+_{i+1,j} + b^+_{i+1,j}a_{i+2,j} - b_{i+1,j}a^+_{i+2,j} + \ldots\right)$$
$$+ \frac{1}{2}J_\perp\left(1+2P\right)\left(\ldots + a^+_{i,j}b_{i,j+1} - a_{i,j}b^+_{i,j+1} + b^+_{i+1,j}a_{i+1,j+1} - b_{i+1,j}a^+_{i+1,j+1} + \ldots\right)$$
$$+ J\left(\ldots - ma^+_{i,j}a_{i,j} + mb^+_{i+1,j}b_{i+1,j} + mb^+_{i+1,j}b_{i+1,j} - ma^+_{i+2,j}a_{i+2,j} + \ldots\right)$$
$$+ J_\perp\left(\ldots - ma^+_{i,j}a_{i,j} + mb^+_{i,j+1}b_{i,j+1} + mb^+_{i+1,j}b_{i+1,j} - ma^+_{i+1,j+1}a_{i+1,j+1} + \ldots\right)$$
$$+ NJQ^2 + NJ_\perp P^2 + N\left(J+J_\perp\right)m^2. \tag{83}$$

Performing the Fourier transformation (60) one gets

$$H = \frac{1}{2}\sum_{\mathbf{k}}\Big(\left(2\left(J+J_\perp\right)m + \mathrm{i}J\left(1+2Q\right)\sin k_x\right)b^+_{\mathbf{k}}a_{\mathbf{k}} + \left(2\left(J+J_\perp\right)m - \mathrm{i}J\left(1+2Q\right)\sin k_x\right)a^+_{\mathbf{k}}b_{\mathbf{k}}$$
$$+ J_\perp\left(1+2P\right)\cos k_y\left(b^+_{\mathbf{k}}b_{\mathbf{k}} - a^+_{\mathbf{k}}a_{\mathbf{k}}\right)\Big) + NJQ^2 + NJ_\perp P^2 + N\left(J+J_\perp\right)m^2$$
$$= \sum_{\mathbf{k}}{}'\Big(\left(2\left(J+J_\perp\right)m + \mathrm{i}J\left(1+2Q\right)\sin k_x\right)b^+_{\mathbf{k}}a_{\mathbf{k}}$$
$$+ \left(2\left(J+J_\perp\right)m - \mathrm{i}J\left(1+2Q\right)\sin k_x\right)a^+_{\mathbf{k}}b_{\mathbf{k}} + J_\perp\left(1+2P\right)\cos k_y\left(b^+_{\mathbf{k}}b_{\mathbf{k}} - a^+_{\mathbf{k}}a_{\mathbf{k}}\right)\Big)$$
$$+ NJQ^2 + NJ_\perp P^2 + N\left(J+J_\perp\right)m^2, \tag{84}$$

where $b^+_{\mathbf{k}} = d^+_{k_x,k_y}$, $a_{\mathbf{k}} = d_{k_x\pm\pi,k_y\pm\pi}$, etc.. Introducing the operators

$$\tilde{a}_{\mathbf{k}} = a_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\delta_{\mathbf{k}}}, \quad \tilde{a}^+_{\mathbf{k}} = a^+_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\delta_{\mathbf{k}}}, \quad \tilde{b}_{\mathbf{k}} = b_{\mathbf{k}}, \quad \tilde{b}^+_{\mathbf{k}} = b^+_{\mathbf{k}};$$
$$2\left(J+J_\perp\right)m \pm \mathrm{i}J\left(1+2Q\right)\sin k_x = \sqrt{4\left(J+J_\perp\right)^2m^2 + J^2\left(1+2Q\right)^2\sin^2 k_x}\;\mathrm{e}^{\pm\mathrm{i}\delta_{\mathbf{k}}} \tag{85}$$

and then the operators

$$\alpha_{\mathbf{k}} = \cos\frac{\omega_{\mathbf{k}}}{2}\tilde{b}_{\mathbf{k}} + \sin\frac{\omega_{\mathbf{k}}}{2}\tilde{a}_{\mathbf{k}}, \quad \beta_{\mathbf{k}} = \sin\frac{\omega_{\mathbf{k}}}{2}\tilde{b}_{\mathbf{k}} - \cos\frac{\omega_{\mathbf{k}}}{2}\tilde{a}_{\mathbf{k}};$$
$$\cos\omega_{\mathbf{k}} = \frac{J_\perp\left(1+2P\right)\cos k_y}{|E_{\mathbf{k}}|}, \quad \sin\omega_{\mathbf{k}} = \frac{\sqrt{4\left(J+J_\perp\right)^2m^2 + J^2\left(1+2Q\right)^2\sin^2 k_x}}{|E_{\mathbf{k}}|},$$
$$|E_{\mathbf{k}}| = \sqrt{4\left(J+J_\perp\right)^2m^2 + J^2\left(1+2Q\right)^2\sin^2 k_x + J_\perp^2\left(1+2P\right)^2\cos^2 k_y} \tag{86}$$

one gets the final form of the Heisenberg Hamiltonian in fermionic language

$$H = \sum_{\mathbf{k}}{}'\Lambda_{\mathbf{k}}\left(\alpha^+_{\mathbf{k}}\alpha_{\mathbf{k}} - \beta^+_{\mathbf{k}}\beta_{\mathbf{k}}\right) + NJQ^2 + NJ_\perp P^2 + N\left(J+J_\perp\right)m^2 \tag{87}$$

with $\Lambda_{\mathbf{k}} = |E_{\mathbf{k}}| \geq 0$ defined by Eq. (86).

The ground state energy per site which follows from (87) reads

$$\frac{E_0}{N} = -\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_x}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k_y}{2\pi}\sqrt{4\left(J+J_\perp\right)^2m^2 + J^2\left(1+2Q\right)^2\sin^2 k_x + J_\perp^2\left(1+2P\right)^2\cos^2 k_y}$$
$$+ JQ^2 + J_\perp P^2 + \left(J+J_\perp\right)m^2, \tag{88}$$

where the values of the introduced parameters are determined by minimizing $\frac{E_0}{N}$ (88) with respect to Q, P and m

$$Q = \frac{1}{2}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \frac{J\sin^2 k_x\,(1+2Q)}{\sqrt{4\,(J+J_\perp)^2\, m^2 + J^2\,(1+2Q)^2 \sin^2 k_x + J_\perp^2\,(1+2P)^2 \cos^2 k_y}}, \tag{89}$$

$$P = \frac{1}{2}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \frac{J_\perp\cos^2 k_y\,(1+2P)}{\sqrt{4\,(J+J_\perp)^2\, m^2 + J^2\,(1+2Q)^2 \sin^2 k_x + J_\perp^2\,(1+2P)^2 \cos^2 k_y}}, \tag{90}$$

$$2m = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \frac{2\,(J+J_\perp)\, m}{\sqrt{4\,(J+J_\perp)^2\, m^2 + J^2\,(1+2Q)^2 \sin^2 k_x + J_\perp^2\,(1+2P)^2 \cos^2 k_y}}. \tag{91}$$

The Néel flux solution (see Eq. (73)) is given by Eqs. (83), (87), (86), (88), (91) in which P and Q are equal to zero.

VIII. THE 2D $S=\frac{1}{2}$ HEISENBERG MODEL: A COMPARISON OF SOME RESULTS

In the final Section we want to list out some problems of the two–dimensional spin models theory which were attacked with the exploiting of the 2D Jordan–Wigner transformation.

The 2D antiferromagnetic Heisenberg model with $J = J' = J_\perp = J'_\perp$ (without external field) was considered in Refs. [5,7,6]. The main results obtained concern the ground state energy [5,7], the specific heat [7] and the Raman spectrum [6,7]. A comparison with some experimental data for La_2CuO_4 was given.

In Ref. [8] the effects of the interchain interaction on the one–dimensional spin–$\frac{1}{2}$ antiferromagnetic Heisenberg model were examined. For this purpose the Heisenberg model with $J = J'$, $J_\perp = J'_\perp$ was considered and the in–phase Néel flux solution was analysed at zero temperature. The author found that the one–dimensional limit is singular, i.e. the staggered magnetization $m \neq 0$ ($2m = 0.513$) when $J_\perp \to +0$ (although we know from exact results that for the antiferromagnetic chain $m = 0$). In the other limiting case $J = J_\perp$ the theory based on the fermionization procedure yields $2m = 0.778$ (the spin wave result is $2m = 0.6$; more accurate calculations predict $m = 0.3074$ (see Ref. [19])). The result of Ref. [8] stays somewhat separately in the estimate of the value of $\frac{J_\perp}{J}$ at which the staggered magnetization appears. Different theories predict $\frac{J_\perp}{J}$ from 0 to 0.2 (for details see Ref. [19]). Ref. [8] predicts $\frac{J_\perp}{J} = 0$, moreover $2m$ jumps from zero to 0.513 for any infinitesimally small $\frac{J_\perp}{J}$.

The antiferromagnetic Heisenberg model on a ladder within the frames of the in–phase Néel phase solution (the ground state energy, the singlet–triplet energy gap) was discussed in Ref. [9]. The effects of the interladder interaction and magnetic field on the susceptibility at nonzero temperatures were studied in Ref. [11].

A consideration of the spin–Peierls state under magnetic field was reported in Ref. [10]. A study of the stepped spin–Peierls transition for the quasi–one–dimensional XY and Heisenberg models using the 2D Jordan–Wigner transformation was reported in Refs. [12,13]. The 2D Jordan–Wigner transformation was applied for a study of the zero temperature spin–Peierls transition for the quasi–one–dimensional XY and Heisenberg systems in Ref. [14]. In particular, the phase diagram between the dimerised and uniform states in the parameter space of interchain interaction and spin–lattice coupling was constructed.

Many more problems may be considered within the frames of the 2D Jordan–Wigner fermionization approach. Probably, the 2D Jordan–Wigner transformation should be of more use for the 2D spin–$\frac{1}{2}$ XY models since for such models no further approximations (except the mean–field–like treatment of the phase factors) are required. Besides, a more sophisticated treatment of the phase factors is desirable.

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ФЕРМІОНІЗАЦІЯ ЙОРДАНА–ВІҐНЕРА ДЛЯ ДВОВИМІРНИХ СПІН–$\frac{1}{2}$ СИСТЕМ: КОРОТКИЙ ОГЛЯД

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Розглянуто систему, що складається зі спінів $\frac{1}{2}$, декартові компоненти яких можуть зображати матриці Паулі, розміщених у вузлах ґратки $\mathbf{i}$. Спінові оператори на різних вузлах $\mathbf{i} \neq \mathbf{j}$ комутують. Перетворення Йордана–Віґнера дозволяє зобразити спінові оператори безспіновими фермі–операторами, а саме:

$$s_{\mathbf{i}}^{+} = \mathrm{e}^{-\mathrm{i}\alpha_{\mathbf{i}}} d_{\mathbf{i}}^{+} = d_{\mathbf{i}}^{+} \mathrm{e}^{-\mathrm{i}\alpha_{\mathbf{i}}}, \quad s_{\mathbf{i}}^{-} = \mathrm{e}^{\mathrm{i}\alpha_{\mathbf{i}}} d_{\mathbf{i}} = d_{\mathbf{i}} \mathrm{e}^{\mathrm{i}\alpha_{\mathbf{i}}}, \quad \alpha_{\mathbf{i}} = \sum_{\mathbf{j}(\neq \mathbf{i})} B_{\mathbf{ij}} d_{\mathbf{j}}^{+} d_{\mathbf{j}}.$$

Тут $B_{\mathbf{ij}}$ є c–числова матриця, що задовольняє умову

$$\mathrm{e}^{\mathrm{i}B_{\mathbf{ij}}} = -\mathrm{e}^{\mathrm{i}B_{\mathbf{ji}}}.$$

Для квадратної ґратки з $\mathbf{i} = i_x \mathbf{n}_x + i_y \mathbf{n}_y$, де $\mathbf{n}_x$ і $\mathbf{n}_y$ є одиничними векторами вздовж горизонтального й вертикального напрямків, можна взяти

$$B_{\mathbf{ij}} = \mathrm{Im} \ln \left(j_x - i_x + \mathrm{i} \left(j_y - i_y \right) \right)$$

(Ванґ, 1991) або

$$B_{\mathbf{ij}} = \pi \left(\Theta \left(i_x - j_x \right) \left(1 - \delta_{i_x, j_x} \right) + \delta_{i_x, j_x} \Theta \left(i_y - j_y \right) \left(1 - \delta_{i_y, j_y} \right) \right)$$

(Азуз, 1993).

Застосування перетворення Йордана–Віґнера до ізотропної XY моделі на квадратній ґратці приводить до моделі сильно зв'язаних безспінових ферміонів з інтеґралами переносу, що містять оператори $d^{+}d$. Якщо ж є і взаємодія Ізинґа, то безспінові ферміони взаємодіють. Вирішальне наближення полягає в наступній заміні у фазових множниках інтеґралів переносу $d^{+}d \to \langle d^{+}d \rangle = \langle s^{z} \rangle + \frac{1}{2} \to \frac{1}{2}$. Використовуючи далі подібність до задачі про рух електронів в однорідному магнетному полі, перпендикулярному до площини, у якій рухаються електрони, фазові множники можна вибрати так, що матимемо модель сильно зв'язаних безспінових ферміонів на ґратці з регулярно знакозмінними інтеґралами переносу (для ізотропної XY моделі), які взаємодіють (для моделі Гайзенберґа). Таким чином, розрахунок термодинамічних функцій двовимірної XY моделі не потребує подальших наближень, а для моделі Гайзенберґа через взаємодію безспінових ферміонів потрібні дальші наближення. Відомо чотири способи трактування доданків, що описують взаємодію безспінових ферміонів, для двовимірної антиферомагнетної моделі Гайзенберґа (Ванґ, 1991). Аналітичний наближений підхід, що ґрунтується на ферміонізації Йордана–Віґнера у двовимірному випадку, було застосовано в дослідженні властивостей спін–$\frac{1}{2}$ антиферомагнетної моделі Гайзенберґа на квадратній ґратці, на драбинкових структурах, у випадку спін–Пайєрлсової димеризації (1991 – 1999).