ЖУРНАЛ ФІЗИЧНИХ ДОСЛІДЖЕНЬ
т. 6, № 1 (2002) с. 46–54

JOURNAL OF PHYSICAL STUDIES
v. 6, No. 1 (2002) p. 46–54

THE $1/N$-EXPANSION IN QUANTUM MECHANICS. HIGH-ORDER APPROXIMATIONS

I. O. Vakarchuk
Ivan Franko National University of Lviv, Department for Theoretical Physics
12 Drahomanov Str., Lviv, 79005, Ukraine
(Received December 29, 2001)

The energy spectrum of a particle moving in a centrosymmetric field in the N-dimensional space is calculated by means of the ordinary Rayleigh–Schrödinger perturbation theory up to $(\hbar/N)^3$. The energy levels for power-law potentials are found as an example.

Key words: anharmonic oscillator, quasiclassical approximation, energy spectrum, $1/N$-expansion.

PACS number(s): 03.65.–w, 03.65.Ge, 03.65.Sq

I. INITIAL EQUATIONS

The perturbation theory called "$1/N$-expansion" found a wide application in different problems of theoretical physics. Most probably, it originates from the works by T. Holstein and H. Primakoff [1] where a transition from spin- to Bose-operators was proposed. It allowed finding exact solutions of problems on spin models in the case of large spin values, i. e., for a large number of degrees of freedom. Later, T. Berlin and M. Kac [2] formulated the so-called spherical model of a ferromagnet of N spins. There, a crucial point in the exact solution consisted in utilization of a saddle-point technique with the condition $N \to \infty$. Both of these problems permit to find corrections to the exact results and to develop a perturbation theory in a small parameter proportional to the inverse number of degrees of freedom. Later the $1/N$-expansion technique found its applications in the problems of nuclear physics, elementary particle physics, in the theory of critical phenomena [3–5]. We do not intend here to adduce a review of a numerous literature on this problem and refer to the work by L. D. Mlodinov and N. Papanicolau [6] where the $1/N$-expansion was studied by means of an algebraic procedure based on the Holstein–Primakoff representation. There, the expansion was considered as an alternative to the standard quantum-mechanical perturbation theory and was utilized to investigate the systems with a certain class of potentials. The method for the solution of the Schrödinger equation in N-dimensional space using the $1/N$ perturbation theory is proposed in [7].

Our task is to obtain within the $1/N$-expansion the higher-order corrections for the energy levels of a particle moving in the field of a centre of attraction and to study some particular models. Such calculations may be also interesting in view of investigations of the so-called quasi-exactly solvable models in frames of the supersymmetric quantum mechanics techniques.

In the N-dimensional space consider a particle of mass m with the Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + NU\left(\frac{x}{\sqrt{N}}\right), \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ is a radius-vector in the Cartesian coordinates, $\hat{\mathbf{p}} = (-i\hbar\partial/\partial x_1, \ldots, -i\hbar\partial/\partial x_N)$ is a momentum operator in the coordinate representation, and the potential energy U depends only on a distance from the force centre $x = (x_1^2 + \ldots + x_N^2)^{1/2}$.

Steady-state Schrödinger equation for the wavefunction $\psi = \psi(\mathbf{x})$

$$\hat{H}\psi(\mathbf{x}) = E\psi(\mathbf{x}) \tag{2}$$

allows a separation of variables in the hyperspherical coordinates in view of a centrosymmetric nature of the force field.

Therefore,

$$\psi(\mathbf{x}) = R(x)Y_{l,m}(\Omega) \tag{3}$$

where the spherical harmonic satisfies the N-dimensional Laplace equation [8]

$$\boldsymbol{\nabla}^2(x^l Y_{l,m}(\Omega)) = 0 \tag{4}$$

with $\Omega = (\vartheta_1, \ldots, \vartheta_{N-2}, \varphi)$, ϑ_j being the jth azimuth and φ denoting the horizontal angle; a quantum number $l = 0, 1, 2, \ldots$, while a compound quantum number $m = (m_1, \ldots, m_{N-2})$, where the integers satisfy the condition $l \geq m_1 \geq m_2 \geq \ldots \geq m_{N-2}$. The spherical harmonic $Y_{l,m}(\Omega)$ is expressed via products of the Hehenbauer polynomials depending on $\cos\vartheta_j$ and has an azimuth factor $e^{\pm i m_{N-2}\varphi}$ [8].

For the radial wavefunction $R(x)$, substituting (3) into the Schrödinger equation (2) and accounting (4), one gets the following equation:

$$-\frac{\hbar^2}{2m}\left(x^{-N+1}\frac{d}{dx}x^{N-1}\frac{d}{dx} - \frac{l(l+N-2)}{x^2}\right)R(x) + NU\left(\frac{x}{\sqrt{N}}\right)R(x) = ER(x). \tag{5}$$

The radial function $R(x)$ is normalized with the weight function being a radial part of the Jacobian from the Cartesian to the hyperspherical coordinates. For the bound states one has:

$$\int_0^\infty x^{N-1}|R(x)|^2\,dx = 1.$$

Let us introduce a Schrödinger wavefunction

$$\chi(x) = x^{\frac{N-1}{2}} R(x)$$

which is normalized without a weight coefficient and satisfies the equation:

$$\Big[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{8mx^2}(N+2l-1)(N+2l-3) + NU\left(\frac{x}{\sqrt{N}}\right)\Big]\chi = E\chi. \tag{6}$$

Changing the variables in equation (6)

$$y = \frac{x}{\sqrt{N}}$$

one finds an equation for the eigenvalues of energy per one degree of freedom:

$$\left[-\frac{\hbar^2}{2m^*}\frac{d^2}{dy^2} + w(y)\right]\chi = \frac{E}{N}\chi, \tag{7}$$
$$0 \le y < \infty,$$

where the effective mass is

$$m^* = mN^2, \tag{8}$$

the effective potential reads:

$$w(y) = \nu\frac{\hbar^2}{8my^2} + U(y), \tag{9}$$

and the parameter is given in the form:

$$\nu = \frac{(N+2l-1)(N+2l-3)}{N^2}. \tag{10}$$

II. PERTURBATION THEORY FORMULATION

In the limit $N \to \infty$, when the effective mass (8) $m^* \to \infty$, the contribution from the kinetic energy operator vanishes. In this case the quantity E/N is found from (7) as an effective potential value at the point of minimum $y = y_0$:

$$\frac{E}{N} = w(y_0), \qquad N \to \infty, \tag{11}$$

where the coordinate y_0 is found from the condition

$$w'(y_0) = 0. \tag{12}$$

Expression (11) is a zero-order approximation in the parameter $1/N$ for the ground state energy. Next approximations are found using the expansion of the effective potential in powers of $z = y - y_0$. Then the equation (7) reads:

$$(\hat{H}_0 + V)\chi = (E/N - w_0)\chi, \tag{13}$$
$$-y_0 < z < \infty,$$

where

$$w_0 = w(y_0),$$

and

$$\hat{H}_0 = -\frac{\hbar^2}{2m^*}\frac{d^2}{dz^2} + \frac{m^*\omega^2}{2}z^2 \tag{14}$$

is the Hamiltonian of a harmonic oscillator with the frequency defined by the second derivative of $w(y)$ at the point $y = y_0$,

$$\omega = \sqrt{\frac{w''(y_0)}{m^*}}. \tag{15}$$

The operator V includes the anharmonicities of all orders starting from the cubic one and is expressed as a series:

$$V = \sum_{k\ge 3} \beta_n z^k \tag{16}$$

with the coefficients

$$\beta_k = \frac{w^{(k)}(y_0)}{k!}, \tag{17}$$

where $w^{(k)}(y_0)$ is the k-th order derivative of $w(y)$ at the point $y = y_0$.

Let us develop a perturbation theory to calculate the energy eigenvalues considering V as a perturbation. In the zeroth order of the theory, when $V = 0$, the energy eigenvalues are the energy levels of a harmonic oscillator with the Hamiltonian:

$$\frac{E_n}{N} = w_0 + \hbar\omega\left(n+\frac{1}{2}\right), \qquad (18)$$

$$n = 0, 1, 2, \ldots .$$

Since the frequency $\omega \sim 1/N$ (see Eq. (15)), in the limit $N \to \infty$ the first term in E_n/N is of the zeroth order of smallness in the parameter $1/N$, while the second term is of the first order. However, the contribution into the first term stems also from w_0 since equation (12) for y_0 contains the parameter ν including terms of the order $1/N$. The next terms in the expansion of the energy in powers of $1/N$ are obtained from the perturbation theory in the operator V.

The quantity $(\hbar/m^*\omega)^{1/2} \sim (\hbar/N)^{1/2}$ is known to be a unit of length in the theory of an oscillator. Therefore, the k-th term in series (16) is proportional to $(\hbar/N)^{k/2}$, $k \geq 3$. Hence, our perturbation theory in the parameter $1/N$ is obviously equivalent to the quasiclassical expansion in powers of $\hbar$.

Before proceeding to specific calculations let us note one more important point. The coordinate z in equation (13) does not span the whole real axis but varies within the ranges $(-y_0 \div \infty)$. Thus, the wavefunctions of the zeroth-order Hamiltonian $\hat{H}_0$ do not coincide with the wavefunctions of a linear harmonic oscillator. However, the condition $N \to \infty$ simplifies the situation again. Indeed, introducing (as in the theory of an oscillator) a dimensionless variable $\xi = z/(\hbar/m^*\omega)^{1/2}$, one obtains the following ranges for its variation: $-N^{1/2}y_0[w''(y_0)m/\hbar^2]^{1/4} \leq \xi < \infty$. When $N \to \infty$ one returns to the ordinary problem of a linear harmonic oscillator. Therefore, when looking for corrections of the power nature, the lower limit gives no contribution. Only the terms $\sim \exp(-Ny_0^2\sqrt{mw''(y_0)}/\hbar)$, generated by the ground state wavefunction, appear. They obviously vanish in the comparison with those proportional to $1/N$ and hence one does not take them into account. Summarizing the above statements, one can see that the task is to calculate the corrections from the operator V anharmonicities utilizing the standard perturbation theory scheme.

Within Rayleigh–Schrödinger perturbation theory, the series for the energy E_n/N in powers of the matrix elements calculated in the basis of a linear harmonic oscillator $\hat{H}_0$ eigenfunctions reads:

$$E_n/N = w_0 + \hbar\omega\left(n+\frac{1}{2}\right) + \sum_{p\geq 1} E^{(p)}/N, \qquad (19)$$

where the first two contributions are

$$E_n^{(1)}/N = V_{nn}, \qquad (20)$$

$$E_n^{(2)}/N = \sum_{n'\neq n} \frac{|V_{nn'}|^2}{\hbar\omega(n-n')}. \qquad (21)$$

Rewrite the expansion (19) in the form of a series in powers of the small parameter $\hbar/N$:

$$E_n/N = w_0 + \hbar\omega\left(n+\frac{1}{2}\right) + \sum_{k\geq 1} \Delta_k E_n/N. \qquad (22)$$

To this end let us note that, according to (16), every k-th term of the expansion of the matrix element V_{nn}, after having been made dimensionless, gives a factor $(\hbar/2m^*\omega)^{k/2} \sim (\hbar/N)^{k/2}$, while the p-th order in the perturbation theory in V leads to a quantity $(\hbar\omega)^{p-1} \sim (\hbar/N)^{p-1}$ in the denominator. Taking into account that the anharmonicities in the perturbation operator V start from the cubic one and the fact that the diagonal matrix elements from the odd powers of z equal to zero, one finds the first non-vanishing contribution in the perturbation theory series in $\hbar/N$ to be quadratic. Indeed, it is obtained from the fourth anharmonicity in the first order (20) and from the cubic one in the second order (21) of the perturbation theory in V:

$$\Delta_1 E_n/N = \Delta_1 E_n^{(1)}/N + \Delta_1 E_n^{(2)}/N,$$

$$\Delta_1 E_n^{(1)}/N = \beta_4 \langle n|z^4|n\rangle,$$

$$\Delta_1 E_n^{(2)}/N = \beta_3^2 \sum_{n'\neq n} \frac{|\langle n|z^3|n'\rangle|^2}{\hbar\omega(n-n')}.$$

These quantities can be easily calculated and have gone down in textbooks on quantum mechanics a long time ago:

$$\frac{\Delta_1 E_n}{N} = \beta_4 \left(\frac{\hbar}{2m^*\omega}\right)^2 3(2n^2+2n+1)$$
$$- \frac{\beta_3^2}{\hbar\omega}\left(\frac{\hbar}{2m^*\omega}\right)^3 30\left(n^2+n+\frac{11}{30}\right). \qquad (23)$$

III. THE $\Delta_2 E_n/N$ CORRECTION

Let us analyze which terms of the series (19) contribute the correction $\Delta_2 E_n/N \sim (\hbar/N)^3$. From the first correction (20) the k-th term in the expansion of V is proportional to $(\hbar/N)^{k/2}$ and gives a contribution at $k = 6$:

$$\Delta_2 E_n^{(1)}/N = \beta_6 \langle n|z^6|n\rangle.$$

A simple calculation of this matrix element leads to

$$\Delta_2 E^{(1)} = N\beta_6 \left(\frac{\hbar}{2m^*\omega}\right)^3 5(4n^3+6n^2+8n+3). \qquad (24)$$

As a next step find a contribution from $E_n^{(2)}/N$ of the order $(\hbar/N)^3$. The product of the matrix elements in expression (21) numerator for $\hbar/N$ gives the power $(k_1+k_2)/2$ (where k_1 and k_2 are the powers of z in the matrix elements $V_{nn'}$ and $V_{n'n}$). Taking into account the denominator one obtains the condition: $(k_1+k_2)/2-1=3$ or $k_1+k_2=8$. This implies three possibilities:

$$k_1=3,\quad k_2=5,$$
$$k_1=5,\quad k_2=3,$$
$$k_1=k_2=4.$$

Extracting corresponding expressions from (21) and uniting the first two of them, on gets:

$$\Delta_2 E_n^{(2)}/N = 2\beta_3\beta_5 \sum_{n'\neq n} \frac{\langle n|x^5|n'\rangle\langle n'|x^3|n\rangle}{\hbar\omega(n-n')},$$

$$\Delta_3 E_n^{(2)}/N = \beta_4^2 \sum_{n'\neq n} \frac{|\langle n|x^4|n'\rangle|^2}{\hbar\omega(n-n')}.$$

After some not too complicated calculations one arrives with the following result:

$$\Delta_2 E_n^{(2)} = -N\frac{\beta_3\beta_5}{\hbar\omega}\left(\frac{\hbar}{2m^*\omega}\right)^4 \times 10(28n^3+42n^2+40n+13) \qquad (25)$$

$$\Delta_3 E_n^{(2)} = -N\frac{\beta_4^2}{\hbar\omega}\left(\frac{\hbar}{2m^*\omega}\right)^4 \times 2(34n^3+51n^2+59n+21). \qquad (26)$$

We now pass to the analysis of the third order of the perturbation theory in V:

$$\frac{E_n^{(3)}}{N} = \sum_{n'\neq n}\sum_{n''\neq n} \frac{V_{nn'}V_{n'n''}V_{n''n}}{(\hbar\omega)^2(n-n')(n-n'')} - V_{nn}\sum_{n'\neq n}\frac{|V_{nn'}|^2}{(\hbar\omega)^2(n-n')^2}. \qquad (27)$$

For the powers of z in the matrix elements of the first term in (27) one can write the following conditions: $(k_1+k_2+k_3)/2-2=3$ or $k_1+k_2+k_3=10$. The possibilities read:

$$k_1=k_2=3,\quad k_3=4,$$
$$k_1=k_3=3,\quad k_2=4,$$
$$k_2=k_3=3,\quad k_1=4.$$

One should notice that the contributions from the first and the third case coincide.

In the second term of the expression (27) one takes a contribution from z^4 in the diagonal term V_{nn} and the cubic one for $V_{nn'}$ in the expression under the sum. Thus, in order to find a contribution from the third order of the perturbation theory, one should calculate the expression:

$$\Delta E_n^{(3)} = 2\Delta_{11}E_n^{(3)} + \Delta_{12}E_n^{(3)} + \Delta_2 E_n^{(3)}, \qquad (28)$$

where

$$\Delta_{11}E_n^{(3)} = N\frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\sum_{n'\neq n}\sum_{n''\neq n}\frac{\langle n|z^3|n'\rangle\langle n'|z^3|n''\rangle\langle n''|z^4|n\rangle}{(n-n')(n-n'')},$$

$$\Delta_{12}E_n^{(3)} = N\frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\sum_{n'\neq n}\sum_{n''\neq n}\frac{\langle n|z^3|n'\rangle\langle n'|z^4|n''\rangle\langle n''|z^3|n\rangle}{(n-n')(n-n'')},$$

$$\Delta_2 E_n^{(3)} = -N\frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\langle n|x^4|n\rangle\sum_{n'\neq n}\frac{|\langle n|z^3|n'\rangle|^2}{(n-n')^2}.$$

After rather cumbersome calculations we arrive with the formulas:

$$\Delta_{11}E_n^{(3)} = N\frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\left(\frac{\hbar}{2m^*\omega}\right)^5(30n^5+75n^4+576n^3+789n^2+666n+204),$$

$$\Delta_{12}E_n^{(3)} = N\frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\left(\frac{\hbar}{2m^*\omega}\right)^5\frac{1}{3}(148n^5+370n^4+3112n^3+4298n^2+3238n+915),$$

$$\Delta_2 E_n^{(3)} = -N\frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\left(\frac{\hbar}{2m^*\omega}\right)^5\frac{1}{3}(328n^5+820n^4+1168n^3+932n^2+430n+87).$$

As a result the overall expression (28) reads:

$$\Delta E_n^{(3)} = N\frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\left(\frac{\hbar}{2m^*\omega}\right)^5 12(150n^3+225n^2+189n+57). \tag{29}$$

Finally, let us turn to the fourth order of the perturbation theory in V:

$$\frac{E_n^{(4)}}{N} = \sum_{n'\neq n}\sum_{n''\neq n}\sum_{n'''\neq n}\frac{V_{nn'}V_{n'n''}V_{n''n'''}V_{n'''n}}{(\hbar\omega)^3(n-n')(n-n'')(n-n''')}$$
$$+ V_{nn}^2\sum_{n'\neq n}\frac{|V_{nn'}|^2}{(\hbar\omega)^3(n-n')^3} - \frac{E_n^{(2)}}{N}\sum_{n'\neq n}\frac{|V_{nn'}|^2}{(\hbar\omega)^2(n-n')^2}$$
$$- 2V_{nn}\sum_{n'\neq n}\sum_{n''\neq n}\frac{V_{nn'}V_{n'n''}V_{n''n}}{(\hbar\omega)^3(n-n')^2(n-n'')}. \tag{30}$$

Similarly to the previous, from the condition $(k_1+k_2+k_3+k_4)/2-3=3$, i. e., $(k_1+k_2+k_3+k_4)=12$, one finds only one possibility yielding a contribution $\sim(\hbar/N)^3$: $k_1=k_2=k_3=k_4=3$.

Since $\langle n|z^3|n\rangle=0$, in (30) terms with diagonal matrix elements are absent. Thus, the contribution $\sim(\hbar/N)^3$ from the fourth order of the perturbation theory in V reads:

$$\Delta E^{(4)} = \Delta_1 E_n^{(4)} + \Delta_2 E_n^{(4)}, \tag{31}$$

where

$$\Delta_1 E_n^{(4)} = N\frac{\beta_3^4}{(\hbar\omega)^3}\sum_{n'\neq n}\sum_{n''\neq n}\sum_{n'''\neq n}\frac{\langle n|z^3|n'\rangle\langle n'|z^3|n''\rangle\langle n''|z^3|n'''\rangle\langle n'''|z^3|n\rangle}{(n-n')(n-n'')(n-n''')},$$

$$\Delta_2 E_n^{(4)} = -\Delta_1 E_n^{(2)}\frac{\beta_3^2}{(\hbar\omega)^2}\sum_{n'\neq n}\frac{|\langle n|x^3|n'\rangle|^2}{(n-n')^2}.$$

Extremely tedious and exhausting calculations yield:

$$\Delta_1 E_n^{(4)} = -N\frac{\beta_3^4}{(\hbar\omega)^3}\left(\frac{\hbar}{2m^*\omega}\right)^6\left(\frac{3109}{3}+\frac{34856}{9}n+5674n^2+\frac{42244}{9}n^3+\frac{4100}{3}n^4+\frac{1640}{3}n^5\right),$$

$$\Delta_2 E_n^{(4)} = N\frac{\beta_3^4}{(\hbar\omega)^3}\left(\frac{\hbar}{2m^*\omega}\right)^6\left(\frac{319}{3}+\frac{5426}{9}n+1444n^2+\frac{16864}{9}n^3+\frac{4100}{3}n^4+\frac{1640}{3}n^5\right),$$

so sum (31) reads:

$$\Delta E_n^{(4)} = -N\frac{\beta_3^4}{(\hbar\omega)^3}\left(\frac{\hbar}{2m^*\omega}\right)^6 30(31+109n+141n^2+94n^3). \tag{32}$$

An analysis implies that higher corrections of the perturbation theory in V do not contribute the order $\sim(\hbar/N)^3$ and therefore the final result for $\Delta_2 E_n$ is as follows:

$$\Delta_2 E_n = \Delta_2 E_n^{(1)} + \Delta_2 E_n^{(2)} + \Delta_3 E_n^{(2)} + \Delta E_n^{(3)} + \Delta E_n^{(4)}.$$

Collecting expressions (24)–(26), (29) and (32) together, we finally arrive with the following:

$$\frac{\Delta_2 E_n}{N} = \beta_6\left(\frac{\hbar}{2m^*\omega}\right)^3 5(4n^3+6n^2+8n+3) - \frac{\beta_4^2}{\hbar\omega}\left(\frac{\hbar}{2m^*\omega}\right)^4 2(34n^3+51n^2+59n+21)$$

$$- \frac{\beta_3\beta_5}{\hbar\omega}\left(\frac{\hbar}{2m^*\omega}\right)^4 10(28n^3+42n^2+40n+13) + \frac{\beta_3^2\beta_4}{(\hbar\omega)^2}\left(\frac{\hbar}{2m^*\omega}\right)^5 12(150n^3+225n^2+189n+57)$$
$$- \frac{\beta_3^4}{(\hbar\omega)^3}\left(\frac{\hbar}{2m^*\omega}\right)^6 30(94n^3+141n^2+109n+31). \qquad (33)$$

This expression, obtained here for the first time, allows to find corrections to eigenvalues of a particle energy per one degree of freedom up to the order $(1/N)^3$. This can be performed by a simple calculation of the effective potential (17) derivatives at the point of the potential minimum.

IV. ENERGY LEVELS IN THE FIELD OF A POWER-LIKE POTENTIALS

Following [6] let us consider a class of potentials with the power dependence on the distance:

$$U = U_0 s y^s, \qquad (34)$$

where the parameter s can be both positive and negative. We adduce some particular cases in the conventional notation:

$$s = 2, \qquad U_0 = \frac{m\omega_0^2}{4} \qquad (35)$$

is a harmonic oscillator with the frequency ω_0;

$$s = -1, \qquad U_0 = \frac{Ze^2}{N\sqrt{N}} \qquad (36)$$

is a particle in the Coulomb attracting field of a force centre with a charge Z (hydrogen atom);

$$s = 4, \qquad U_0 = \frac{\alpha}{4}, \qquad \alpha = \text{const} \qquad (37)$$

is an oscillator "x^4";

$$s = 1, \qquad U_0 = \frac{\beta}{2\sqrt{N}}, \qquad \beta = \text{const} \qquad (38)$$

is a linear potential ("x"-oscillator).

One can find from (9), (12), and (34) that the effective potential

$$w(y) = \nu\frac{\hbar^2}{8my^2} + U_0 s y^s \qquad (39)$$

reaches its minimum at the point

$$y_0 = \left(\frac{\nu\hbar^2}{4mU_0 s^2}\right)^{\frac{1}{s+2}}. \qquad (40)$$

The minimal value of the potential reads:

$$w_0 = \nu\frac{s+2}{s}\frac{\hbar^2}{8my_0^2}, \qquad (41)$$

the frequency (15) is written as follows:

$$\omega = \frac{\hbar\sqrt{\nu(s+2)}}{2my_0^2 N}. \qquad (42)$$

Inserting expressions (41), (42) into (22) and taking into account the value (40) for y_0, within the accepted approximation we truncate the series of the correction to E_n at $\Delta_2 E_n$:

$$E_n = N\frac{s+2}{s}\frac{\hbar^2}{8m}\left(\frac{4mU_0s^2}{\hbar^2}\right)^{\frac{2}{s+2}}\left[\nu^{\frac{s}{s+2}} + \frac{4s}{N\sqrt{s+2}}\,\nu^{\frac{s-2}{2(s+2)}}\left(n+\frac{1}{2}\right)\right] + \Delta_1 E_n + \Delta_2 E_n. \qquad (43)$$

Let us now write the coefficients β_k necessary for the calculation of the corrections $\Delta_1 E_n$ from (23) and $\Delta_2 E_3$ from (33). These coefficients determine the intensity of the corresponding order anharmonicities in the perturbation potential V. One can find from (17):

$$\beta_3 = \frac{\nu\hbar^2}{8my_0^5}\left[\frac{(s-1)(s-2)}{3} - 4\right],$$

$$\beta_4 = \frac{\nu\hbar^2}{8my_0^6}\left[\frac{(s-1)(s-2)(s-3)}{3\cdot 4} + 5\right],$$

$$\beta_5 = \frac{\nu\hbar^2}{8my_0^7}\left[\frac{(s-1)(s-2)(s-3)(s-4)}{3\cdot 4\cdot 5} - 6\right],$$

$$\beta_6 = \frac{\nu\hbar^2}{8my_0^8}\left[\frac{(s-1)(s-2)(s-3)(s-4)(s-5)}{3\cdot 4\cdot 5\cdot 6} + 7\right].$$

Substituting the above expressions into (23) and (33) and accounting (40) one obtains:

$$\frac{\Delta_1 E_n}{N} = -\frac{\hbar^2}{24mN^2}\left(\frac{4mU_0 s^2}{\hbar^2}\right)^{\frac{2}{s+2}} \nu^{-\frac{2}{s+2}} \times \left[n(n+1)(s-2)(s-11) + \frac{s^2-19s+16}{6}\right], \tag{44}$$

$$\frac{\Delta_2 E_n}{N} = \frac{\hbar^2}{8mN^3}\left(\frac{4mU_0 s^2}{\hbar^2}\right)^{\frac{2}{s+2}} \nu^{-\frac{s+6}{2(s+2)}} \frac{(s-2)}{\sqrt{s+2}}\left[n^3\left(-\frac{461}{54} + \frac{67}{36}s - \frac{2}{9}s^2 + \frac{5}{108}s^3\right)\right.$$

$$\left. + n^2\left(-\frac{461}{36} + \frac{67}{24}s - \frac{s^2}{3} + \frac{5}{72}s^3\right) + \left(-\frac{487}{108} + \frac{97}{72}s - \frac{s^2}{9} + \frac{7}{216}s^3\right) + \left(-\frac{13}{108} + \frac{5}{24}s + \frac{s^3}{216}\right)\right]. \tag{45}$$

In order to obtain an explicit $1/N$-expansion of the energy (22), in expressions (43), (44), (45) one should additionally expand the parameter ν (10) in the corresponding powers:

$$\nu^{\frac{s}{s+2}} = 1 + \frac{4s(l-1)}{N(s+2)} + \frac{s[(s+2)(2l-1)(2l-3) - 16(l-1)^2]}{N^2(s+2)^2}$$

$$+ \frac{8s(l-1)[8(s+4)(l-1)^2 - 3(s+2)(2l-1)(2l-3)]}{3N^3(s+2)^3} + O\left(\frac{1}{N^4}\right),$$

$$\nu^{\frac{s-2}{2(s+2)}} = 1 + \frac{2(s-2)(l-1)}{N(s+2)} + \frac{(s-2)[(s+2)(2l-1)(2l-3) - 4(s+6)(l-1)^2]}{2N^2(s+2)^2} + O\left(\frac{1}{N^3}\right),$$

$$\nu^{-\frac{s}{s+2}} = 1 - \frac{8(l-1)}{N(s+2)} + O\left(\frac{1}{N^2}\right).$$

In expression (45) for $\Delta_2 E$ the parameter ν equals one. Substituting these expansions into formula (43) for the energy E_n and collecting terms at corresponding powers of the parameter $1/N$, we finally find (with explicit noting the dependence of the energy on two quantum numbers, radial n and orbital l):

$$\frac{E_{n,l}}{N} = \frac{\hbar^2}{2m}\left(\frac{4mU_0 s^2}{\hbar^2}\right)^{\frac{2}{s+2}}\left[\varepsilon_0 + \frac{\varepsilon_1}{N} + \frac{\varepsilon_2}{N^2} + \frac{\varepsilon_3}{N^3} + O\left(\frac{1}{N^4}\right)\right], \tag{46}$$

where

$$\varepsilon_0 = \frac{s+2}{4s},$$

$$\varepsilon_1 = n\sqrt{s+2} + l + \frac{1}{2}\sqrt{s+2} - 1,$$

$$\varepsilon_2 = \frac{s-2}{s+2}\left[\frac{106+15s-s^2}{72} - \sqrt{s+2} + l\sqrt{s+2} + l(l-2) + 2n(l-1)\sqrt{s+2} - \frac{n(n+1)(s-11)(s+2)}{12}\right],$$

$$\varepsilon_3 = \frac{(s-2)}{4(s+2)}\left\{ -\frac{13}{108} + \frac{5s}{24} + \frac{s^3}{216} + \frac{232}{9(s+2)^{3/2}} - \frac{4s(s-15)}{9(s+2)^{3/2}} - \frac{s+18}{s+2} + \frac{n^3}{108}\left[(5s^2-24s+201)s - 922\right]\right.$$

$$+ \frac{4l}{s+2}\left(8 + \frac{s^2-15s-106}{9\sqrt{s+2}}\right) + \frac{16l^2}{s+2}\left(\frac{2}{\sqrt{s+2}} - 1\right) - \frac{32l^3}{3(s+2)^{3/2}} \tag{47}$$

$$+ n^2\left[\frac{8(s-11)(l-1)}{3\sqrt{s+2}}+\frac{(5s^2-24s+201)s-922}{72}\right]$$

$$+ n\left[\frac{(7s^2-24s+291)s-974}{216}-\frac{8(s-11)}{3\sqrt{s+2}}-\frac{2(s+18)}{s+2}+8l\left(\frac{8}{s+2}+\frac{s-11}{3\sqrt{s+2}}\right)-\frac{32l^2}{s+2}\right]\Bigg\}.$$

The first terms of the expansion (46) were found in an algebraic technique in [6]. Also, they were obtained in [9] by solving the N-dimensional Schrödinger equation for the power-dependent potential. The expression for ε_3 is a new result.

Let us check out the results treating models that permit an exact solution. Namely, consider harmonic oscillator and hydrogen atom. For the oscillator, according to (35), one finds: $\varepsilon_2 = \varepsilon_3 = 0$, so the formula (46) yields:

$$E_{n,l} = \hbar\omega_0\left(2n+l+\frac{N}{2}\right)+O\left(\frac{1}{N^3}\right).$$

Thus, as it should be, the perturbation theory series in $1/N$ reconstructs the exact result.

For the hydrogen atom from formulas (46) and (47) taking into account (36) one obtains:

$$E_{n,l} = \frac{8mZ^2e^4}{\hbar^2N^2}\left[-\frac{1}{4}+\frac{(n+l-1/2)}{N}\right. \tag{48}$$
$$\left.-\frac{3}{4N^2}(2n+2l-1)^2+\frac{(2n+2l-1)^3}{N^3}+O\left(\frac{1}{N^4}\right)\right].$$

One can easily see that these four terms reconstruct an expansion of the exact formula for the energy levels of the N-dimensional hydrogen atom correctly:

$$E_{n,l} = -\frac{2mZ^2e^4}{\hbar^2(2n+2l+N-1)^2}.$$

In particular, for the ground state in the case $N = 3$ expression (48) gives:

$$E_{0,0}\Big/\frac{me^4Z^2}{2\hbar^2} = -0.9547$$

which is very close to the exact value.

V. THE OSCILLATOR MODELS

Let us study here the ground state of an anharmonic oscillator "x^4". From expressions (46), (47), taking into account (37) one gets:

$$\frac{E_{0,0}}{N} = \left(\frac{2\alpha\hbar^4}{m^2}\right)^{1/3}\left[\frac{3}{8}+\frac{1}{N}\left(\sqrt{\frac{3}{2}}-1\right)\right.$$
$$-\frac{1}{N^2}\left(\sqrt{\frac{2}{3}}-\frac{25}{36}\right) \tag{49}$$
$$\left.+\frac{1}{N^3}\left(\frac{17}{27}-\frac{287}{432}\sqrt{\frac{2}{3}}\right)+O\left(\frac{1}{N^4}\right)\right].$$

As an example write the energy value in the case $N = 1$ when the parameter $1/N$ is no longer small:

$$E_{0,0}\Big/\left(\frac{2\alpha\hbar^4}{m^2}\right)^{1/3} = 0.5648813.$$

The exact value of this quantity is well-known, it equals 0.5301810. Thus, the expansion (49) reproduces this result quite well. The result is unexpectedly good taking into account that for $N = 1$ and $l = 0$ the effective potential w, as one can see from (9), (10), equals U, so the passage to the limit $N \to 1$ is "illegal".

Let us turn now to the "x"-oscillator model (38). For the ground state from (46), (47) one finds:

$$\frac{E_{0,0}}{N} = \left(\frac{\hbar^2\beta^2}{2mN}\right)^{1/3}\left[\frac{3}{4}+\frac{1}{N}\left(\frac{\sqrt{3}}{2}-1\right)\right.$$
$$+\frac{1}{3N^2}\left(\sqrt{3}-\frac{5}{3}\right) \tag{50}$$
$$\left.+\frac{1}{9N^3}\left(\frac{337}{72}\sqrt{3}-8\right)+O\left(\frac{1}{N^4}\right)\right].$$

The numerical value for $N = 1$ obtained from (50) reads:

$$E_{0,0} = 0.649705\left(\frac{\hbar^2\beta^2}{2m}\right)^{1/3},$$

while the exact value of this numerical coefficient equals 0.641799. One can see the $1/N$-expansion (50) to be in a good agreement with the exact result.

VI. CONCLUSIONS

In the frames of the ordinary perturbation theory for an anharmonic oscillator as an auxiliary problem, we have found with the $(\hbar/N)^3$ accuracy the energy levels of a particle moving in the N-dimensional space in the centrosymmetric field. These expressions are presented by formulas (22), (23), and (33). The obtained expressions permit to find the energy levels for any potential provided corresponding derivatives at the minimum point exist. In particular, we have studied the validity of our techniques in the case of potentials with the power dependence on the distance. Even in the worst case $N = 1$ the agreement with the exact result for the anharmonic oscillator appeared quite good. Possessing several terms of a series, it is possible to improve the convergence of the $1/N$ expansion by means of Padé-approximants or continued fractions. Using these formulae one can find criteria of the bound states existence and disappearance depending on the potential parameters. An estimate for this approximate approach regarding the excited states may be given by the so-called quasi-exact solutions obtained within the supersymmetric quantum mechanics. The general expressions found here permit to find the energy levels of a two-particle cluster which are necessary in the investigation of many-particle system thermodynamic functions within the cluster approaches.

ACKNOWLEDGEMENTS

I express a gratitude to my colleagues V. Tkachuk and A. Rovenchak for the discussions.

[1] T. Holstein, H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[2] T. H. Berlin, M. Kac, Phys. Rev. **86**, 821 (1952).
[3] H. J. Lipkin, M. Meshkov, A. J. Glick, Nucl. Phys. **62**, 188, 211 (1965).
[4] H. E. Stanley, Phys. Rev. **176**, 718 (1968).
[5] G. t'Hooft, Nucl. Phys. B **72**, 461 (1974); B **75**, 461 (1974).
[6] L. D. Mlodinov, N. Papanicolau, Ann. Phys. **128**, 314 (1980).
[7] L. D. Mlodinow, M. P. Shatz, J. Math. Phys. **25**, 943 (1984).
[8] H. Bateman, A. Erdelyi, *Higher Transcendental Function. V. 2* (McGraw-Hill Book Co., New York, Toronto, London, 1953).
[9] U. Sukhatme, T. Imbo, Phys. Rev. D **28**, 418 (1983).

$1/N$-РОЗКЛАД У КВАНТОВІЙ МЕХАНІЦІ. ВИЩІ НАБЛИЖЕННЯ

І. О. Вакарчук
Львівський національний університет імені Івана Франка,
фізичний факультет, кафедра теоретичної фізики
вул. Драгоманова, 12, Львів, 79005, Україна

Для частинки, що рухається в центральносиметричному полі в N-вимірному просторі, методами звичайної теорії збурень Релея–Шредінґера для ангармонічного осцилятора обчислено спектр енерґій з точністю до $(\hbar/N)^3$ включно. Як приклад знайдено рівні енерґії для потенціялів зі степеневою залежністю від відстані.