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NONLINEAR THEORY OF STOCHASTIC RESONANCE

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Theory of nonlinear resonance, including stochastic one, is developed on the basis of the statistical field theory and using variables action-angle. Explicit expressions of action, proper frequency and nonlinearity parameter as functions of the system energy and the external signal frequency are found for the cases of nonlinear pendulum and double well potential.

Key words: nonlinear resonance; variables action–angle

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I. INTRODUCTION

Since the late 1980s, a variety of theoretical and experimental papers appears devoted to the study of stochastic resonance and discovering its new applications in different fields of science and engineering [1]. Nowadays, the stochastic resonance is a well established phenomenon displayed in a bistabile system simultaneously driven by noise and a periodic signal. There appears to be an optimal noise level at which the system exhibits almost periodic transitions from one state to other with the frequency of the coherent signal. The enhancement of a weak input signal has been suggested to characterize by the signal-to-noise ratio (SNR) that takes a maximum value at an optimal noise level, i. e., a behaviour which is reminiscent of a usual resonance phenomenon.

The observed enhancement is not due to the matching of two frequencies, but rather to a cooperative effect of the coherent signal and the noise. This peculiarity is reflected formally as follows. The kinematic condition

$$\pi r_K = \Omega \tag{1}$$

relates the frequency of the coherent signal Ω to the Kramers rate r_K, which is the function of the noise level type of temperature T:

$$r_K = \frac{\omega_K}{2\pi} \exp\left(-\frac{\Delta F}{T}\right), \quad \omega_K \equiv \frac{\sqrt{F''(x_0)|F''(x_b)|}}{m\gamma}. \tag{2}$$

Here, the prime denotes the derivation with respect to a generalized coordinate x, $F(x)$ is a related dependence of the effective energy, which has a minimum at the point x_0, a maximum at x_b and a barrier of the height $\Delta F \equiv F(x_b) - F(x_0)$; m is a particle mass, γ is a kinetic coefficient. Physically, the kinematic condition means that for every frequency Ω, the stochastic system could pick out suitable noise level T. However, the value SNR [1]

$$SNR \propto T^{-2} r_K(T), \tag{3}$$

being found on the basis of the statistical theory, displays the stochastic resonance maximum at the temperature

$$T_m = \frac{1}{2}\Delta F, \tag{4}$$

whose value is fixed by the energy barrier ΔF, but not the external signal frequency Ω.

It is easy to see that the reason of this contradiction is using the linear approach for the SNR determination. In this paper, we develop the theory of nonlinear resonance, including the stochastic one. The main ingredients of our approach are: (i) the noise accounting by means of introduction of a generalized momentum and (ii) passage to the variables action-angle, being usual at studying nonlinear phenomena [2]. Section II is devoted to the first of these approaches on the basis of the statistical field theory [3]. Section III contains details of examination of nonlinear resonance in terms of variables action–angle. The latter are tested on the example of the simplest model of a harmonic oscillator. Then, explicit expressions of action, proper frequency and nonlinearity parameter as functions of the system energy are found for the cases of nonlinear pendulum, double well potential and nonlinear pendulum under constant external field. Assuming the proper frequency be identical to the frequency of external signal, we examine resonance conditions of the nonlinear pendulum in Section IV and the case of the stochastic resonance in Section V. Conclusion in Section VI shows that the above mentioned contradiction is resolved because the nonlinear resonance condition fixes the proper frequency as function of the external one, but not the noise level, as in the linear case.

II. BASIC EQUATIONS

Let us study a hydrodynamic mode amplitude $x(\mathbf{r}, t)$ which space-time dependence is determined by the Langevin equation

$$\dot{x}(\mathbf{r},t) = \gamma f(x,t) + \zeta(\mathbf{r},t), \tag{5}$$

which can be interpreted in the Ito sense. Here the dot indicates the differentiation with respect to time t, $\mathbf{r}$ is coordinate, γ is a kinetic coefficient, $f(x,t)$ is a force conjugated to the stochastic variable x, and $\zeta(\mathbf{r},t)$ is a stochastic term having the form of white noise:

$$\langle\zeta(\mathbf{r},t)\rangle = 0, \quad \langle\zeta(\mathbf{r},t)\zeta(0,0)\rangle = T\delta(\mathbf{r})\delta(t), \tag{6}$$

where the angle brackets denote averaging and T is a noise intensity of temperature type [4]. For the system under consideration, the total force

$$f(x,t) = f_0(x) + f_{\rm ext}(t) \tag{7}$$

consists of the usual internal term

$$f_0(x(\mathbf{r},t)) = -\frac{\delta\mathcal{F}}{\delta x(\mathbf{r},t)},$$
$$\mathcal{F}\{x(\mathbf{r},t)\} = \int\left[F(x) + \frac{\beta}{2}|\nabla x|^2\right]\mathrm{d}\mathbf{r}, \tag{8}$$

where $F(x)$ is a system potential per unit volume, $\beta > 0$ is a constant, $\nabla \equiv \partial/\partial\mathbf{r}$, and external harmonic term

$$f_{\rm ext}(t) = A\cos(\Omega t + \varphi) \tag{9}$$

being determined by an amplitude A, frequency Ω, and an initial phase φ. It is now convenient to go over to dimensionless quantities by referring the coordinate $\mathbf{r}$ to a characteristic spacing a, the time t and the inverse frequency Ω^{-1} to the scale $a^3/T\gamma$, the amplitude A, the internal force f_0, and the quantity F to T/a^3, and the fluctuation ζ to $T\gamma/a^3$. In this case, Eq. (5) can be written as follows:

$$\dot{x} = [\nabla^2 x + f(x,t)] + \zeta(t),$$
$$f(x,t) = f_0(x) + f_{\rm ext}(t);$$
$$f_0 \equiv -\partial F/\partial x,$$
$$f_{\rm ext}(t) = A\cos(\Omega t + \varphi). \tag{10}$$

The range of applicability of the Ginzburg–Landau approximation (8) is determined by the condition according to which the scale a is much smaller than the correlation length $\xi = \beta^{1/2}|\partial^2 F/\partial x^2|_{x=0}^{-1/2}$ [6]. Averaging Eq. (5) and disregarding correlations, we obtain the Landau–Khalatnikov equation for the order parameter $\langle x(\mathbf{r},t)\rangle$.

The standard field scheme [3] is based on the investigation of the generating functional corresponding to the stochastic equation (10). It is a functional Laplace transform

$$Z\{u(\mathbf{r},t)\} = \int Z\{x(\mathbf{r},t)\}\exp\left(\int ux\,\mathrm{d}\mathbf{r}\mathrm{d}t\right)Dx(\mathbf{r},t) \tag{11}$$

for the generalized partition function

$$Z\{x(\mathbf{r},t)\} = \left\langle\prod_{(\mathbf{r},t)}\delta\left\{\dot{x}(\mathbf{r},t) - \nabla^2 x(\mathbf{r},t) - f(\mathbf{r},t) - \zeta(\mathbf{r},t)\right\}\det\left|\frac{\delta\zeta(\mathbf{r},t)}{\delta x(\mathbf{r},t)}\right|\right\rangle_\zeta. \tag{12}$$

Here, the argument of the δ-function reduces to the Langevin equation (10), and the determinant, providing the passage from the continual integration over $\zeta(\mathbf{r},t)$ to $x(\mathbf{r},t)$, is equal to the unity within the Ito calculus.

In the framework of the standard approach [3], the n-fold variation of the functional (11) with respect to the auxiliary field $u(\mathbf{r},t)$ allows one to find the n-th order correlator for the hydrodynamic mode amplitude $x(\mathbf{r},t)$ and to construct the perturbation theory. However, we shall proceed from expression (12) for the conjugated functional $Z\{x(\mathbf{r},t)\}$, variation of which leads to the most probable realization of the stochastic field $x(\mathbf{r},t)$. Obviously, in the framework of the mean-field approximation functional (12) reduces to the dependence $Z\{\langle x(\mathbf{r},t)\rangle\}$, which corresponds to the Landau free energy $F\{\langle x(\mathbf{r},t)\rangle\} = -T\ln Z\{\langle x(\mathbf{r},t)\rangle\}$ [6].

Passing to the consideration of functional (12), we represent the δ-function in the integral form

$$\delta\{x(\mathbf{r},t)\} = \int_{-i\infty}^{i\infty}\exp\left(-\int px\,\mathrm{d}\mathbf{r}\mathrm{d}t\right)Dp. \tag{13}$$

Then, averaging over the noise ζ with using the Gauss distribution

$$P_0\{\zeta\} \propto \exp\left\{-\frac{1}{2}\int\zeta^2(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\mathrm{d}t\right\}, \tag{14}$$

which corresponds to condition (6), and taking into account Eq. (13), we reduce functional (12) to the standard form

$$Z\{x(\mathbf{r},t)\} = \int P\{x(\mathbf{r},t), p(\mathbf{r},t)\} Dp, \quad P \equiv e^{-S}. \quad (15)$$

Here, probability distribution $P\{x,p\}$ is given by action $S = \int \mathcal{L}\,\mathrm{d}\mathbf{r}\mathrm{d}t$, where Lagrangian is

$$\mathcal{L} = p\left(\dot{x} - \nabla^2 x - f\right) - p^2/2. \quad (16)$$

Further, we use Euler equations

$$\frac{\partial \mathcal{L}}{\partial \mathrm{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathrm{x}}} - \nabla \frac{\partial \mathcal{L}}{\partial \nabla \mathrm{x}} + \nabla^2 \frac{\partial \mathcal{L}}{\partial \nabla^2 \mathrm{x}} = \frac{\partial \mathcal{R}}{\partial \dot{\mathrm{x}}}, \quad \mathrm{x} \equiv \{x, p\}, \quad (17)$$

where dissipative function is $\mathcal{R} = \frac{1}{2}\dot{x}^2$. As a result, equations for the most probable realizations of the stochastic fields $x(\mathbf{r},t)$, $p(\mathbf{r},t)$ take the form:

$$\dot{x} = (\nabla^2 x + f) + p, \quad (18)$$

$$\dot{p} = -\nabla^2 p - f' p - \dot{x}, \quad (19)$$

where prime stands for derivation with respect to variable x. A comparison of (18) with the stochastic equation (10), having the same form, shows that the field $p(\mathbf{r},t)$ is the most probable value of the fluctuations $\zeta(\mathbf{r},t)$ of the conjugate force. Differentiating Eq. (18) with respect to the time and inserting result into Eq. (19), we obtain the equation of motion as follows:

$$\ddot{x} + (1 + f')\dot{x} = 2f'\nabla^2 x + \dot{f} + f f', \quad (20)$$

where only terms of the lowest order of spatiotemporal derivations are kept.

Using the described field theory allows us to pass from the differentiation stochastic equation of motion (5) of the first order to the equivalent system of two differentiation equations (18), (19) of the same order, or to the single differentiation equation (20) of the second order. Further, we need in using Hamiltonian $\mathcal{H} = p\dot{x} - \mathcal{L}$ that depends on the field variable x and the conjugate momentum p. According to Eq. (16), Hamiltonian can be written in the form

$$\begin{aligned} &\mathcal{H}(x,p;t) = \mathcal{H}_0(x,p) + \mathcal{H}_1(p,t); \\ &\mathcal{H}_0 = -\nabla x \nabla p + \frac{1}{2}p^2 + p f_0, \\ &\mathcal{H}_1 = A p \cos(\Omega t + \varphi). \end{aligned} \quad (21)$$

It is easy to see that these expressions, being inserted into dissipative Hamilton equations

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} - \nabla \frac{\partial \mathcal{H}}{\partial \nabla p}, \quad (22)$$

$$\dot{p} = -\left(\frac{\partial \mathcal{H}}{\partial x} - \nabla \frac{\partial \mathcal{H}}{\partial \nabla x}\right) - \frac{\partial \mathcal{R}}{\partial \dot{x}},$$

lead to the equations of motion (18), (19).

It is very important to take into account further that the most probable amplitude p of fluctuations varies near the magnitude $-f_0$, so that we ought to pass to oscillating momentum: $p + f_0 \to p$. Moreover, as it is seen from the Fourier transformation (13), the momentum p takes imaginary magnitudes, so that the power f_0 has to be considered as the imaginary one, as well, and the sign in front of the last term of $\mathcal{H}_0$ in Eqs. (21) must be reversed [5]:

$$\begin{aligned} &\mathcal{H}(x,p;t) = \mathcal{H}_0(x,p) + \mathcal{H}_1(p,t); \\ &\mathcal{H}_0 = \frac{1}{2}p^2 + \frac{1}{2}f_0^2, \\ &\mathcal{H}_1 = -A f_0 \cos(\Omega t + \varphi), \end{aligned} \quad (23)$$

where gradient terms are suppressed for brevity.

III. NONLINEAR RESONANCE IN TERMS OF VARIABLES ACTION–ANGLE

To analyze the set of equations (18), (19), it is convenient to use the phase portrait method. However, in our case such a portrait flows in the course of the time due to the appearance of the time-dependent external force (9). To avoid this time-variation in phase portrait we need to pass from the above used variables x, p to the new ones: action I and angle ϑ defined as follows [2]:

$$\begin{aligned} &I(\mathcal{H}) \equiv \frac{1}{2\pi}\oint p(x,\mathcal{H})\,\mathrm{d}x, \\ &\vartheta \equiv \frac{\partial S(x,I)}{\partial I}, \\ &S(x,I) \equiv \int_0^x p(x', \mathcal{H}(I))\,\mathrm{d}x', \end{aligned} \quad (24)$$

where the shorted action $S(x,I)$ plays a role of the generalized function. The convenience of the so-introduced variables is that the zero Hamiltonian $\mathcal{H}_0$ in Eqs. (23) does not depend on the angle ϑ, so that the corresponding phase portrait is stable in the course of the time. The equations of motion for the generalized coordinate ϑ and the conjugate momentum I read (cf. Eqs. (23))

$$\dot{\vartheta} = \frac{\partial \mathcal{H}}{\partial I}, \quad \dot{I} = -\frac{\partial \mathcal{H}_1}{\partial \vartheta}, \quad (25)$$

where non-homogeneity and dissipation effects are suppressed. According to the second of these equations the action I is a constant if an external perturbation is absent.

A. Harmonic oscillator

To recall a physical meaning of the variables ϑ, I introduced, let us consider firstly the simplest case of harmonic oscillator. In this case, the internal power in Eq. (23)

$$f_0 = -\omega_0 x \tag{26}$$

is linear and fixed by a proper frequency ω_0. Then, the Hamilton equations (23) lead to the equation of damping oscillation under external power:

$$\ddot{x} + \dot{x} + \omega_0^2 x = -A\omega_0 \cos(\Omega t + \varphi). \tag{27}$$

This equation differs crucially from Eq. (20) because the former corresponds to the momentum origin $p = 0$, whereas the latter — to $p = -f_0$. According to Eq.(27), dissipation shifts resonance frequency from proper magnitude ω_0 to value

$$\varpi = \sqrt{\omega_0^2 - 2^{-2}}, \tag{28}$$

whereas a maximum real part of characteristic relation x/A relates to frequency

$$\omega_{\max} = \sqrt{\omega_0^2 - 2^{-1}}. \tag{29}$$

Such a character of the dissipation influence keeps at accounting for anharmonicity effects if under parameter ω_0 one means a proper frequency $\omega(\mathcal{H})$ of nonlinear oscillations depending on the system energy.

To demonstrate advantages of using variables action-angle, let us calculate now their magnitudes at the condition that external power in Hamiltonian (23) is switched off. Then, the first Eqs. (24), (25) give immediately

$$I = \frac{\mathcal{H}}{\omega_0}, \quad \dot{\vartheta} = \omega_0. \tag{30}$$

Respectively, the shorted action and angle takes the form:

$$S = I\left[\arcsin\left(\frac{x}{x_0}\right) + \frac{x}{x_0}\sqrt{1 - \left(\frac{x}{x_0}\right)^2}\,\right];$$

$$\vartheta = \arcsin\left(\frac{x}{x_0}\right), \quad x_0^2 \equiv \frac{2I}{\omega_0}. \tag{31}$$

The last of Eqs. (30) gives the usual relation between the angle and the time

$$\vartheta = \omega_0 t + \vartheta_0, \tag{32}$$

the using of which arrives at the harmonic laws of motion

$$x = (2\mathcal{H}/\omega_0^2)^{1/2}\sin(\omega_0 t + \vartheta_0),$$

$$p = (2\mathcal{H})^{1/2}\cos(\omega_0 t + \vartheta_0). \tag{33}$$

B. Nonlinear pendulum

The simplest model of the system with the possibility of the barrier overcoming is known to be the nonlinear pendulum (in this Subsection, we consider the pendulum without a friction and an external perturbation). Here, Hamiltonian takes the form

$$\mathcal{H}_0 = \frac{1}{2}p^2 + 2\omega_0^2 \sin^2(x/2), \tag{34}$$

corresponding to the power $f_0 = -2\omega_0 \sin(x/2)$ in Eqs. (23). Hamilton equations (23) arrive at the system

$$\dot{x} = p, \quad \dot{p} = -\omega_0^2 \sin x. \tag{35}$$

Combination of these equations gives the nonlinear one:

$$\ddot{x} + \omega_0^2 \sin x = 0. \tag{36}$$

This equation is non-solvable in analytical form and we ought to use the phase portrait method. The form of this portrait follows from Eqs. (35) to be shown in Fig. 1a. It is seen that the system behaviour is governed by energy $\mathcal{H}$ with respect to the critical value $\mathcal{H}_c \equiv 2\omega_0^2$. At condition $\mathcal{H} < \mathcal{H}_c$, the system moves finitely, whereas with overcoming the critical energy $\mathcal{H}_c$ it passes to infinite motion. Let us describe such a behaviour quantitatively.

In this line, the simplest topic is the solution corresponding to separatrix, for which the energy is critical one: $\mathcal{H} = \mathcal{H}_c$. In such a case, the definition (34) gives the separatrix form as follows:

$$p = \pm 2\omega_0 \cos(x/2). \tag{37}$$

Then, the first of Eqs. (35) arrives at the separatrix law of motion

$$x = 4\arctan\exp(\pm\omega_0 t) - \pi, \tag{38}$$

where the different signs correspond to upper and lower branches of the separatrix. This dependence can be written in much more elegant form of $\cos(x/2) = [\cosh(\pm\omega_0 t)]^{-1}$, the insertion of which into Eq. (37) arrives at the famous soliton dependence

$$p = \pm\frac{2\omega_0}{\cosh(\omega_0 t)}, \tag{39}$$

where choice of signs corresponds to solitons moving to right or left sides. In accordance with the motion laws (38), (39), the system behaviour on the separatrix (37) is as follows: at time $t = -\infty$ the system is located in the saddles $S_{\mp}$, where the coordinate $x = \mp\pi$ and the momentum $p = 0$. In the course of the time within the domain $-\infty < t < \infty$, the former arises monotonously from $-\pi$ to π, whereas the latter increases at $t < 0$ and decreases at $t > 0$. It is characteristical that coordinate variation and finite magnitudes of the momentum take place within the domain $\Delta t \sim \omega_0^{-1}$ located near the time $t = 0$. The forms of the corresponding kink $x(t)$ and soliton $p(t)$ are depicted in Fig. 2.

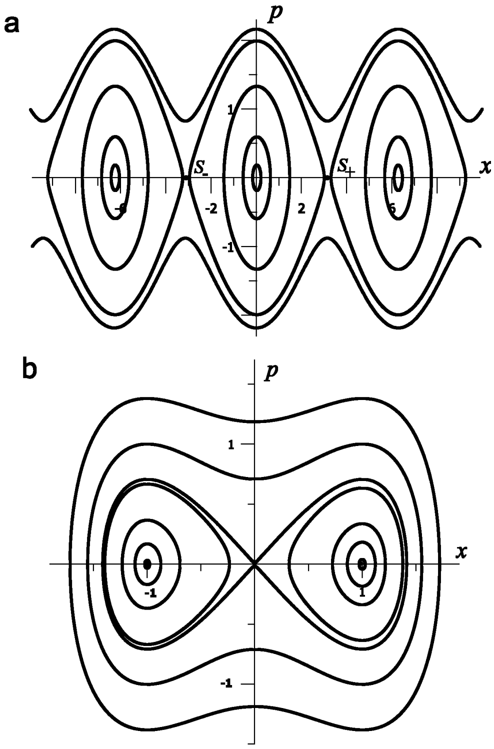

Fig. 1. Phase portraits for nonlinear pendulum (a) and double well potential (b).

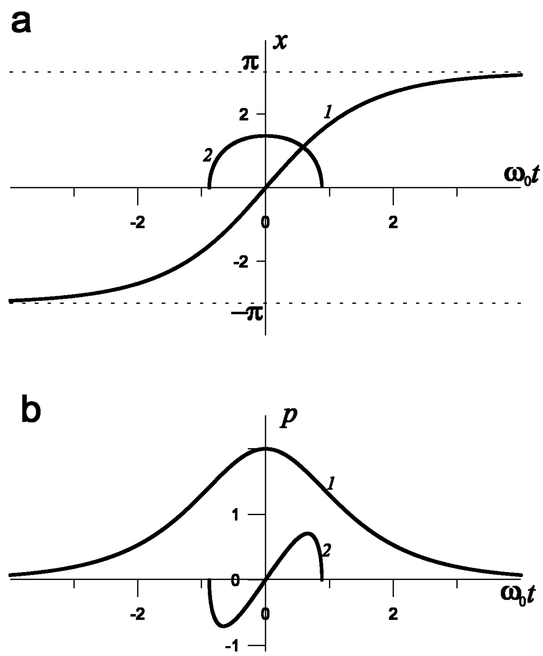

Fig. 2. (a) Motion laws for nonlinear pendulum (curve 1) and double well potential (curve 2); (b) Corresponding time dependencies of the conjugate momentum.

General solution of Eqs. (35) can be obtained with using the variables action-angle defined by Eqs. (24). It is convenient to introduce a parameter

$$\kappa^2 \equiv \frac{1}{2}\frac{\mathcal{H}}{\omega_0^2} \equiv \frac{\mathcal{H}}{\mathcal{H}_c}, \quad \mathcal{H}_c \equiv 2\omega_0^2, \tag{40}$$

taking the magnitude $\kappa = 1$ at critical energy $\mathcal{H} = \mathcal{H}_c$, and a new variable ξ defined by equalities

$$\sin\xi \equiv \begin{cases} \kappa^{-1}\sin(x/2) & \text{at} \quad \kappa \le 1, \\ \sin(x/2) & \text{at} \quad \kappa \ge 1. \end{cases} \tag{41}$$

Then, the generating function is expressed in terms of the incomplete Jacobian elliptic integrals $F(\xi,\kappa)$, $E(\xi,\kappa)$ of the first and second orders [7] as follows:

$$S(x,I) = 4\omega_0 \begin{cases} \left[E(\xi,\kappa) - (1-\kappa^2)F(\xi,\kappa)\right] & \text{at}\,\kappa \le 1, \\ \kappa E(\xi, 1/\kappa) & \text{at}\,\kappa \ge 1. \end{cases} \tag{42}$$

Differentiation of these equalities with respect to I arrives at expressions for the angle ϑ that generalyzes the last equality (31) (we supress these expressions because of their very complicated form).

Fortunately, formulas for action $I \equiv 4S(\xi = \pi/2)/2\pi$ follow from Eqs. (42) immediately and are expressed by means of the complete Jacobian elliptic integrals $K(\kappa) \equiv F(\xi = \pi/2, \kappa)$, $E(\kappa) \equiv E(\xi = \pi/2, \kappa)$. Taking into account corresponding dependences shown in Fig. 3, we shall need further in using asymptotics of these integrals [7]

$$K(\kappa) \approx \begin{cases} \frac{\pi}{2}\left(1+\frac{\kappa^2}{4}\right) & \text{at} \quad \kappa \ll 1, \\ \ln\frac{4}{\sqrt{1-\kappa^2}} & \text{at} \quad 1-\kappa^2 \ll 1; \end{cases} \qquad (43)$$

$$E(\kappa) \approx \begin{cases} \frac{\pi}{2}\left(1-\frac{\kappa^2}{4}\right) & \text{at} \quad \kappa \ll 1, \\ 1+\frac{1-\kappa^2}{2}\ln\frac{4}{\sqrt{1-\kappa^2}} & \text{at} \quad 1-\kappa^2 \ll 1. \end{cases} \qquad (44)$$

Resulting dependence $I(\mathcal{H})$ depicted in Fig. 4 shows monotonic increase from $I = 0$ at $\mathcal{H} = 0$ to infinity with the logarithmical inflection at the critical energy $\mathcal{H}_c$. This behaviour is characterized by the following asymptotics:

$$I \approx 2\omega_0 \begin{cases} \mathcal{H}/\mathcal{H}_c & \text{at} \quad \mathcal{H} \ll \mathcal{H}_c, \\ \frac{4}{\pi}\left(1-\frac{1-\mathcal{H}/\mathcal{H}_c}{4}\ln\frac{16}{1-\mathcal{H}/\mathcal{H}_c}\right) & \text{at} \quad 0<\mathcal{H}_c-\mathcal{H} \ll \mathcal{H}_c, \\ \frac{4}{\pi}\left(1+\frac{\mathcal{H}/\mathcal{H}_c-1}{4}\ln\frac{16}{\mathcal{H}/\mathcal{H}_c-1}\right) & \text{at} \quad 0<\mathcal{H}-\mathcal{H}_c \ll \mathcal{H}_c, \\ 2\sqrt{\frac{\mathcal{H}}{\mathcal{H}_c}}\left(1-\frac{\mathcal{H}_c}{4\mathcal{H}}\right) & \text{at} \quad \mathcal{H} \gg \mathcal{H}_c. \end{cases} \qquad (45)$$

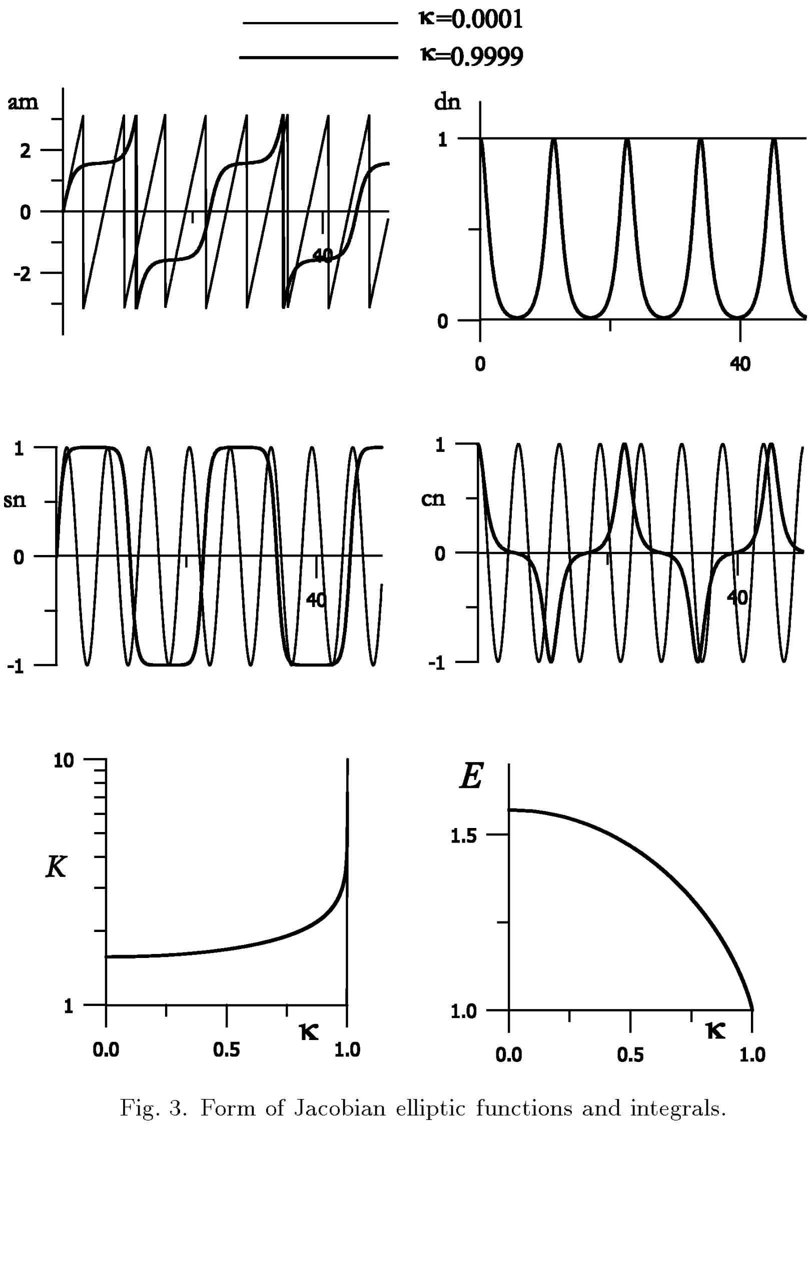

Fig. 3. Form of Jacobian elliptic functions and integrals.

Accounting the properties of the elliptic integrals [7], we obtain for the proper frequency $\omega \equiv \dot{\vartheta}$ determined by the first equality (25) in the following form:

$$\omega = \frac{\pi}{2} \frac{\omega_0}{K(\bar{\kappa})} \begin{cases} 1 & \text{at} \quad \kappa \leq 1, \\ \kappa & \text{at} \quad \kappa \geq 1, \end{cases} \tag{46}$$

where

$$\bar{\kappa} \equiv \begin{cases} \kappa & \text{at} \quad \kappa \leq 1, \\ \kappa^{-1} & \text{at} \quad \kappa \geq 1. \end{cases} \tag{47}$$

As it is seen in Fig. 4, the proper frequency falls down from the bare magnitude ω_0 at the minimal energy $\mathcal{H} = 0$ to zero at $\mathcal{H} = \mathcal{H}_c$ and then, after an infinitely sharp cusp, the value ω increases monotonously. According to Eqs. (44), such a behaviour is presented by asymptotics:

$$\omega \approx \omega_0 \begin{cases} 1 - \frac{\mathcal{H}}{4\mathcal{H}_c} & \text{at}\, \mathcal{H} \ll \mathcal{H}_c, \\ \pi \left(\ln \frac{16}{|1 - \mathcal{H}/\mathcal{H}_c|} \right)^{-1} & \text{at}\, |\mathcal{H} - \mathcal{H}_c| \ll \mathcal{H}_c, \\ \sqrt{\frac{\mathcal{H}}{\mathcal{H}_c}} \left(1 - \frac{\mathcal{H}_c}{4\mathcal{H}} \right) & \text{at}\, \mathcal{H} \gg \mathcal{H}_c. \end{cases} \tag{48}$$

On the other hand, definitions (34), (41) arrive at time-dependencies of the momentum:

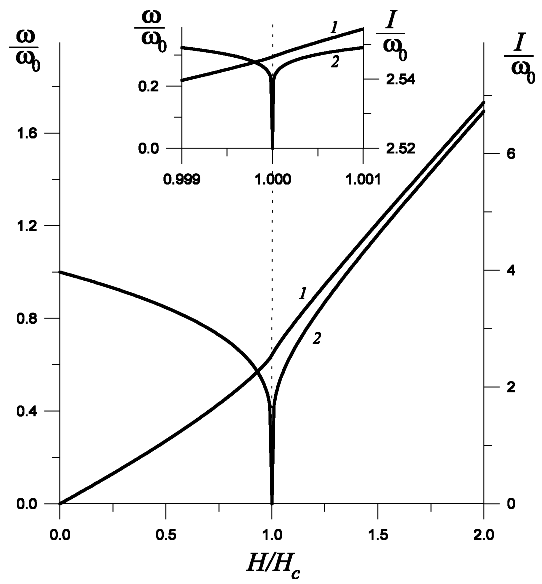

Fig. 4. Energy dependencies of the action and the proper frequency for nonlinear pendulum.

$$p = \pm 2\omega_0 \kappa \begin{cases} \cos\xi = \mathrm{cn}(t, \kappa) & \text{at} \quad \mathcal{H} \leq \mathcal{H}_c, \\ \sqrt{1 - \kappa^{-2}\sin^2\xi} = \mathrm{dn}(t, \kappa^{-1}) & \text{at} \quad \mathcal{H} \geq \mathcal{H}_c, \end{cases} \tag{49}$$

where $\kappa^2 \equiv \mathcal{H}/\mathcal{H}_c$; $\mathrm{cn}(t, \kappa)$, $\mathrm{dn}(t, \kappa^{-1})$ are the Jacobian elliptic functions shown in Fig. 3. With accounting for Eqs. (41), these expressions pass to Eqs. (37), (39) at $\mathcal{H} = \mathcal{H}_c$.

To elucidate the system behaviour with energy increase, let us expand the dependences (49) into Fourier series [2]

$$p = \pm 8\omega \begin{cases} \sum\limits_{n=1}^{\infty} a_n \cos[(2n-1)\omega t] & \text{at}\, \mathcal{H} \leq \mathcal{H}_c, \\ \frac{1}{4} + \sum\limits_{n=1}^{\infty} a_n \cos(n\omega t) & \text{at}\, \mathcal{H} \geq \mathcal{H}_c, \end{cases} \tag{50}$$

where one denotes

$$a_n \equiv \begin{cases} \frac{k^{n-1/2}}{1+k^{2n-1}} & \text{at}\, \mathcal{H} \leq \mathcal{H}_c, \\ \frac{k^n}{1+k^{2n}} & \text{at}\, \mathcal{H} \geq \mathcal{H}_c; \end{cases} \tag{51}$$

$$k \equiv \exp\left(-\pi \frac{K'}{K}\right),$$

$$K' \equiv K(\sqrt{1-\bar{\kappa}^2}), \quad K \equiv K(\bar{\kappa}),$$

parameter $\bar{\kappa}$ is determined by Eq. (47). According to Eqs. (44), one has asymptotics

$$k \approx \begin{cases} \kappa^2/32 & \text{at}\, \kappa \ll 1, \\ k \approx \exp(-\pi/N) & \text{at}\, 1 - \kappa^2 \ll 1, \end{cases} \tag{52}$$

where number $N \equiv \omega_0/\omega$ is asymptotically as follows:

$$N \approx \begin{cases} 1 & \text{at}\, \mathcal{H} \ll \mathcal{H}_c, \\ \frac{1}{\pi} \ln \frac{16\mathcal{H}_c}{|\mathcal{H} - \mathcal{H}_c|} & \text{at}\, |\mathcal{H} - \mathcal{H}_c| \ll \mathcal{H}_c. \end{cases} \tag{53}$$

Thus, at low energies a single harmonics prevails to correspond to the coefficient of the Fourier series (50)

$$a_n \approx \left(\frac{\mathcal{H}}{32\mathcal{H}_c} \right)^{n-1/2}, \qquad \mathcal{H} \ll \mathcal{H}_c. \tag{54}$$

Respectively, near the critical energy, where the Fourier series gets the harmonics number $N \gg 1$, one obtaines

$$a_n \approx 8\omega \begin{cases} 1 & \text{at} \quad 1 < n < N, \\ \exp[-\pi(n/N)] & \text{at} \quad n > N. \end{cases} \tag{55}$$

As is known [2], the above described behaviour is characterized by nonlinearity parameter

$$\alpha \equiv \left| \frac{\mathrm{d}\ln\omega}{\mathrm{d}\ln I} \right|. \tag{56}$$

According to (42), (46), this parameter is determined by the following equation:

$$\alpha = \begin{cases} \frac{1-\kappa^2}{\kappa^2} \left[\frac{1}{1-\kappa^2} \frac{E(\kappa)}{K(\kappa)} - 1 \right]^2 & \text{at} \quad \kappa \le 1, \\ \frac{\kappa^2}{\kappa^2-1} \left(\frac{E(\kappa^{-1})}{K(\kappa^{-1})} \right)^2 & \text{at} \quad \kappa \ge 1. \end{cases} \tag{57}$$

As it is shown in Fig. 5a, the non-linearity parameter takes on the value $\alpha = 0$ at $\mathcal{H} = 0$ and then goes to infinity at the critical energy $\mathcal{H}_c$, tending to magnitude $\alpha = 1$ at $\mathcal{H} \to \infty$. Such a behaviour is characterized by the following asymptotics:

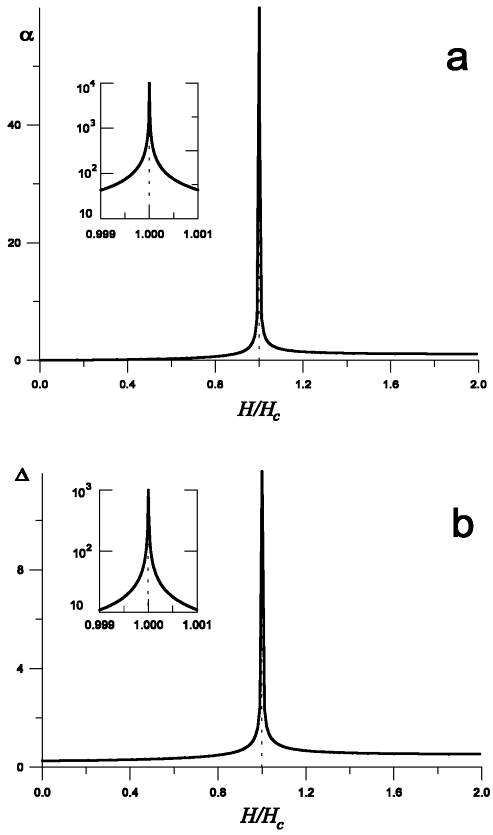

Fig. 5. Energy dependencies of the nonlinearity parameter (a) and curvature of the dependence $\mathcal{H}(I)$ (b) for nonlinear pendulum.

$$\alpha \approx \begin{cases} \frac{1}{4}(\mathcal{H}/\mathcal{H}_c) & \text{at} \quad \mathcal{H} \ll \mathcal{H}_c, \\ 4\left(1 - \frac{\mathcal{H}}{\mathcal{H}_c}\right)^{-1} \left(\ln\frac{16}{1-\mathcal{H}/\mathcal{H}_c}\right)^{-2} \left[1 - \frac{1-\mathcal{H}/\mathcal{H}_c}{2} \ln\frac{16}{1-\mathcal{H}/\mathcal{H}_c}\right] & \text{at } 0 < \mathcal{H}_c - \mathcal{H} \ll \mathcal{H}_c, \\ 4\left(\frac{\mathcal{H}}{\mathcal{H}_c} - 1\right)^{-1} \left(\ln\frac{16}{\mathcal{H}/\mathcal{H}_c-1}\right)^{-2} \left[1 + \frac{\mathcal{H}/\mathcal{H}_c-1}{2} \ln\frac{16}{\mathcal{H}/\mathcal{H}_c-1}\right] & \text{at } 0 < \mathcal{H} - \mathcal{H}_c \ll \mathcal{H}_c, \\ 1 - \mathcal{H}_c/\mathcal{H} & \text{at} \quad \mathcal{H} \gg \mathcal{H}_c. \end{cases} \tag{58}$$

As a result, the observed picture of nonlinear oscillation is as follows. At low energy, when $\mathcal{H} \ll \mathcal{H}_c$, only single harmonic with the frequency $\omega \approx \omega_0$ keeps in the Fourier series (50), so that the low-energy limit reduces to above considered case of harmonic oscillation (see Subsection III.A). With the energy increase, the harmonics number N arises in a manner of the dependence $K(\kappa)$ shown in Fig. 3, taking logarithmically large magnitudes (53) near the critical value $\mathcal{H}_c \equiv 2\omega_0^2$. On the other hand, the oscillation frequency (48) and the harmonic amplitudes (55) decrease monotonously to zero. Thus, one can mean in a coarse manner that, with energy increase in the domain $0 \le \mathcal{H} \le \mathcal{H}_c$, the single harmonic oscillation transforms to a harmonics superposition, whose number N increases monotonously to infinity, whereas frequency ω and amplitude $a_n \sim 8\omega$ decrease to zero. Just under the critical energy ($0 < \mathcal{H}_c - \mathcal{H} \ll \mathcal{H}_c$) the system behaviour is characterized by a set of solitons of different signs, whereas just above $\mathcal{H}_c$ the signs of these solitons coincide (see Eqs. (49)). Remarkable peculiarity of such soliton set is that the width of a single soliton is reduced to $\Delta t \sim \omega_0^{-1}$, whereas the distance between them is $N\Delta t \sim \omega^{-1} \gg \omega_0^{-1}$. Just for critical energy $\mathcal{H}_c$, the system behaviour is caracterized by the separatrix solution (39) that reduces to the single soliton.

C. Double well potential

Taking into account the use in the stochastic resonance problem [1], let us consider now the oscillation in a double well potential. We will show that, in terms of the variables action–angle, the difference with the above studied case of the nonlinear pendulum is quite quantitative, but not qualititative.

The basic model is presented by the potential

$$F = \frac{\omega_0^2}{4}(1-x^2)^2, \tag{59}$$

counting off the energy at the equilibrium positions $x = \pm 1$, ω_0 is bare frequency. In this case, Hamilton equations (23) take the forms type of Eqs. (35):

$$\dot{x} = p, \quad \dot{p} = \omega_0^2 x(1-x^2). \tag{60}$$

Near the saddle point $x = 0$, $p = 0$, corresponding phase portrait (see Fig. 1b) has the form differing from the one depicted in Fig. 1a for a nonlinear pendulum. This form is characterized by the separatrix (cf. Eq. (37))

$$p = \pm\omega_0 x\sqrt{1 - x^2/2}, \tag{61}$$

corresponding to the condition $\mathcal{H} = (\omega_0/2)^2$. Similarly to the case of nonlinear pendulum, the first of Eqs. (60) arrives at the separatrix law of motion:

$$x = \pm 2^{1/2}\sqrt{1 - [\sinh(\omega_0 t)]^2}. \tag{62}$$

For the time-dependence of the momentum, one has double-soliton solution (cf. Eq. (39))

$$p = \mp 2^{1/2}\omega_0 \sinh(\omega_0 t)\sqrt{1 - [\sinh(\omega_0 t)]^2}, \tag{63}$$

where the choice of signs corresponds to solitons moving to the right or left sides. According to the motion laws (62), (63) — on the one hand, and Eqs. (38), (39) — on the other, the difference between the separatrix solutions for double well potential and nonlinear pendulum is that, in the course of the time near the point $t = 0$, the momentum gains two peaks of different signs in the former case and the single peak in the latter.

To introduce the variables action–angle, it is convenient to use parameter κ of the type given by Eq. (40) and a new variable ξ determined by the equality type of (41):

$$\kappa^2 \equiv \frac{\mathcal{H}}{\mathcal{H}_c}, \quad \mathcal{H}_c \equiv \left(\frac{\omega_0}{2}\right)^2; \tag{64}$$

$$x^2 \equiv 1 - \begin{cases} \kappa \sin\xi & \text{at} \quad \kappa \le 1, \\ \sin\xi & \text{at} \quad \kappa \ge 1. \end{cases} \tag{65}$$

Moreover, we shall need in using integrals

$$\mathcal{I}_n(\alpha,\kappa) \equiv \int_0^\alpha \frac{(1-\kappa\xi)^{n-\frac{1}{2}}}{\sqrt{1-\xi^2}}\mathrm{d}\xi, \ \alpha \le 1,$$

$$n = 0, \pm 1, \pm 2, \ldots \quad \text{at} \quad \kappa \le 1;$$

$$\mathcal{J}_n(\alpha,\kappa) \equiv \int_0^\alpha \frac{(1-\kappa^{-2}\xi^2)^{\frac{1}{2}-n}}{\sqrt{1-\xi}}\mathrm{d}\xi, \ \alpha \le 1,$$

$$n = 0, 1, 2, \ldots \quad \text{at} \quad \kappa \ge 1. \tag{66}$$

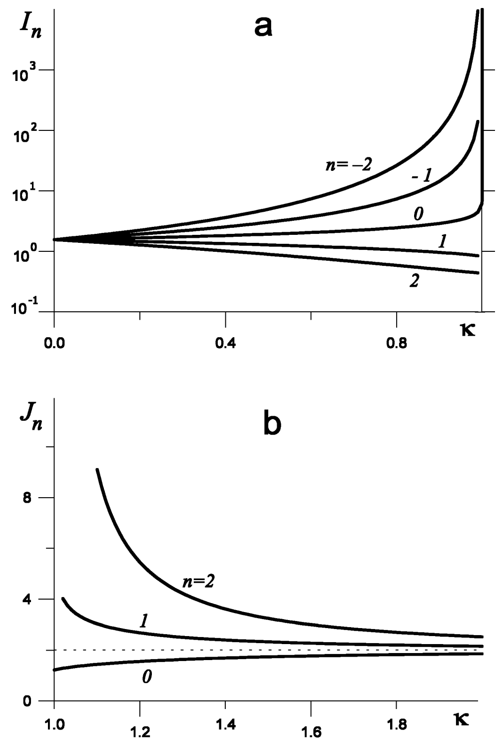

Fig. 6. Form of the integrals (66).

The form of dependencies $\mathcal{I}_n(\kappa) \equiv \mathcal{I}_n(\alpha = 1, \kappa)$, $\mathcal{J}_n(\kappa) \equiv \mathcal{J}_n(\alpha = 1, \kappa)$ is depicted in Fig. 6, the corresponding asymptotics read:

$$\mathcal{I}_n(\kappa) \approx \frac{\pi}{2} - \frac{1}{2}(2n-1)\kappa, \quad \kappa \ll 1;$$

$$\mathcal{J}_n(\kappa) \approx 2 + \frac{8}{15}(2n-1)\kappa^{-2}, \quad \kappa \gg 1. \tag{67}$$

The integrals are subjected to simple derivation rules

$$\kappa\frac{\mathrm{d}\mathcal{I}_n}{\mathrm{d}\kappa} = \left(n - \frac{1}{2}\right)(\mathcal{I}_n - \mathcal{I}_{n-1}),$$

$$\kappa\frac{\mathrm{d}\mathcal{J}_n}{\mathrm{d}\kappa} = (2n-1)(\mathcal{J}_n - \mathcal{J}_{n-1}), \tag{68}$$

where the arguments α, κ are supressed for brevity.

As a result, the last of definitions (24) arrives at the expression (cf. Eq. (42))

$$S = \frac{\omega_0}{2\sqrt{2}} \begin{cases} -(1-\kappa^2)\mathcal{I}_0 + 2\mathcal{I}_1 - \mathcal{I}_2 & \text{at} \quad \kappa \le 1, \\ \kappa \mathcal{J}_0 & \text{at} \quad \kappa \ge 1. \end{cases} \tag{69}$$

The action $I \equiv (2/\pi)S$ follows from this at $\alpha = 1$. Respectively, for the proper frequency one obtains instead of Eq. (46)

$$\omega = \sqrt{2}\pi\omega_0\kappa^2 \begin{cases} \left[-(1-\kappa^2)\mathcal{I}_{-1} + (3\kappa^2-1)\mathcal{I}_0 + 5\mathcal{I}_1 - 3\mathcal{I}_2\right]^{-1} & \text{at} \quad \kappa \le 1, \\ \left(2\kappa\mathcal{J}_1\right)^{-1} & \text{at} \quad \kappa \ge 1. \end{cases} \tag{70}$$

Energy dependences $I(\mathcal{H})$, $\omega(\mathcal{H})$, following from Eqs. (69), (70), are depicted in Fig. 7. It is seen that these take the form type of the corresponding dependences for a nonlinear pendulum (see Fig. 4). Fourier spectrum of the time dependence $p(t)$ of the momentum behaves in analogous manner as in Eqs. (50): with the energy increase in the domain $0 \le \mathcal{H} \le \mathcal{H}_c$, single harmonic oscillation transforms to a harmonic superposition, whose number increases monotonously to infinity, whereas frequencies and amplitudes decrease to zero. Such a behaviour is characterized by the nonlinearity parameter (56), taking the following form (cf. Eq. (57)):

$$\alpha = \begin{cases} 6\frac{\left[(1-\kappa^2)\mathcal{I}_{-2} - 2\mathcal{I}_{-1} + \kappa^2\mathcal{I}_0 + 2\mathcal{I}_1 - \mathcal{I}_2\right]\left[-(1-\kappa^2)\mathcal{I}_0 + 2\mathcal{I}_1 - \mathcal{I}_2\right]}{\left[-(1-\kappa^2)\mathcal{I}_{-1} + (3\kappa^2-1)\mathcal{I}_0 + 5\mathcal{I}_1 - 3\mathcal{I}_2\right]^2} & \text{at} \quad \kappa \le 1, \\ \frac{\mathcal{J}_0\mathcal{J}_2}{\mathcal{J}_1^2} & \text{at} \quad \kappa \ge 1. \end{cases} \tag{71}$$

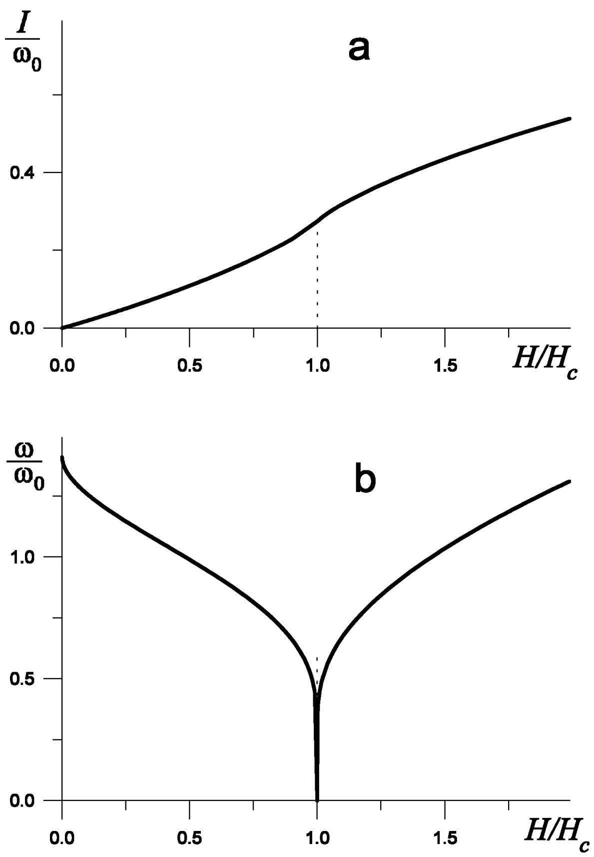

Fig. 7. Energy dependencies of the action (a) and the proper frequency (b) for double well potential.

Respectively, double curvature $\Delta \equiv 2\frac{d\omega}{dI}$ of the dependence $\mathcal{H}(I)$ is connected with the parameter α as $\Delta \equiv 2(\omega/I)\alpha$ to read

$$\Delta = 2\pi^2 \begin{cases} 6\kappa^2 \frac{(1-\kappa^2)\mathcal{I}_{-2} - 2\mathcal{I}_{-1} + \kappa^2\mathcal{I}_0 + 2\mathcal{I}_1 - \mathcal{I}_2}{[-(1-\kappa^2)\mathcal{I}_{-1} + (3\kappa^2-1)\mathcal{I}_0 + 5\mathcal{I}_1 - 3\mathcal{I}_2]^3} & \text{at} \quad \kappa \le 1, \\ \frac{\mathcal{J}_2}{\mathcal{J}_1^3} & \text{at} \quad \kappa \ge 1. \end{cases} \tag{72}$$

D. Nonlinear pendulum under constant external field

Before studying the effect of periodical external field, let us anounce main peculiatities of the constant perturbation following supersymmetry theory [8]. In this case, the potential energy

$$F = 2\omega_0^2 \sin(x/2) - \mathcal{E}x \tag{73}$$

is characterized, besides the proper frequency ω_0, by a field strength $\mathcal{E}$. Switching such bias field arrives at the expression for the flux $j \equiv \langle x \rangle / t$ as follows:

$$j = \gamma \frac{2\pi T}{\mathcal{I}^2(\mathcal{E})} \sinh \frac{\pi \mathcal{E}}{T}, \tag{74}$$

where one introduces the integral

$$\mathcal{I}(\mathcal{E}) = \int_{x_e}^{x_e+2\pi} \exp\left\{\frac{F(x)}{T}\right\} \mathrm{d}x, \tag{75}$$

$$\sin x_e \equiv \frac{\mathcal{E}}{\omega_0^2},$$

T is the temperature. In ergodic systems, the diffusion coefficient determined by equality

$$\langle (x - jt)^2 \rangle \equiv Dt \tag{76}$$

takes the form

$$D \equiv T \frac{\partial j}{\partial \mathcal{E}} = \gamma \frac{2\pi^2 T}{\mathcal{I}^2(\mathcal{E})} \mathrm{ch} \frac{\pi \mathcal{E}}{T}. \tag{77}$$

However, non-ergodicity effects arrive at a much more complicated form of the diffusion coefficient [8]

$$D = j \left[\pi \coth \frac{\pi \mathcal{E}}{T} + \frac{1}{2} \arcsin \frac{\mathcal{E}}{\omega_0^2} + \frac{\mathcal{I}\mathcal{E}}{2(\omega_0^4 - \mathcal{E}^2)}\right], \tag{78}$$

where $\mathcal{I}$ is characteristic magnitude of the integral (76). Dependences $j(\mathcal{E})$, $D(\mathcal{E})$ related to Eqs.(74), (78) are depicted in Fig. 8.

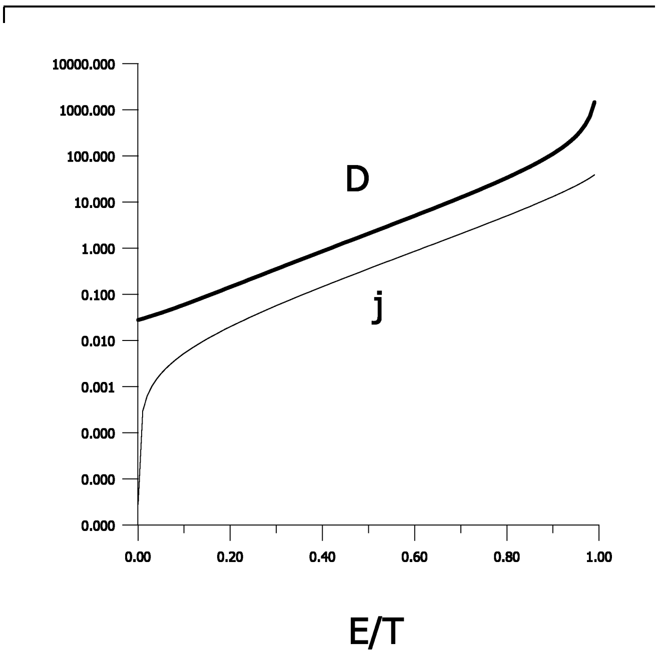

Fig. 8. Dependence of the flux j and the diffusion coefficient D on the force E of external field for nonlinear pendulum.

IV. NONLINEAR RESONANCE CONDITIONS

With accounting dissipation effects shifting the bare frequency ω_0 according to Eq. (28), the resonance condition reads

$$m_n \varpi = \Omega, \quad \varpi \equiv \sqrt{[\omega(\mathcal{H})]^2 - 2^{-2}}, \quad n = 1, 2, \ldots, \tag{79}$$

where Ω is the frequency of external signal, $\omega(\mathcal{H})$ is the energy-dependent proper frequency determined by Eqs. (48), (46), m_n is a resonance multiplicity being a rational number related to the natural one n. Formally, this condition means that we have to consider the phase portrait plane, which is revolved with angle velocity $m_n \varpi$. Then, the external addition of the Hamiltonian (23) is written in the form

$$\mathcal{H}_1 = -A f_{00} \cos\vartheta, \quad \vartheta \equiv (\Omega - m_n \varpi)\, t, \tag{80}$$

where the internal power f_{00} corresponds to resonance conditions (79). Respectively, the zero term of the Hamiltonian (23) can be expanded near the resonant action I_0 as follows:

$$\mathcal{H}_0 \approx \mathcal{H}_{00} + \omega_{00}\, \delta I + \frac{\Delta_0}{4} (\delta I)^2, \quad \delta I \equiv I - I_0. \tag{81}$$

Here, the resonant action I_0, as well as other resonant values

$$\mathcal{H}_{00} \equiv \mathcal{H}_0(I = I_0), \ \omega_{00} \equiv \left.\frac{\mathrm{d}\mathcal{H}_0}{\mathrm{d}I}\right|_{I=I_0},$$

$$\Delta_0 \equiv 2\left.\frac{\mathrm{d}\omega}{\mathrm{d}I}\right|_{I=I_0} = 2\frac{\omega_{00}}{I_0}\alpha_0, \ \alpha_0 \equiv \alpha(I = I_0) \qquad (82)$$

are given by resonance condition (79). It is worthwhile noting that curvature $\Delta/2$ is determined by the nonlinearity parameter (56) taken at resonance condition $I = I_0$ (see Figs. 5). The energy-dependence of the curvature is characterized by the following asymptotics:

$$\Delta \approx \pi^2 \begin{cases} (2\pi)^{-2}\left(1 - \frac{1}{4}\frac{\mathcal{H}}{\mathcal{H}_c}\right) & \text{at} \quad \mathcal{H} \ll \mathcal{H}_c, \\ \left(1 - \frac{\mathcal{H}}{\mathcal{H}_c}\right)^{-1}\left(\ln\frac{16}{1-\mathcal{H}/\mathcal{H}_c}\right)^{-3}\left[1 - \frac{1-\mathcal{H}/\mathcal{H}_c}{4}\ln\frac{16}{1-\mathcal{H}/\mathcal{H}_c}\right] & \text{at } \mathcal{H}_c - \mathcal{H} \ll \mathcal{H}_c, \\ \left(\frac{\mathcal{H}}{\mathcal{H}_c} - 1\right)^{-1}\left(\ln\frac{16}{\mathcal{H}/\mathcal{H}_c-1}\right)^{-3}\left[1 + \frac{\mathcal{H}/\mathcal{H}_c-1}{4}\ln\frac{16}{\mathcal{H}/\mathcal{H}_c-1}\right] & \text{at } \mathcal{H} - \mathcal{H}_c \ll \mathcal{H}_c, \\ \frac{1}{2\pi^2}\left(1 - \frac{\mathcal{H}_c}{\mathcal{H}}\right) & \text{at} \quad \mathcal{H} \gg \mathcal{H}_c. \end{cases} \qquad (83)$$

As a result, Hamilton equations (25) arrive at the equation type of Eq. (36) for nonliniar pendulum:

$$\ddot{\vartheta} + \omega_m^2 \sin\vartheta = 0. \qquad (84)$$

Here, instead of the proper frequency ω_0, the value ω_m stands for modulation frequency being determined as follows:

$$\omega_m = \left(\frac{A\Delta f_{00}}{2}\right)^{1/2}. \qquad (85)$$

Respectively, the maximum value of the resonant energy variation and corresponding magnitude for the action

$$\delta\mathcal{H}_m = Af_{00}, \quad \delta I_m = \left(\frac{4Af_{00}}{\Delta}\right)^{1/2} \qquad (86)$$

are determined by Eqs. (80), (81) to characterize a resonance window.

Thus, we obtain the following picture of nonlinear resonance. At given magnitudes of the frequency Ω of external signal, the condition (79) fixes the system energy $\mathcal{H}$ as follows:

$$\frac{\pi}{\sqrt{\omega_0^{-2} + (2\Omega/m_n\omega_0)^2}} = K(\kappa)\begin{cases} 1 & \text{at} \quad \kappa \le 1, \\ \kappa^{-1} & \text{at} \quad \kappa \ge 1. \end{cases} \qquad (87)$$

According to Fig. 9, the corresponding dependence $\mathcal{H}_{00}(\Omega)$ has two branches, the lower of which relates to the finite motion, the upper — to the infinite one. Energy $\mathcal{H}_{00}$ related to the former falls down monotonously from the upper magnitude fixed by condition $K(\kappa) = \pi\omega_0$ to zero within interval $0 < \Omega < \Omega_m$, where maximum frequency is

$$\Omega_m = m_n\omega_0\sqrt{1 - (2\omega_0)^{-2}}. \qquad (88)$$

Respectively, energy of the infinite motion arises monotonously from minimal magnitude fixed at $\Omega = 0$ by condition $\kappa^{-1}K(\kappa^{-1}) = \pi\omega_0$ to $\mathcal{H}_{00} \to \infty$ at $\Omega \to \infty$.

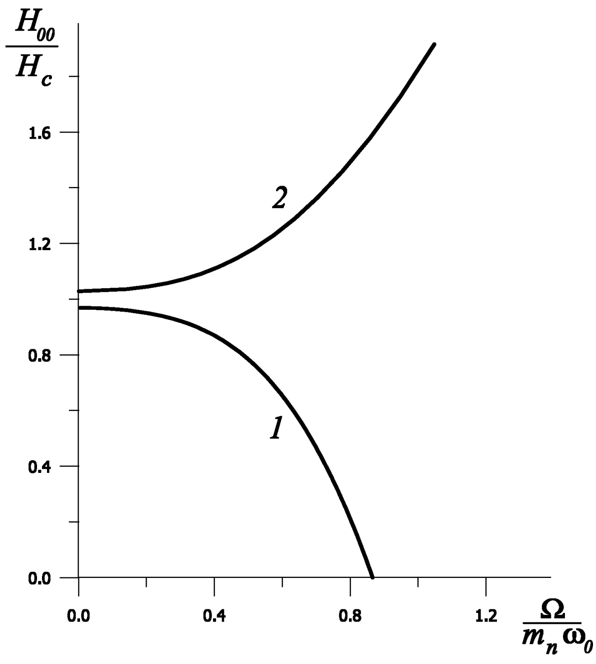

Fig. 9. Dependence of the nonlinear pendulum energy on the external signal frequency: curve 1 relates to finite motion, curve 2 — to infinite one.

Using abbreviated notation

$$\tilde{\Omega}^2 \equiv 1 + (2/m_n)^2\Omega^2, \tag{89}$$

one obtains the following asymptotics:

$$\mathcal{H}_{00} \approx \mathcal{H}_c \begin{cases} 1 - 16\exp\left[-2\pi(\omega_0/\tilde{\Omega})\right] & \text{at} \quad \Omega \ll \Omega_m, \\ 4\left[1-(2\omega_0)^{-2}\right]\frac{\Omega_m - \Omega}{\Omega_m} & \text{at} \quad 0 < \Omega_m - \Omega \ll \Omega_m \end{cases} \tag{90}$$

for the finite motion and

$$\mathcal{H}_{00} \approx \mathcal{H}_c \begin{cases} 1 + 16\exp\left[-2\pi(\omega_0/\tilde{\Omega})\right] & \text{at} \quad \Omega \ll \Omega_m, \\ \left(\frac{\tilde{\Omega}}{2\omega_0}\right)^2 & \text{at} \quad \Omega \gg \Omega_m \end{cases} \tag{91}$$

for the infinite one. Frequency-dependencies of the resonant magnitudes $I_0(\Omega)$, $\omega_{00}(\Omega)$, $\alpha(\Omega)$, $\Delta_0(\Omega)$ of the action, the proper frequency, the nonlinearity parameter and the double curvature of curve $\mathcal{H}_0(I)$ are depicted in Figs. 10, 11. In the case of the finite motion, these dependencies are characterized by the following asymptotics:

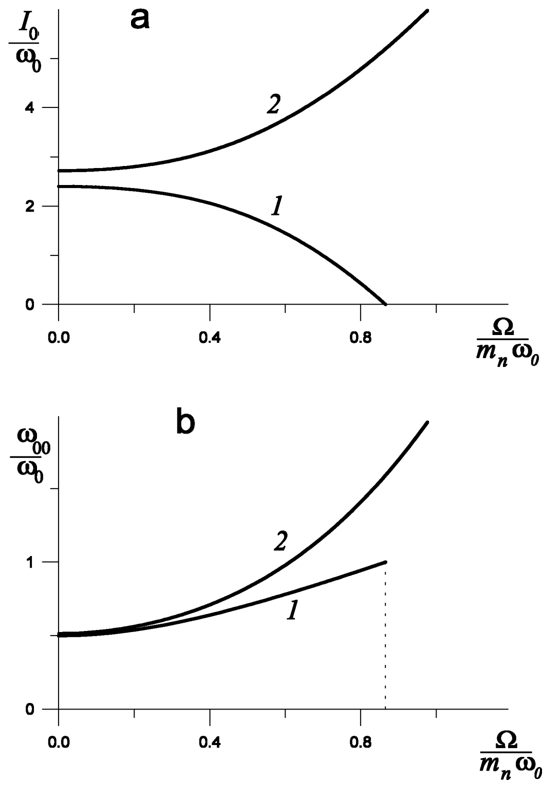

Fig. 10. Frequency dependencies of the action (a) and the proper frequency (b) for nonlinear pendulum: curve 1 relates to finite motion, curve 2 — to infinite one.

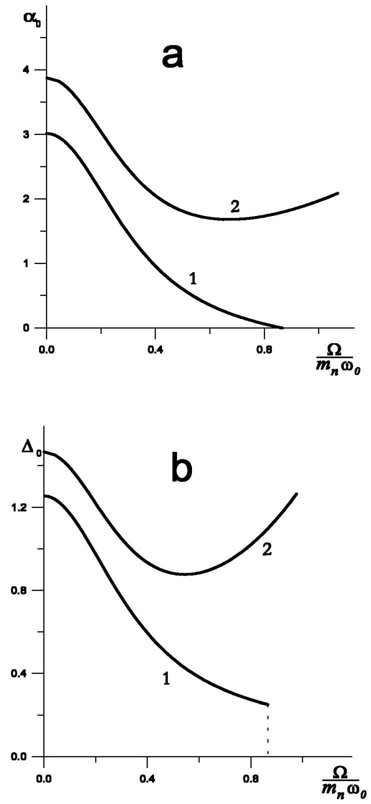

Fig. 11. Frequency dependencies of the nonlinearity parameter (a) and curvature of the dependence $\mathcal{H}(I)$ (b) for nonlinear pendulum: curve 1 relates to finite motion, curve 2 — to infinite one.

$$I_0 \approx 8\omega_0 \begin{cases} \pi^{-1}\left\{1 - 8\pi(\omega_0/\tilde{\Omega})\exp\left[-2\pi(\omega_0/\tilde{\Omega})\right]\right\} & \text{at} \quad \Omega \ll \Omega_m, \\ \left[1-(2\omega_0)^{-2}\right]\frac{\Omega_m - \Omega}{\Omega_m} & \text{at} \quad 0 < \Omega_m - \Omega \ll \Omega_m; \end{cases} \tag{92}$$

$$\omega_{00} \approx \omega_0 \begin{cases} \frac{\tilde{\Omega}}{2\omega_0} & \text{at} \quad \Omega \ll \Omega_m, \\ 1 - \left[1 - (2\omega_0)^{-2}\right] \frac{\Omega_m - \Omega}{\Omega_m} & \text{at} \quad 0 < \Omega_m - \Omega \ll \Omega_m; \end{cases} \tag{93}$$

$$\Delta_0 \approx \frac{1}{4} \begin{cases} \left(\frac{\tilde{\Omega}}{2\omega_0}\right)^2 \left\{\frac{\tilde{\Omega}}{8\pi\omega_0} \exp\left[2\pi(\omega_0/\tilde{\Omega})\right] - 1\right\} & \text{at} \quad \Omega \ll \Omega_m, \\ 1 - \left[1 - (2\omega_0)^{-2}\right] \frac{\Omega_m - \Omega}{\Omega_m} & \text{at} \quad 0 < \Omega_m - \Omega \ll \Omega_m. \end{cases} \tag{94}$$

Respectively, for the infinite motion one has:

$$I_0 \approx 2\omega_0 \begin{cases} \frac{4}{\pi} \left\{1 + 8\pi(\omega_0/\tilde{\Omega}) \exp\left[-2\pi(\omega_0/\tilde{\Omega})\right]\right\} & \text{at} \quad \Omega \ll \Omega_m, \\ \tilde{\Omega}/\omega_0 & \text{at} \quad \Omega \gg \Omega_m; \end{cases} \tag{95}$$

$$\omega_{00} \approx \tilde{\Omega}/2 \quad \text{at} \quad \Omega \ll \Omega_m \quad \text{and} \quad \Omega \gg \Omega_m; \tag{96}$$

$$\Delta_0 \approx \frac{1}{2} \begin{cases} \frac{1}{2} \left(\frac{\tilde{\Omega}}{2\omega_0}\right)^2 \left\{\frac{\tilde{\Omega}}{8\pi\omega_0} \exp\left[2\pi(\omega_0/\tilde{\Omega})\right] + 1\right\} & \text{at} \quad \Omega \ll \Omega_m, \\ 1 - (2\omega_0/\tilde{\Omega})^2 & \text{at} \quad \Omega \gg \Omega_m. \end{cases} \tag{97}$$

At $\omega_0 \gg 1/2$, when $\Omega_m \approx m_n\omega_0$, effective frequency $\tilde{\Omega}$ may be replaced by the short-cut term $(2/m_n)\Omega \gg 1$.

If the oscillations were linear in character, the resonance might realize within window located between energies $\mathcal{H}_{00} - \delta\mathcal{H}_m$ and $\mathcal{H}_{00} + \delta\mathcal{H}_m$. In this window, the oscillations would have single frequency ϖ determined by Eq. (79) and would be modulated with the frequency ω_m given by Eq. (85). However, nonlinearity effects described in Subsection III.B arrive at expanding the harmonics number to magnitude $N_0 = \omega_0/\omega_{00} > 1$ and narrowing the energy window to the width $\mathcal{H}_m \equiv 2\omega_m^2$. So, nonlinear resonance is realized for a share of nonlinear oscillations determined by the ratio

$$\frac{\mathcal{H}_m}{\delta\mathcal{H}_m} = \Delta_0 = 2\frac{\omega_{00}}{I_0}\alpha_0, \tag{98}$$

where Eqs.(82), (85), (86) are taken into account. As it is seen from Fig. 11, the double curvature $\Delta(\Omega)$ decreases monotonously with the external frequency growth within the interval $0 < \Omega < p\Omega_m$. This means that preference of the nonlinear resonance decreases with this frequency growth.

V. STOCHASTIC RESONANCE CONDITIONS

The stochastic resonance is known to be observed at condition (79), where $m_n = (2n)^{-1}$, $n = 1, 2, \ldots$. This condition means that during a period $T = 2\pi/\Omega$ of the external oscillation the stochastic system has a time to overcome an energy barrier ΔF even times $2n$ [1]. The proper frequency of stochastic resonance $\omega \equiv 2\pi r_K$ is reduced to the Kramers' rate r_K given by Eq. (2). This case differs from the above considered case of nonlinear pendulum by only inserting temperature T instead of the system energy $\mathcal{H}$. In the case of the double well potential considered in Subsection III.C, the energy barrier $\Delta F = (\omega_0/2)^2$ is related to the temperature by characteristic parameter $\kappa^2 \equiv 4T/\omega_0^2$. Then, the frequency of the barier overcoming is defined by equation

$$\omega = \omega_K \exp(-\kappa^{-2}), \quad \kappa^2 \equiv 4T/\omega_0^2 \tag{99}$$

instead of the corresponding equation (46) for nonlinear pendulum. Taking into consideration dissipation effects, we obtaine the following condition of the stochastic resonance:

$$T_0 = \frac{\omega_0^2}{2} \left[\ln \frac{(2\omega_K)^2}{1 + (4n)^2\Omega^2}\right]^{-1}. \tag{100}$$

As is shown in Fig. 12, the resonant temperature $T_0(\Omega)$ arises monotonously, taking indefinite values at characteristic frequency

$$\Omega_m = \frac{\omega_K}{2n}\sqrt{1 - (2\omega_K)^{-2}}. \tag{101}$$

Basing on the dependence $T_0(\Omega)$ replacing the above used relation $\mathcal{H}(\Omega)$, we are in position to consider the stochastic resonance conditions in analogy with the nonlinear pendulum.

Here, instead of Eqs. (40), (42), (46), the equalities (65), (69), (70) determine the resonant magnitudes I_0, ω_{00}, α, Δ of the action, the proper frequency, the nonlinearity parameter and the double curvature of the resonant dependence $\mathcal{H}_{00}(I)$. Corresponding dependencies on the temperature are depicted in Figs. 13, 14 to show the indefinite increase of the proper frequency and the action with tending to the characteristic magnitude (101). Related values of the nonlinearity parameter and the curvature decrease thereby.

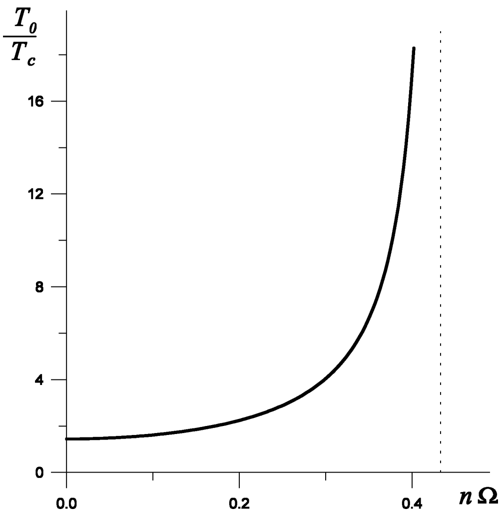

Fig. 12. Dependence of the stochastic resonance temperature on the external signal frequency.

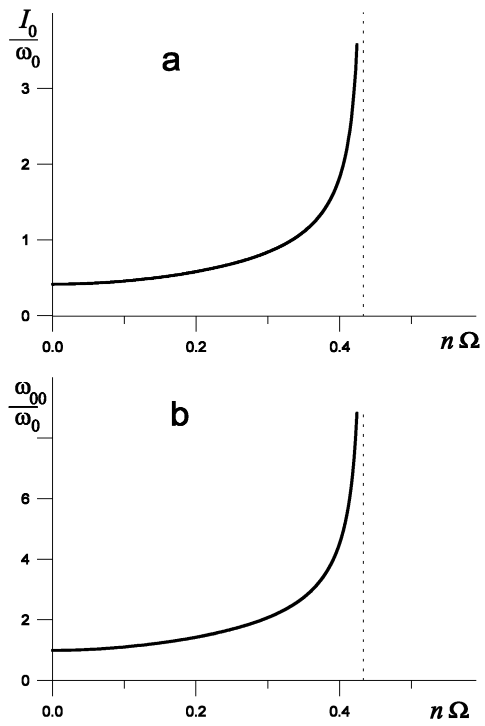

Fig. 13. Frequency dependencies of the action (a) and the proper frequency (b) of the stochastic resonance.

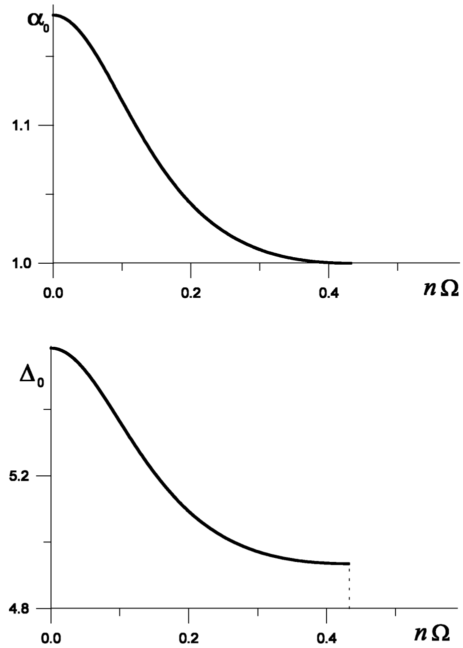

Fig. 14. Frequency dependencies of the nonlinearity parameter α_0 and curvature Δ_0 of the dependence $\mathcal{H}(I)$ at stochastic resonance.

VI. CONCLUSION

The observed picture of nonlinear oscillations shows that with the energy increase the single harmonic oscillation transforms to a harmonics superposition, whose number increases monotonously to infinity, whereas frequency and amplitude decrease to zero. In other words, with tending to a threshold energy, transition of the harmonic oscillations into a set of solitons is observed. We have shown that such a behaviour, being typical for both the nonlinear pendulum and the double well potential, is characterized by the nonlinearity parameter (56).

Our choice of the action S and angle ϑ as principle variables is caused by the fact that Hamiltonian of free nonlinear oscillations does not depend on the angle ϑ. In accordance with Eq. (84), switching on external harmonic signal arrives at a modulation of nonlinear oscillations with the characteristic frequency (85) and nonlinearity parameter (82). It appears that the nonlinearity effect narrows the resonance window to the width fixed by ratio (98) that is reduced to the curvature Δ of the dependence $\mathcal{H}(I)$. According to the dependence $\Delta_0(\Omega)$ depicted in Fig. 14, this window is shrunk with the external frequency growth. As a result, preference of the stochastic resonance decreases with the growth of frequency Ω of the external signal.

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[4] The accepted choice of the coefficient in correlator (6) makes it possible to preserve the standard form of the Focker–Planck equation in which the diffusion term appears with the coefficient 1/2 (accordingly, the Gaussian distribution has the standard dispersion). The doubling of the coefficient leads to the standard Onsager form of the diffusion component of the flux probability.

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НЕЛІНІЙНА ТЕОРІЯ СТОХАСТИЧНОГО РЕЗОНАНСУ

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Теорія нелінійного стохастичного резонансу збудована на основі статистичної теорії поля, яка використовує змінні "дія–кут". Для нелінійного маятника та двоямного потенціялу знайдено явні вирази для дії, власної частоти та параметра нелінійности залежно від енерґії системи та частоти зовнішнього сиґналу.