NILPOTENT ELEMENTS IN PHYSICS*

A. M. Frydryszak†
Institute of Theoretical Physics, University of Wroclaw,
pl. M. Borna 9, 50–204 Wroclaw, Poland
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We briefly review some issues of the nilpotent objects in theoretical physics using simple models as an illustration. Nilpotent elements appear at quantum and classical level in several ways. On the one hand there are celebrated BRST operators, external derivative and related with them cohomology theory, on the other hand there are dual numbers which are a less known structure than complex numbers but important in many approaches, then there are commuting nilpotent variables, somehow generalizing the nilpotent but anticommuting Grassmann variables used in super-mathematics and SUSY models. As an interesting fact we note that nilpotent variables demand the use of the generalized light-cone geometry with the metric of the null signature.
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I. INTRODUCTION

Despite the fact that nilpotent objects are more difficult to handle than regular invertible ones, they play an important role in theoretical physics. It used to be a customary way in constructing structures, to get rid of nilpotents. Null divisors complicate life. That is why in the standard approaches in mathematics or physics we avoid nilpotents. However, there are spectacular theories where nilpotent elements are crucial. The BRST formalism is the best example were the nontrivial nilpotent operator Q,

$$Q^2 = 0 \tag{1}$$

together with the appropriate space allows to quantize a gauge field theory system. This highly nontrivial formalism was being intensively developed in the 90s of the last century. We can say that the meaning of the trivial, at the first sight, nilpotency relation strongly depends on the way it is realized. In the case of the BRST theory we construct a sophisticated robust structure to represent a simple relation of nilpotency. Obviously there are other strong demands like hermiticity, etc. Another well known example gives the supersymmetry theory, where the anticommuting variables are commonly used. Here nilpotency is a byproduct of the anticommutativity of the Grassmannian variables representing fermionic degrees of freedom.

$$\theta\theta' = -\theta'\theta \Rightarrow \theta^2 = 0. \tag{2}$$

A reach branch of mathematics i. e. super-mathematics (superalgebra, superanalysis, supergometry) and a vivid bundle of physical theories, like super-Yang–Mills, supergravity, supersymmetric quantum mechanics, have been developed since the 50s of the 20th century when the first appearance of Grassmanian variables in theoretical physics was noted [1–3]. However, one can look for the realization of the nilpotency condition in the simple commutative structure, not in the graded-commutative or noncommutative one. A possibility is given by the construction invented by Clifford [4]. He introduced the "nilpotent unit"

$$\iota^2 = 0 \tag{3}$$

and following the construction resembling complex numbers defined the so-called dual numbers $\mathcal{D}$

$$z = x + \iota\, y, \qquad x, y \in \mathbb{R}, \mathbb{C}, \tag{4}$$

$$z \cdot z' = x \cdot x' + \iota(x \cdot y' + x' \cdot y), \qquad z \in \mathcal{D}, \tag{5}$$

$$\bar{z} = x - \iota y, \qquad z^{-1} = |z|^{-2}\bar{z}, \qquad |z| = x, \tag{6}$$

$$z = x(1 + \iota\frac{y}{x}) = r(1 + \iota\varphi), \qquad \mathrm{Arg}(z) = \frac{y}{x}. \tag{7}$$

Such numbers and their generalization to the dual quaternions have been known since the 80s of the 19th century and they are related to the rotations and translations in the three dimensional Euclidean space. In the case when rotation is composed with translation we get the so-called screw motion [5, 6]. Dual numbers are a common tool in the differential algebra formalism [7]. There is a very interesting model constructed by Wald *et al.* [8–10] to study quantization of gravity where dual numbers are present. This model avoids the obstruction coming from the Coleman–Mandula theorem [11], so not only supersymmetry is a solution for this obstruction.
In the following we will review some applications of the dual numbers and discuss new issues of the commuting nilpotent variables, with the nilpotency of the second order. We do not review here the generalization known

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as paragrassmann relations where the nilpotency is of a higher n-order i. e. $\vartheta^n = 0$. We will not touch upon usual supersymmetry and BRST formalism. The main goal of this text is to call attention to the less known aspects of the presence of nilpotent objects in physical theories.

II. DUAL NUMBERS IN THEORETICAL PHYSICS

In the introduction we have recalled basic definitions related to the dual numbers. As is easily seen, due to the nilpotency of ι the geometry of the dual plane is different from that of the complex plane. It is interesting to observe that a circle on the dual plane with the centre in z_0 consists of the points of two vertical parallel lines located symmetrically on both sides of z_0. Dual numbers are obviously nonarchimedean. Namely, let $z = (x, y) \in D^+$, when the first nonzero coordinate is positive. Then:

- $z = 0$, or $z \in D^+$, or $-z \in D^+$,
- $x, y \in D^+ \Rightarrow x + y \in D^+$.

This allows to define the order in $\mathcal{D}$ in the following sense: $x < y \Leftrightarrow y - x \in D^+$. Now with respect to the order defined by the sets D^+ we see that the ι is infinitesimal (infinitely small) in the sense that

$$\iota < 1 \Rightarrow n \cdot \iota < 1, \quad n \in \mathbb{N} \tag{8}$$

(n is a natural number). This observation gives justification to the calculus of infintesimals used frequently in physics when making linear approximations (product of two infinitesimals is negligible). Dual numbers are a formal structure behind this popular practice. This fact is frequently neglected.

A. Dual functions

One can define dual (number) functions analogously to the complex functions [12, 13]

$$\begin{aligned} &f(z) = u(x, y) + \iota v(x, y), \\ &f(z + h) - f(z) = f'(z) \cdot h + \mathcal{O} \end{aligned} \tag{9}$$

and in this case analogs of the Cauchy–Riemann equations have the following form

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = 0, \tag{10}$$

and the dual differentiable function has the following simple form

$$f(z) = u(x) + \iota(v(x) + y \cdot u'(x)). \tag{11}$$

For example, the dual exponent is $e^z = e^x(1 + \iota y)$ and the trigonometric function is $\sin(z) = \sin(x) + \iota y \cos(x)$

B. Numerical differential algebra

Despite the simplicity of the form of dual differentiable function these objects are of great use in practical explicit calculations of derivatives for functions of real variables. The area of applications of this kind is named Numerical Differential Algebra and is used in accelerator physics and optics for solving highly nonlinear systems [14]. In the most general terms one considers an equation of the type

$$z_f = M(z_i, a), \tag{12}$$

where a denotes a set of parameters and z relevant fields (z_i — initial, z_f — final). Parameters are from chosen intervals and are known with some precision. The transition matrix $\partial_z M$ and sensitivity $\partial_a M$ are usually obtained using computer analysis and the so-called TPSA — Truncated Power Series Algebra. To explain this idea let us consider a very simple example, namely Veronese Algebra which is denoted by ${}_1D_1$. Let for the elements of ${}_1D_1$ the multiplication be of the dual numbers' type

$$(a_0, a_1)(b_0, b_1) = (a_0 b_0, a_0 b_1 + a_1 b_0) \tag{13}$$

and the derivation be given by

$$(D, \partial): \ \ \partial(a_0, a_1) = (0, a_1). \tag{14}$$

We introduce a notion of standard functions F in the form

$$F(a_0, a_1) = (f(a_0), a_1 f'(a_0)) \tag{15}$$

hence choosing a specific point $(x, 1)$ for evaluation we get

$$F(x, 1) = (f(x), f'(x)). \tag{16}$$

This gives a practical algoritm for the "algebraic" differentiation.

Example: let $f(x) = \frac{1}{1+\frac{1}{x}}$, then

$$\begin{aligned} &id(x, 1) \Rightarrow ((x, 1)^{-1} + (x, 1))^{-1} \\ &= ((\frac{1}{x}, -\frac{1}{x^2}) + (x, 1))^{-1} \\ &= \left(\frac{1}{1+\frac{1}{x}}, -\frac{1-\frac{1}{x^2}}{(1+\frac{1}{x^2})^2}\right). \end{aligned} \tag{17}$$

This idea can be generalized to the case of n-variables and m^{th} order of the derivative i. e. to the algebra ${}_mD_n$:

$$(f, \partial f, \partial^2 f, \ldots, \partial^m f), \tag{18}$$

which is a very practical tool in computer analysis [14].

C. Rotations and robotics

Kinematics is one of the oldest topics studied. For many centuries it was regarded as one of the basic sciences that explained observed physical phenomena. It was used to engineer machines. Its recent developments yield the application of the modern robot kinematics to study biological systems and their functions at the microscopic level and to the engineering of new diagnostic tools. Dual numbers are essential in describing and solving the movements of rigid bodies especially in constructing the parallel or serial robots, were special conditions on pivots and spatial constraints are assumed. The finite screw displacement — combination of translation and rotation is a basic tool in the theoretical kinematics and its representation involves dual numbers. A considerable theoretical development of the description of rigid body displacements was taking place from the early 19th century up till the late 20th century. One of the classical results, the Chasles theorem states that the most general rigid body displacement can be produced by a translation along the line followed (or preceded) by rotation about that line. Such a displacement is called a screw motion. Other essential constructions involve Cayley's formula, Hamilton's quaternions, the Clifford dual quaternions and Study's dual angle [5, 6]. The dual angle $\hat{\theta}$, proposed in the late 19th century, is used in the realization of the screw motion in $\mathbb{R}^3$

$$\begin{aligned} \vec{p}' &= \vec{p} + s\vec{u}, \\ \vec{s} &= s\vec{u}, \\ \hat{\theta} &= \theta + \iota s, \end{aligned} \tag{19}$$

where $\vec{s}$ denotes the translation vector from $\vec{p}$ to $\vec{p'}$ and s is its length, $\vec{u}$ is the unit vector and θ rotation angle. The dual number $\hat{\theta}$ is called a dual angle. The screw transformation is given in unique way by $\hat{\theta}$ and $\vec{u}$. The composition of screw transformations is realized by dual number multiplication. The explicit realization of the idea of a screw motion is with the use of dual quaternions [4]. Let $\mathbb{Q}$ denote a set of quaternions. By the dual quaternion we understand a dual number constructed over quaternion algebra

$$\mathbf{q} = q_0 + \iota q_1, \qquad q_0, q_1 \in \mathbb{Q}. \tag{20}$$

The norm of dual quaternion can be written as

$$\| \mathbf{q} \| = \| q_0 \| + \iota \frac{\langle q_0, q_1 \rangle}{\| q_0 \|}. \tag{21}$$

As is well known, rotation in the $\mathbb{R}^3$ can be represented by a unit quaternion, translations are naturally embeded into the dual quaternions

$$\mathbb{R}^3 \ni v \longrightarrow \mathbf{v} = 1 + \iota(v_0\mathbf{i} + v_1\mathbf{j} + v_2\mathbf{k}), \tag{22}$$

where $v = (v_0, v_1, v_2)$ and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are quaternion units. Now, the dual unit quaternion $\mathbf{t} = 1 + \frac{1}{2}\iota(t_0\mathbf{i} + t_1\mathbf{j} + t_2\mathbf{k})$ represents translation. Again a composition of pure rotation given by the unit quaternion $\mathbf{q}_0$ and pure translation $\mathbf{t}$ gives a screw motion

$$\mathbf{t}\mathbf{q}_0 = \mathbf{q}_0 + \frac{1}{2}\iota(t_0\mathbf{i} + t_1\mathbf{j} + t_2\mathbf{k})\, \mathbf{q}_0, \tag{23}$$

that can be compactly written in the following form

$$\mathbf{q} = \cos\frac{\hat{\theta}}{2} + \mathbf{s}\,\sin\frac{\hat{\theta}}{2}, \tag{24}$$

where $\hat{\theta} = \theta_0 + \iota\theta_1$ is, as before, dual angle and $\mathbf{s} = \mathbf{s}_0 + \iota\mathbf{s}_1$ is a dual unit vector. Geometrically $\frac{1}{2}\theta_0$ is the angle of rotation around the axis $\mathbf{s}_0$ and $\frac{1}{2}\theta_1$ is the translation along this axis. The $\mathbf{s}_1$ is a unique moment of the axis, giving position of the axis in affine space, for the given position vector $\mathbf{p}$ we have $\mathbf{s}_1 = \mathbf{p} \times \mathbf{s}_0$. A formalism based on dual quaternions is widely used in robotics and in computer 3D-simulations of body motion, it is easier to implement, eliminates artifacts and is much faster.

There is a rich area of applications of a dual valued object in practical computer analysis calculations and simulations of movements of complicated rigid bodies [15–17]. But the formalism with the use of dual quaternions is applied as well in classical electrodynamics [18].

D. Dual numbers in field theory

As is well known, in local quantum field theory it is impossible within the conventional structure, to combine nontrivially space-time symmetries with internal ones. This fact was recognized in 1967 in the famous Coleman and Mandula theorem [11]. A new kind of symmetries outside Lie group type were to be introduced. They were uper symmetries. The work of Haag–Łopuszański–Sohnius [19] showed that the graded algebra of generators of symmetries of the S matrix should be admitted in QFT. It is little known that despite supersymmetry there was considered another possibility to go around the obstruction of the Coleman–Mandula theorem. In a series of papers of Wald *et al.* [8–10] in the context of the gravity theory a simplified model of the scalar $\lambda\phi^4$ with the fields taking values in the dual numbers was suggested.

$$\phi(x) = \psi(x) + \iota\chi(x). \tag{25}$$

The model given by the action

$$S[\phi, \chi] = \int (-\partial_a\psi\partial^a\chi - m^2\psi\chi - \lambda\psi^3\chi) \tag{26}$$

exhibits interesting symmetry properties. The field equations for component fields are of the form

$$\begin{cases} \Box\psi - m^2\psi - \lambda\psi^3 = 0 \\ \Box\chi - (m^2 + 3\lambda\psi^2)\chi = 0, \end{cases} \tag{27}$$

where a nontrivial extension of the Poincaré group is introduced.

$$\begin{cases} \psi \;\rightarrow\; \mathcal{P}_g\psi \\ \chi \;\rightarrow\; \mathcal{P}_g\chi + \pounds_h\mathcal{P}_g\psi \end{cases}. \tag{28}$$

This theory is consistent classically. However, as the authors state: “Thus to summarize the result [...], we find that the conventional S-matrix theory, constructed using the Feynman rules, for our model quantum field theory does not make sense, and there do not appear to be any promising alternative approaches. This strongly suggests that a similar conclusion will apply to the new class of gauge theories for massless spin-2 fields, even though at the classical level many of these theories have well-posed initial value formulation and have asymptotically well-behaved solutions to the equations of motion”. This result indirectly supports the distinguished role of supersymmetry whose presence especially at the quantum level is particularly fruitfull.

E. Dual mechanics

Dual number valued mechanical systems were considered in [20] as an on-shell limit of a supersymmetric system. A more general formalism of dual classical mechanics was considered there as well. Let us briefly discuss its elements.

1. Lagrangian formalism

A general Lagrangian of a dual classical mechanical system can be taken in the following form $\mathbf{L} = L_0 + \boldsymbol{\iota} L_N$. Assuming that time is double number valued $\mathbf{T} = t + \boldsymbol{\iota}\, t_N$, one obtains the following general expansion for coordinate functions

$$\mathbf{x}(\mathbf{T}) = x_0(\mathbf{T}) + \boldsymbol{\iota}\, x_N(\mathbf{T}). \tag{29}$$

and expansion of dual valued velocity

$$\frac{d\mathbf{x}(\mathbf{T})}{d\mathbf{T}} = \dot{x}_0(t) + \boldsymbol{\iota}\, \left[\dot{x}_N(t) + \ddot{x}_0(t)\, t_N\right]. \tag{30}$$

Finally the expanded Lagrangian has the form

$$\mathbf{L} = \frac{1}{2}\left(\frac{d\mathbf{x}(\mathbf{T})}{d\mathbf{T}}\right)^2 - \mathbf{U}(\mathbf{x}(\mathbf{T})), \quad \mathbf{U}(\mathbf{x}) = U_0(\mathbf{x}) + \boldsymbol{\iota}\, U_N(\mathbf{x}). \tag{31}$$

When we restrict to the case with $t_N = 0$ i. e., to the theories with the usual time parameter, the above formulas simplify to the following form

$$L_0 = \frac{1}{2}\dot{x}_0^2(t) - U_0(x_0(t)), \tag{32}$$

$$L_N = \dot{x}_0(t)\,\dot{x}_N(t) - x_N(t)\, U_0'(x_0(t)) - U_N(x_0(t)). \tag{33}$$

The relevant equations of motion take the form

$$\ddot{x}_0 + U_0'(x_0) = 0, \tag{34}$$

$$\ddot{x}_N + x_N U_0''(x_0) + U_N'(x_0) = 0. \tag{35}$$

The nilpotent component can be integrated in particular

$$x_N(x_0) = \sqrt{E_0 - U_0(x_0)} \times \left(\frac{1}{2}\int dx_0 \frac{E_N - U_N(x_0)}{(E_0 - U_0(x_0))^{3/2}} + C\right). \tag{36}$$

2. Hamiltonian formalism

For a dual mechanical system it is possible to perform a passage to phase space. The dual number valued Hamiltonian $\mathbf{H} = H_0 + \boldsymbol{\iota} H_N$ has two components and there are three dual component momenta p_0, p_1, p_N therefore the analogue of the Legendre transformation is more complicated and we obtain

$$p_0 = \frac{\partial L_0}{\partial \dot{x}_0} = \dot{x}_0 = \frac{\partial L_N}{\partial \dot{x}_N} = p_1, \quad p_N = \frac{\partial L_N}{\partial \dot{x}_0} = \dot{x}_N, \tag{37}$$

$$H_0 = p_0\dot{x}_0 - L_0 = \frac{p_0^2}{2} + U_0(x_0), \tag{38}$$

$$H_N = p_N\dot{x}_0 + p_0\dot{x}_N - L_N = p_0 p_N + x_N U_0' \times (x_0) + U_N(x_0). \tag{39}$$

Hence finally we obtain six Hamiltonian equations of motion

$$\dot{x}_0 = \tfrac{\partial H_0}{\partial p_0}, \quad \dot{x}_0 = \tfrac{\partial H_N}{\partial p_N}, \quad \dot{x}_N = \tfrac{\partial H_N}{\partial p_0}, \quad \dot{p}_0 = -\tfrac{\partial H_0}{\partial x_0}, \; \dot{p}_N = -\tfrac{\partial H_N}{\partial x_0}, \; \dot{p}_0 = -\tfrac{\partial H_N}{\partial x_N}. \tag{40}$$

One can introduce generalized Poisson brackets for both sectors of the phase space [20]:

$$\{A, B\}_0 = \left(\frac{\partial A}{\partial x_0}\frac{\partial B}{\partial p_0} - \frac{\partial A}{\partial p_0}\frac{\partial B}{\partial x_0}\right) \tag{41}$$

$$\{A, B\}_N = \left(\frac{\partial A}{\partial x_0}\frac{\partial B}{\partial p_N} - \frac{\partial A}{\partial p_N}\frac{\partial B}{\partial x_0}\right), + \left(\frac{\partial A}{\partial x_N}\frac{\partial B}{\partial p_0} - \frac{\partial A}{\partial p_0}\frac{\partial B}{\partial x_N}\right). \tag{42}$$

In the “nilpotent” sector of the phase space the Poisson bracket is related to an unusual conjugation of canonical variables

$$x_0 \leftrightarrow p_N \text{ and } x_N \leftrightarrow p_0. \tag{43}$$

The second type of the Poisson bracket $\{\,,\,\}_N$ needs an additional quantization rule with dual partner $\hbar_N$ of the Planck constant $\hbar$.

III. NILPOTENT MECHANICS

A recently introduced formalism of the nilpotent classical mechanics is based on the nilpotent commuting variables η [22, 23]. Contrary to the Grassmannian variables nilpotency is not automatical here and a relevant differential calculus for the functions of η-variables differs from the usual one. In particular, the Leibniz rule is with an additional term and then the properties of the generalized Poisson brackets are not a direct consequence of derivations. To obtain a nontrivial theory with η-variables it is necessary to consider a particular form of hyperbolic geometry [22], which is related directly to the light cone formalism in spaces with the metric of a zero signature i. e. diag $(+,+,\ldots+,-,-,\ldots,-)$ with the same number of pluses and minuses. However, the only admissible form of a metric in the so-called s-spaces is off-diagonal e. g.

$$s = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}; \quad s^2 = \mathbb{I}_{2n}; \quad s^T = s. \tag{44}$$

The above matrix is strictly traceless and symmetric and analogously to the antisymmetric matrix in superspace it is a good "metric" in linear space (module) for vectors with nilpotent commuting coordinates.

A. Configuration space

To define a mechanical system one introduces a Lagrangian with the terms being analogs of kinetic and potential energy. Analogs only, because the s-form is not positively definite and η coordinates are nilpotent and commuting

$$L = \frac{m}{2} s(\dot\eta, \dot\eta) - V(\eta), \tag{45}$$

$$V(x) = \frac{m\omega}{2} s(\eta, \eta) = \frac{m\omega}{2} s_{ij}\eta^i\eta^j. \tag{46}$$

One can obtain the Euler–Lagrange equations of motion for the nilpotent mechanical system in the form [23]

$$\frac{\partial L}{\partial \eta^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\eta^i}\right) = 0. \tag{47}$$

For the η-harmonic oscillator it gives

$$\ddot\eta_i = -\omega^2\eta_i. \tag{48}$$

B. Phase space

It appears that it is possible to make a passage to the phase space via the Legendre transformation with the definition of momenta

$$p^k = \frac{\partial L}{\partial \dot\eta_k}. \tag{49}$$

The generalization of the Hamiltonian function to the nilpotent phase-space $\mathcal{P}_{\mathcal{N}}$ is conventional

$$H = p^k\dot\eta_k - L \tag{50}$$

and we get for this formalism the equations of motion analogous to the standard Hamilton's equations of motion

$$\dot p^k = -\frac{\partial H}{\partial \eta_k}, \tag{51}$$

$$\dot\eta_k = \frac{\partial H}{\partial p^k}, \tag{52}$$

that for the η-harmonic oscillator gives

$$\begin{cases} \dot x^i = \frac{1}{m}(s^{-1})^{ij}p_j \\ \dot p_i = -m\omega^2 s_{ij}x^j \end{cases}. \tag{53}$$

The extension of the time derivative to the phase-space functions $f(\eta, p)$ is given in the following form (we omit here explicit time dependence)

$$\frac{d}{dt} = \dot\eta_k\partial^k + \dot p^k\bar\partial_k, \quad \text{where } \partial^k = \frac{\partial}{\partial\eta_k} \quad \text{and } \bar\partial_k = \frac{\partial}{\partial p^k}. \tag{54}$$

Defining the η-Poisson bracket as

$$\{f(\eta,p), g(\eta,p)\}_0 = \bar\partial_k f(\eta,p)\cdot\partial^k g(\eta,p) - \partial^k f(\eta,p)\cdot\bar\partial_k g(\eta,p) \tag{55}$$

we can realize the time derivative in the following form

$$\frac{d}{dt} f(\eta,p) = \{H, f(\eta,p)\}_0. \tag{56}$$

It is worth noting that for η-functions time derivative does not fulfill the usual Leibniz rule. The η-Poisson bracket has an analogous property,

$$\text{(i)} \quad \{f,g\}_0 = -\{g,f\}_0, \tag{57}$$

$$\text{(ii)} \quad \{f_1+f_2, g\}_0 = \{f_1,g\}_0 + \{f_2,g\}_0, \tag{58}$$

$$\text{(iii)} \quad \{f, g\cdot h\}_0 = \{f,g\}_0\cdot h + g\cdot\{f,h\}_0 - 2\Diamond(f|g,h), \tag{59}$$

$$\text{(iv)} \quad \sum_{cycl}\{f,\{g,h\}_0\}_0 = 0, \tag{60}$$

where the para-Leibniz term is of the form

$$\Diamond(f|g,h) = \bar\partial^k f\cdot\nabla_k g\cdot\partial_k h - \partial_k f\cdot\bar\nabla^k g\cdot\bar\partial^k h. \tag{61}$$

It is worth noting that one of the consequences of the "non-derivative" character of this bracket is violation of the Jacobi identity. The Jacobiator $J(f,g,h)$ is, in general, not vanishing. It has the following form

$$J(f,g,h) = 2\sum_{\text{cycl}}\sum_i(\eta^i\{\partial_i f, \partial_i g\}\bar\partial^i h - p_i\{\bar\partial^i f, \bar\partial^i g\}\partial_i h). \tag{62}$$

CONCLUSIONS

We have briefly reviewed selected areas of theoretical physics were the nilpotent object plays an essential role and is not something that one usually wants to get rid of. For the physical content it is very important how the nilpotency is realized, the underlying structure can be very complicated like in the case of the BRST theory or fairy simple as for dual numbers. Nilpotent mechanics provides an interesting example of the theory having some properties of the supersymmetry, where only the exclusion principle for fermions is realized by nilpotency of relevant η-fields, but the anticommutativity condition is relaxed.

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НІЛЬПОТЕНТНІ ЕЛЕМЕНТИ У ФІЗИЦІ

А. М. Фридришак
Інститут теоретичної фізики Вроцлавського університету,
пл. М. Борна, 9, 50–204, м. Вроцлав, Польща

Подано короткий огляд деяких питань, що стосуються нільпотентних об'єктів у теоретичній фізиці, для ілюстрацій використано прості моделі. Нільпотентні об'єкти виникають на квантовому і класичному рівнях у деяких випадках. З одного боку, існують відомі BRST-оператори, зовнішня похідна та пов'язані з ними когомологічні теорії. З іншого — дуальні числа, які є менш відомими структурами порівняно з комплексними числами. Також є комутуючі нільпотентні змінні, що певним чином узагальнюють нільпотентні, або некомутуючі, ґрассманові змінні, які використовують у суперматематиці та суперсиметричних моделях. Як цікавий факт відзначено, що нільпотентні змінні вимагають використання узагальненої геометрії світлового конуса з метрикою з нульовою сиґнатурою.