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QUASINORMAL MODES OF THE SCHWARZSCHILD BLACK HOLE AND HIGHER ORDER WKB APPROACH

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The low lying characteristic (quasinormal) modes of black holes can be evaluated with the help of the semi-analytic method based on a modified WKB approach. We extend the 3rd order WKB formula of Iyer and Will to the 6th order beyond the eikonal approximation and attain thereby an accurate formula for computing QN modes. With the help of the obtained formula we find quasinormal modes corresponding to decay of scalar, dirac, electromagnetic, gravitino, and graviton fields in the Schwarzschild background. For Schwarzschild BH the 6th order WKB lower overtone modes, corresponding to scalar, elctromagnetic and gravitational perturbations, coincide with numerical results already for $l \geq 1$, while the 3rd order formula does the same at $l \geq 4$.
Key words: quasinormal modes, black holes, WKB approach.

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I. INTRODUCTION AND SUMMARY

When perturbing a black hole, it undergoes damping oscillations which are characterized by some complex eigenvalues of the wave equations. These values (*quasinormal frequencies*) are complex, and their real parts give the oscillation frequencies, while the imaginary ones determine the damping rates of the modes. The quasinormal modes (QN) of black holes (BH's) depend only on a black hole parameters and not on the way in which they were excited. QN's are called, therefore, “footprints” of a black hole. Quasinormal modes are of considerable interest now: they are studied within the context of Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence (see for example [1–11] and references therein), because of the possibility of observing quasinormal ringing of astrophysical BH's (see [12] for a review), when considering thermodynamic properties of black holes [13–15]. QNM's of near extremal black branes have been investigated recently in [9].

First analytical method for calculations of BH QNMs was apparently proposed by Bahram Mashhoon who used the Poschl–Teller potential to estimate the QN frequencies [16]. In [17] there was proposed a semi-classical method for computing QNM's based on the WKB treatment [18]. Then it was extended to the third order beyond the eikonal approximation in [18]. The accuracy of the 3rd order WKB formula (see Eq. (1.5) in [18]) is the better, the more multipole number l and the less overtone n. For Schwarzschild BH it practically coincides with accurate numerical results of Leaver [19] at $l \geq 4$ when being restricted by lower overtones. For fewer multipoles, however, accuracy is worse, and may reaches 10 per cents at $l = 0$, $n = 0$. Numerical approach [19], on contrary, is very accurate, but, dealing with numerical integration or systems of recurrence relations, very cumbersome, and, often, require modification to be applied to different black holes. At the same time WKB approach lets us to obtain QNM's for a full range of parameters giving thereby some fields of work for intuition as to physical behaviour of a system. Being a most economic way for calculation of the modes, WKB approach is a most useful in astrophysical situations where only lower overtones are excited, while higher overtones are highly damping.

Both advantages and deficiencies of the WKB approach motivated us to extend the existent 3rd order WKB formula up to the 6th order. As a result we obtained an economic, compared to numerical, procedures, and accurate formula for straightforward calculation of QNM frequencies. The 6th order formula applied to the Schwarzschild BH is as accurate at the first multipole as the 3rd order is at the forth multipole. When being restricted by astrophysically relevant situations, i. e. lower overtones n and not very large multipoles l, it is sufficient, we believe, to use the 6th order WKB formula proposed in this paper, in order to find QN frequencies which should insignificantly differ from their true values. We show it here on example of QNM's corresponding to perturbations of fields of different spins: scalar ($s = 0$), dirac ($s = 1/2$), electromagnetic ($s = 1$), gravitino ($s = 3/2$), and graviton (gravitational) ($s = 2$). Note that the 6th order formula provides good accuracy for Kerr BH as well if one considers not large values of m [20].

In Sec. II we propose some preliminaries of the WKB approach in the context of the QNM problem, and extend the WKB formula to the sixth order. In Sec. III we compare the numerical, 3rd WKB order, and 6th WKB order QN values for the perturbations of the Schwarzschild BH.

II. WKB APPROACH

The perturbation equations of a black hole can be reduced to the Schrödinger wave-like equation:

$$\frac{d^2\psi}{dx^2} + Q(x)\psi(x) = 0, \qquad (1)$$

where "the potential" $-Q(x)$ is constant at the event horizon ($x = -\infty$) and at the infinity ($x = +\infty$) and it rises to maximum at some intermediate $x = x_0$. Let us consider radiation of a given frequency ω incident on the black hole from infinity and let $R(\omega)$ and $T(\omega)$ be the reflection and transmission amplitudes respectively. We extend $R(\omega)$ to the complex frequency plane such that $\mathrm{Re}(z) \neq 0$, and $T(z)/R(z)$ is regular. Then the quasinormal modes correspond to the singularities of $R(z)$. We have a direct analogy with the problem of scattering near the peak of the potential barrier in quantum mechanics, where ω^2 plays a role of energy, and the two turning points divide the space into three regions at which boundaries the corresponding solutions should be matched.

To extend the 3rd order WKB formula of [18] we used the techniques of Iyer and Will. Note that since the coefficients M_{ij} that connect amplitudes near the horizon with those at infinity depend only on ν (related to the overtone number n), they may be found to higher orders, simply by solving the interior (between the turning points) problem to higher orders, and without performing an explicit match of the solutions to WKB solutions in the exterior (outside turning points) regions to the same order. The result has a simple form:

$$\frac{\imath Q_0}{\sqrt{2Q_0''}} - \Lambda_2 - \Lambda_3 - \Lambda_4 - \Lambda_5 - \Lambda_6 = n + \frac{1}{2}, \qquad (2)$$

where the correction terms Λ_4, Λ_5, Λ_6 can be found in the Appendix. Note that Λ_4 coincides with preliminary formula (A3) of [18] in proper designations.

An alternative, pure algebraic approach to finding higher order WKB corrections was proposed by O. Zaslavskii [21], using a quantum anharmonic oscillator problem where WKB correction terms come from perturbation theory corrections to the potential anharmonicity. When considering BH QNMs the WKB approach was effectively used also in many problems (see for example [22]- [30] [22–30] and references therein).

III. QNMS OF A SCHWARZSCHILD BLACK HOLE

"The potential" $Q(x)$ in case of a Schwarzschild black hole has the form

$$Q(x) = \omega^2 - \left(1 - \frac{1}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{1-s^2}{r^3}\right), \qquad (3)$$

where $s = 0$ corresponds to scalar perturbations, $s = 1/2$ to neutrino perturbations, $s = 1$ to electromagnetic perturbations, $s = 3/2$ to gravitino perturbations, $s = 2$ to gravitational perturbations. The tortoise coordinate $x = \ln(r-1)+r$, where we put the mass of the black hole to be $M = 1$. The effective potential $V = -Q(x) + \omega^2$ as a function of r is shown on Fig. 1. The quasinormal modes satisfies the boundary conditions:

$$\phi(x) \sim c_{\pm}e^{\mp\omega x i} \qquad \text{as} \qquad x \to \pm\infty. \qquad (4)$$

We use the 6th order WKB formula to re-obtain the QNMs of a Schwarzschild black hole and compare the results with the 3rd order WKB technique and with numerical results given in [18], [19] which are expected to be exact. One can check out that each extra WKB order gives the values of ω^2 which are closer to their numerical values than the previous order formula for all the modes we have considered. Thus we are sure that the 6th order WKB formula, obtained here, is correct. It is known that the WKB technique is the more accurate, the less the overtone number n, and the more the multipole number l. The 3rd order technique gives the same results as numerical ones for $l \geq 4$ at lower overtones. We see from the following tables that the 6th order formula does the same already for $l = 1$. Thus, extended to the 6th order, the WKB method makes it possible to calculate the low laying QN frequencies as accurately as numerical methods, at least for a Schwarzschild black hole. It is understood that when considering more complex black holes we shall have greater error, nevertheless even for a Kerr black hole we obtain reasonable agreement with numerical results [20] except for an extremal parameters of the angular momentum. For an extremal Reissner–Nordstrem black hole, the WKB formula, quite unexpectedly, gives the results which are more accurate than the unmodified Leaver method [30].

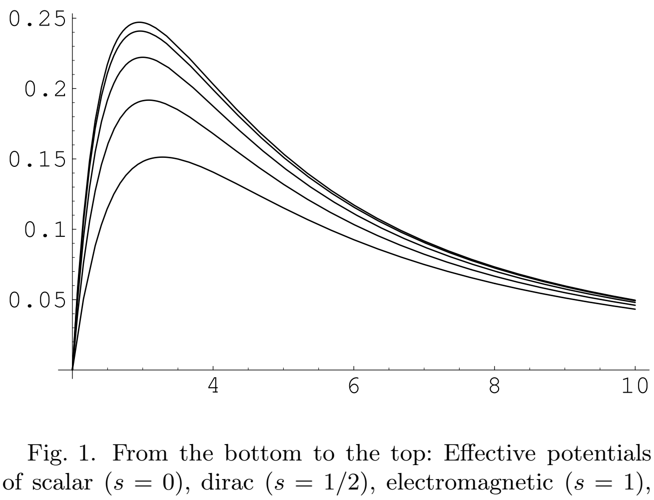

Fig. 1. From the bottom to the top: Effective potentials of scalar ($s = 0$), dirac ($s = 1/2$), electromagnetic ($s = 1$), gravitino ($s = 3/2$), graviton ($s = 2$) perturbations of $l = 2$ multipole.

multipole, overtone	numerical	3rd order WKB	6th order WKB
$l = 0,\, n = 0$	$0.1105 - 0.1049i$	$0.1046 - 0.1152i$	$0.1105 - 0.1008i$
$l = 1,\, n = 0$	$0.2929 - 0.0977i$	$0.2911 - 0.0980i$	$0.2929 - 0.0977i$
$l = 1,\, n = 1$	$0.2645 - 0.3063i$	$0.2622 - 0.3074i$	$0.2645 - 0.3065i$
$l = 2,\, n = 0$	$0.4836 - 0.0968i$	$0.4832 - 0.0968i$	$0.4836 - 0.0968i$
$l = 2,\, n = 1$	$0.4639 - 0.2956i$	$0.4632 - 0.2958i$	$0.4638 - 0.2956i$
$l = 2,\, n = 2$	$0.4305 - 0.5086i$	$0.4317 - 0.5034i$	$0.4304 - 0.5087i$

Table 1. Schwarzschild QN frequencies for scalar perturbations ($s = 0$).

multipole, overtone	numerical	3rd order WKB	6th order WKB
$l = 1,\, n = 0$	–	$0.2803 - 0.0969i$	$0.2822 - 0.0967i$
$l = 1,\, n = 1$	–	$0.2500 - 0.3049i$	$0.2525 - 0.3040i$
$l = 2,\, n = 0$	–	$0.4768 - 0.9639i$	$0.4772 - 0.0963i$
$l = 2,\, n = 1$	–	$0.4565 - 0.2947i$	$0.4571 - 0.2945i$
$l = 2,\, n = 2$	–	$0.4244 - 0.5016i$	$0.4231 - 0.5070i$
$l = 3,\, n = 0$	–	$0.6706 - 0.0963i$	$0.6708 - 0.0963i$
$l = 3,\, n = 1$	–	$0.6557 - 0.2917i$	$0.6560 - 0.2917i$
$l = 3,\, n = 2$	–	$0.6299 - 0.4931i$	$0.6286 - 0.4950i$
$l = 3,\, n = 3$	–	$0.5970 - 0.6997i$	$0.5932 - 0.7102i$

Table 2. Schwarzschild QN frequencies for Dirac perturbations ($s = 1/2$).

multipole, overtone	numerical	3rd order WKB	6th order WKB
$l = 1,\, n = 0$	$0.2483 - 0.0925i$	$0.2459 - 0.0931i$	$0.2482 - 0.0926i$
$l = 1,\, n = 1$	$0.2145 - 0.2937i$	$0.2113 - 0.2958i$	$0.2143 - 0.2941i$
$l = 2,\, n = 0$	$0.4576 - 0.0950i$	$0.4571 - 0.0951i$	$0.4576 - 0.0950i$
$l = 2,\, n = 1$	$0.4365 - 0.2907i$	$0.4358 - 0.2910i$	$0.4365 - 0.2907i$
$l = 2,\, n = 2$	$0.4012 - 0.5016i$	$0.4023 - 0.4959i$	$0.4009 - 0.5017i$
$l = 3,\, n = 0$	$0.6569 - 0.0956i$	$0.6567 - 0.0956i$	$0.6569 - 0.0956i$
$l = 3,\, n = 1$	$0.6417 - 0.2897i$	$0.6415 - 0.2898i$	$0.6417 - 0.2897i$
$l = 3,\, n = 2$	$0.6138 - 0.4921i$	$0.6151 - 0.4901i$	$0.6138 - 0.4921i$
$l = 3,\, n = 3$	$0.5779 - 0.7063i$	$0.5814 - 0.6955i$	$0.5775 - 0.7065i$

Table 3. Schwarzschild QN frequencies for electromagnetic perturbations ($s = 1$).

multipole, overtone	numerical	3rd order WKB	6th order WKB
$l = 1,\, n = 0$	–	$0.1817 - 0.0866i$	$0.1739 - 0.08357i$
$l = 1,\, n = 1$	–	$0.1354 - 0.2812i$	$0.1198 - 0.2813i$
$l = 2,\, n = 0$	–	$0.4231 - 0.926i$	$0.4236 - 0.0925i$
$l = 2,\, n = 1$	–	$0.4000 - 0.2842i$	$0.4007 - 0.2838i$
$l = 2,\, n = 2$	–	$0.3636 - 0.4853i$	$0.3618 - 0.4919i$
$l = 3,\, n = 0$	–	$0.6332 - 0.0945i$	$0.6333 - 0.0944i$
$l = 3,\, n = 1$	–	$0.6173 - 0.2864i$	$0.6175 - 0.2863i$
$l = 3,\, n = 2$	–	$0.5898 - 0.4846i$	$0.5884 - 0.4868i$
$l = 3,\, n = 3$	–	$0.5547 - 0.6882i$	$0.5505 - 0.7000i$

Table 4. Schwarzschild QN frequencies for gravitino perturbations ($s = 3/2$).

multipole, overtone	numerical	3rd order WKB	6th order WKB
$l=2$, $n=0$	$0.3737-0.0890i$	$0.3732-0.0892i$	$0.3736-0.0890i$
$l=2$, $n=1$	$0.3467-0.2739i$	$0.3460-0.2749i$	$0.3463-0.2735i$
$l=2$, $n=2$	$0.3011-0.4783i$	$0.3029-0.4711i$	$0.2985-0.4776i$
$l=3$, $n=0$	$0.5994-0.0927i$	$0.5993-0.0927i$	$0.5994-0.0927i$
$l=3$, $n=1$	$0.5826-0.2813i$	$0.5824-0.2814i$	$0.5826-0.2813i$
$l=3$, $n=2$	$0.5517-0.4791i$	$0.5532-0.4767i$	$0.5516-0.4790i$
$l=3$, $n=3$	$0.5120-0.6903i$	$0.5157-0.6774i$	$0.5111-0.6905i$
$l=4$, $n=0$	$0.8092-0.0942i$	$0.8091-0.0942i$	$0.8092-0.0942i$
$l=4$, $n=1$	$0.7966-0.2843i$	$0.7965-0.2844i$	$0.7966-0.2843i$
$l=4$, $n=2$	$0.7727-0.4799i$	$0.7736-0.4790i$	$0.7727-0.4799i$
$l=4$, $n=3$	$0.7398-0.6839i$	$0.7433-0.6783i$	$0.7397-0.6839i$
$l=4$, $n=4$	$0.7015-0.8982i$	$0.7072-0.8813i$	$0.7006-0.8985i$

Table 5. Schwarzschild QN frequencies for graviton perturbations ($s=2$).

IV. CONCLUSION

We have derived the 6th order WKB formula for calculations of lower overtone quasinormal modes and checked it on example of a Schwarzschild black hole. It proved out that the 6th order WKB formula is much more accurate than the 3rd order one, and within lower overtone gives the results practically coinciding with those found by numerical methods for all multipoles $l \geq 1$. The calculation of scalar quasinormal modes of charged rotating dilatonic black holes with the help of the above formula [20] makes it possible to describe quasinormal behaviour with good accuracy unless the angular momentum of the black hole is near its extremal value.

V. APPENDIX

Here we shall follow the designations: Q_0 means the value of the potential Q at its peak, while Q_i is the ith derivative of Q with the respect to the tortoise coordinate x. Then Q_i^j is the jth power of the ith derivative of Q.

$$\Lambda_4 = \frac{1}{597196800\sqrt{2}Q_2^7\sqrt{Q_2}}(2536975Q_3^6 - 9886275Q_2Q_3^4Q_4 + 5319720Q_2^2Q_3^3Q_5$$
$$- 225Q_2^2Q_3^2(-40261Q_4^2 + 9688Q_2Q_6) + 3240Q_2^3Q_3(-1889Q_4Q_5 + 220Q_2Q_7)$$
$$- 729Q_2^3(1425Q_4^3 - 1400Q_2Q_4Q_6 + 8Q_2(-123Q_5^2 + 25Q_2Q_8)))$$
$$+ \frac{(n+1/2)^2}{4976640\sqrt{2}Q_2^7\sqrt{Q_2}}(348425Q_3^6 - 1199925Q_2Q_3^4Q_4 + 57276Q_2^2Q_3^3Q_5$$
$$- 45Q_2^2Q_3^2(-20671Q_4^2 + 4552Q_2Q_6) + 1980Q_2^3Q_3(-489Q_4Q_5 + 52Q_2Q_7)$$
$$- 27Q_2^3(2845Q_4^3 - 2360Q_2Q_4Q_6 + 56Q_2(-31Q_5^2 + 5Q_2Q_8)))$$
$$+ \frac{(n+1/2)^4}{2488320\sqrt{2}Q_2^7\sqrt{Q_2}}(192925Q_3^6 - 581625Q_2Q_3^4Q_4 + 234360Q_2^2Q_3^3Q_5$$
$$- 45Q_2^2Q_3^2(-8315Q_4^2 + 1448Q_2Q_6) + 1080Q_2^3Q_3(-161Q_4Q_5 + 12Q_2Q_7)$$
$$- 27Q_2^3(625Q_4^3 - 440Q_2Q_4Q_6 + 8Q_2(-63Q_5^2 + 5Q_2Q_8))), \tag{5}$$

$$\Lambda_5 = \frac{(n+1/2)}{57330892800Q_2^{10}}(2768256Q_{10}Q_2^7 - 1078694575Q_3^8 + 5357454900Q_2Q_3^6Q_4$$

$$
\begin{aligned}
&- 2768587920Q_2^2Q_3^5Q_5 + 90Q_2^2Q_3^4(-88333625Q_4^2 + 12760664Q_2Q_6)\\
&- 4320Q_2^3Q_3^3(-1451425Q_4Q_5 + 91928Q_2Q_7) - 27Q_2^4(7628525Q_4^4 - 9382480Q_2Q_4^2Q_6\\
&+ 64Q_2^2(19277Q_2^6 + 37764Q_2Q_7) + 576Q_2Q_4(-21577Q_5^2 + 2505Q_2Q_8))\\
&+ 540Q_2^3Q_3^2(6515475Q_4^3 - 3324792Q_2Q_4Q_6 + 16Q_2(-126468Q_5^2 + 12679Q_2Q_8))\\
&- 432Q_2^4Q_3(5597075Q_4^2Q_5 - 854160Q_2Q_4Q_7 + 8Q_2(-145417Q_5Q_6 + 6685Q_2Q_9)))\\
&+ \frac{(n+1/2)^3}{477757440Q_2^{10}}(31104Q_{10}Q_2^7 - 42944825Q_3^8 + 193106700Q_2Q_3^6Q_4\\
&- 90039120Q_2^2Q_3^5Q_5 + 30Q_2^2Q_3^4(-8476205Q_4^2 + 1102568Q_2Q_6)\\
&- 4320Q_2^3Q_3^3(-41165Q_4Q_5 + 2312Q_2Q_7) - 9Q_2^4(445825Q_4^4 - 472880Q_2Q_4^2Q_6\\
&+ 64Q_2^2(829Q_2^6 + 1836Q_2Q_7) + 4032Q_2Q_4(-179Q_5^2 + 15Q_2Q_8))\\
&+ 180Q_2^3Q_3^2(532615Q_4^3 - 241224Q_2Q_4Q_6 + 16Q_2(-9352Q_5^2 + 799Q_2Q_8))\\
&- 144Q_2^4Q_3(392325Q_4^2Q_5 - 51600Q_2Q_4Q_7 + 8Q_2(-8853Q_5Q_6 + 335Q_2Q_9)))\\
&+ \frac{(n+1/2)^5}{1194393600Q_2^{10}}(10368Q_{10}Q_2^7 - 66578225Q_3^8 + 272124300Q_2Q_3^6Q_4\\
&- 112336560Q_2^2Q_3^5Q_5 + 9450Q_2^2Q_3^4(-33775Q_4^2 + 3656Q_2Q_6)\\
&- 151200Q_2^3Q_3^3(-1297Q_4Q_5 + 56Q_2Q_7) - 27Q_2^4(89075Q_4^4 - 83440Q_2Q_4^2Q_6\\
&+ 64Q_2^2(131Q_2^6 + 396Q_2Q_7) + 576Q_2Q_4(-343Q_5^2 + 15Q_2Q_8))\\
&+ 540Q_2^3Q_3^2(188125Q_4^3 - 71400Q_2Q_4Q_6 + 16Q_2(-3052Q_5^2 + 177Q_2Q_8))\\
&- 432Q_2^4Q_3(118825Q_4^2Q_5 - 11760Q_2Q_4Q_7 + 8Q_2(-2303Q_5Q_6 + 55Q_2Q_9))),
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\Lambda_6 = &\frac{-i}{202263389798400Q_2^{12}\sqrt{2Q_2}}(-171460800Q_{12}Q_2^9 + 1714608000Q_{11}Q_2^8Q_3\\
&- 10268596800Q_{10}Q_2^7Q_3^2 + 970010662775Q_3^{10} + 3772137600Q_{10}Q_2^8Q_4\\
&- 6262634175525Q_2Q_3^8Q_4 + 13782983196150Q_2^2Q_3^6Q_4^2 - 11954148125850Q_2^3Q_3^4Q_4^3\\
&+ 3449170577475Q_2^4Q_3^2Q_4^4 - 144528059025Q_2^5Q_4^5 + 3352602187200Q_2^2Q_3^7Q_5\\
&- 12300730092000Q_2^3Q_3^5Q_4Q_5 + 11994129604800Q_2^4Q_3^3Q_4^2Q_5 - 2624788605600Q_2^5Q_3Q_4^3Q_5\\
&+ 2580769643760Q_2^4Q_3^4Q_5^2 - 3453909784416Q_2^5Q_3^2Q_4Q_5^2 + 438440697072Q_2^6Q_4^2Q_5^2\\
&+ 260524397952Q_2^6Q_3Q_5^3 - 1475306441280Q_2^3Q_3^6Q_6 + 4329682610400Q_2^4Q_3^4Q_4Q_6\\
&- 2865128172480Q_2^5Q_3^2Q_4^2Q_6 + 233443879200Q_2^6Q_4^3Q_6 - 1660199804928Q_2^5Q_3^3Q_5Q_6\\
&+ 1281705296256Q_2^6Q_3Q_4Q_5Q_6 - 87403857408Q_2^7Q_5^2Q_6 + 231105873600Q_2^6Q_3^2Q_6^2\\
&- 68412859200Q_2^7Q_4Q_6^2 + 552968700480Q_2^4Q_3^5Q_7 - 1231789749120Q_2^5Q_3^3Q_4Q_7\\
&+ 470726303040Q_2^6Q_3Q_4^2Q_7 + 413953400448Q_2^6Q_3^2Q_5Q_7 - 126242178048Q_2^7Q_4Q_5Q_7\\
&- 91489305600Q_2^7Q_3Q_6Q_7 + 5619715200Q_2^8Q_7^2 - 175752294480Q_2^5Q_3^4Q_8\\
&+ 271759652640Q_2^6Q_3^2Q_4Q_8 - 39736040400Q_2^7Q_4^2Q_8 - 73378363968Q_2^7Q_3Q_5Q_8
\end{aligned}
$$

$$+ 9773265600Q_2^8Q_6Q_8 + 47107126080Q_2^6Q_3^3Q_9 - 43345290240Q_2^7Q_3Q_4Q_9 + 7400248128Q_2^8Q_5Q_9)$$

$$- \frac{(n+1/2)^2 i}{687970713600Q_2^{12}\sqrt{2Q_2}}(-4551552Q_{12}Q_2^9 + 60279552Q_{11}Q_2^8Q_3$$

$$- 425036160Q_{10}Q_2^7Q_3^2 + 73727194625Q_3^{10} + 116743680Q_{10}Q_2^8Q_4$$

$$- 443649208275Q_2Q_3^8Q_4 + 901144103850Q_2^2Q_3^6Q_4^2 - 711096726150Q_2^3Q_3^4Q_4^3$$

$$+ 182164306725Q_2^4Q_3^2Q_4^4 - 6289615575Q_2^5Q_4^5 + 222467624400Q_2^2Q_3^7Q_5$$

$$- 746418445200Q_2^3Q_3^5Q_4Q_5 + 653423900400Q_2^4Q_3^3Q_4^2Q_5 - 124319674800Q_2^5Q_3Q_4^3Q_5$$

$$+ 143980943040Q_2^4Q_3^4Q_5^2 - 169712521920Q_2^5Q_3^2Q_4Q_5^2 + 18188188416Q_2^6Q_4^2Q_5^2$$

$$+ 11240861184Q_2^6Q_3Q_5^3 - 91198200240Q_2^3Q_3^6Q_6 + 241513732080Q_2^4Q_3^4Q_4Q_6$$

$$- 140030897040Q_2^5Q_3^2Q_4^2Q_6 + 9200103120Q_2^6Q_4^3Q_6 - 84218693760Q_2^5Q_3^3Q_5Q_6$$

$$+ 55248386688Q_2^6Q_3Q_4Q_5Q_6 - 3173043456Q_2^7Q_5^2Q_6 + 10464952896Q_2^6Q_3^2Q_6^2$$

$$- 2403421632Q_2^7Q_4Q_6^2 + 31637744640Q_2^4Q_3^5Q_7 - 62649953280Q_2^5Q_3^3Q_4Q_7$$

$$+ 20409822720Q_2^6Q_3Q_4^2Q_7 + 18860532480Q_2^6Q_3^2Q_5Q_7 - 4693344768Q_2^7Q_4Q_5Q_7$$

$$- 3625731072Q_2^7Q_3Q_6Q_7 + 188054784Q_2^8Q_7^2 - 9155635200Q_2^5Q_3^4Q_8$$

$$+ 12238024320Q_2^6Q_3^2Q_4Q_8 - 1405278720Q_2^7Q_4^2Q_8 - 2866700160Q_2^7Q_3Q_5Q_8$$

$$+ 303295104Q_2^8Q_6Q_8 + 2210705280Q_2^6Q_3^3Q_9 - 1685525760Q_2^7Q_3Q_4Q_9 + 235488384Q_2^8Q_5Q_9)$$

$$- \frac{(n+1/2)^4 i}{20065812480Q_2^{12}\sqrt{2Q_2}}(-66528Q_{12}Q_2^9 + 1245888Q_{11}Q_2^8Q_3$$

$$- 11158560Q_{10}Q_2^7Q_3^2 + 4668804525Q_3^{10} + 2116800Q_{10}Q_2^8Q_4$$

$$- 25898331375Q_2Q_3^8Q_4 + 47959232650Q_2^2Q_3^6Q_4^2 - 33861927750Q_2^3Q_3^4Q_4^3$$

$$+ 7454763225Q_2^4Q_3^2Q_4^4 - 184988475Q_2^5Q_4^5 + 11891917800Q_2^2Q_3^7Q_5$$

$$- 36105463800Q_2^3Q_3^5Q_4Q_5 + 27953667000Q_2^4Q_3^3Q_4^2Q_5 - 4457716200Q_2^5Q_3Q_4^3Q_5$$

$$+ 6285855240Q_2^4Q_3^4Q_5^2 - 6471756144Q_2^5Q_3^2Q_4Q_5^2 + 565259688Q_2^6Q_4^2Q_5^2$$

$$+ 380939328Q_2^6Q_3Q_5^3 - 4375251160Q_2^3Q_3^6Q_6 + 10317018600Q_2^4Q_3^4Q_4Q_6$$

$$- 5113813320Q_2^5Q_3^2Q_4^2Q_6 + 238888440Q_2^6Q_4^3Q_6 - 3203871552Q_2^5Q_3^3Q_5Q_6$$

$$+ 1758685824Q_2^6Q_3Q_4Q_5Q_6 - 88566912Q_2^7Q_5^2Q_6 + 335466432Q_2^6Q_3^2Q_6^2$$

$$- 55073088Q_2^7Q_4Q_6^2 + 1351294560Q_2^4Q_3^5Q_7 - 2341442880Q_2^5Q_3^3Q_4Q_7$$

$$+ 626542560Q_2^6Q_3Q_4^2Q_7 + 619520832Q_2^6Q_3^2Q_5Q_7 - 123524352Q_2^7Q_4Q_5Q_7$$

$$- 96574464Q_2^7Q_3Q_6Q_7 + 4048704Q_2^8Q_7^2 - 341160120Q_2^5Q_3^4Q_8$$

$$+ 386210160Q_2^6Q_3^2Q_4Q_8 - 30837240Q_2^7Q_4^2Q_8 - 78073632Q_2^7Q_3Q_5Q_8$$

$$+ 5848416Q_2^8Q_6Q_8 + 70415520Q_2^6Q_3^3Q_9 - 43424640Q_2^7Q_3Q_4Q_9 + 5255712Q_2^8Q_5Q_9)$$

$$- \frac{(n+1/2)^6 i}{300987187200Q_2^{12}\sqrt{2Q_2}}(-72576Q_{12}Q_2^9 + 1886976Q_{11}Q_2^8Q_3$$

$$
\begin{aligned}
&- 22135680Q_{10}Q_2^7Q_3^2 + 27463538375Q_3^{10} + 2903040Q_{10}Q_2^8Q_4 \\
&- 141448688325Q_2Q_3^8Q_4 + 240655765350Q_2^2Q_3^6Q_4^2 - 152907158250Q_2^3Q_3^4Q_4^3 \\
&+ 28724479875Q_2^4Q_3^2Q_4^4 - 413669025Q_2^5Q_4^5 + 59058073200Q_2^2Q_3^7Q_5 \\
&- 164264209200Q_2^3Q_3^5Q_4Q_5 + 113654696400Q_2^4Q_3^3Q_4^2Q_5 - 15166342800Q_2^5Q_3Q_4^3Q_5 \\
&+ 26061194880Q_2^4Q_3^4Q_5^2 - 23876233920Q_2^5Q_3^2Q_4Q_5^2 + 1767189312Q_2^6Q_4^2Q_5^2 \\
&+ 1292433408Q_2^6Q_3Q_5^3 - 18902165520Q_2^3Q_3^6Q_6 + 40256773200Q_2^4Q_3^4Q_4Q_6 \\
&- 17116974000Q_2^5Q_3^2Q_4^2Q_6 + 483582960Q_2^6Q_4^3Q_6 - 11384150400Q_2^5Q_3^3Q_5Q_6 \\
&+ 5285056896Q_2^6Q_3Q_4Q_5Q_6 - 246903552Q_2^7Q_5^2Q_6 + 992779200Q_2^6Q_3^2Q_6^2 \\
&- 101860416Q_2^7Q_4Q_6^2 + 4966859520Q_2^4Q_3^5Q_7 - 7661606400Q_2^5Q_3^3Q_4Q_7 \\
&+ 1683037440Q_2^6Q_3Q_4^2Q_7 + 1861574400Q_2^6Q_3^2Q_5Q_7 - 316141056Q_2^7Q_4Q_5Q_7 \\
&- 235146240Q_2^7Q_3Q_6Q_7 + 8895744Q_2^8Q_7^2 - 1042372800Q_2^5Q_3^4Q_8 \\
&+ 1016789760Q_2^6Q_3^2Q_4Q_8 - 52436160Q_2^7Q_4^2Q_8 - 189060480Q_2^7Q_3Q_5Q_8 \\
&+ 9217152Q_2^8Q_6Q_8 + 175190400Q_2^6Q_3^3Q_9 - 87816960Q_2^7Q_3Q_4Q_9 + 10378368Q_2^8Q_5Q_9). \qquad (7)
\end{aligned}
$$

All six WKB corrections printed in MATHEMATICA are available from the author in electronic form upon request.

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КВАЗІНОРМАЛЬНІ МОДИ ЧОРНОЇ ДІРИ ШВАРЦШІЛЬДА ТА ВКБ-ПІДХІД ВИЩОГО ПОРЯДКУ

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Нижчі характеристичні (квазінормальні) моди чорних дір можна обчислити за допомогою напіваналітичного методу, заснованого на модифікованому ВКБ-підході. Ми продовжили ВКБ-формулу Аєра й Вілла з третього порядку до 6-го порядку після ейконального наближення й одержали, таким чином, прецизійну формулу для обчислення КН-мод. За допомогою отриманої формули знайдено КН-моди, які відповідають загасанню скалярного, діраківського, електромагнетного, ґравітаційного й ґравітонного полів на фоні шварцшільдової чорної діри. Для чорної діри Шварцшільда ВКБ-значення для нижчих обертонів збігається з чисельними результатами вже для $l \geq 1$, тоді як ВКБ-формула в 3-му порядку дає таку ж точність лише при $l \geq 4$.