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HAMILTONIAN WITH DELTA TYPE INTERACTION: PERTURBATION BY DYNAMICS

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We study the Hamiltonian in two dimensional system with an interaction supported by a set of codimension one. The perturbation in our model is given by the appropriated operator. We derive the formula for the eigenvalues and the corresponding eigenfunctions. Moreover we analyze the functions counting the number of discrete and embedded points of the spectrum depending on coupling constants involved in the model. Finally, we study generalized eigenfunctions.
Key words: singular perturbations, discrete spectrum, embedded eigenvalues.

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I. INTRODUCTION

We are interested in quantum systems with a very short range of interaction which can be modelled by the so called delta potentials. The old-fashioned problem belonging to this line of research is usually called point interaction and corresponds to the heuristical Hamiltonian

$$-\Delta - \alpha\delta(x)\,, \quad \alpha \in \mathbb{R}.$$

It is well known that the quantum system governed by this Hamiltonian acting in $L^2(\mathbb{R})$ has one eigenvalue

$$\epsilon_0 = -\frac{\alpha^2}{4} \tag{1.1}$$

provided the interaction is attractive, i. e. $\alpha > 0$. A lot of interesting results concerning the delta type potentials and their applications can be found in the monograph [1] (see also references therein).

The model we study in this paper addresses a more general situation. To explain this let us consider an open set $\Omega \subseteq \mathbb{R}^{>}$ with the C^1 boundaries and a set $\Sigma \subset \Omega$ of a lower dimension. The Hamiltonian of our system can be heuristically written

$$-\Delta + V\delta(x-\Sigma)\,,$$

where Δ is the Laplace operator acting in $L^2(\Omega)$, V is a self-adjoint operator in $L^2(\Sigma)$ and $\delta(\cdot - \Sigma)$ denotes the Dirac delta with support on Σ. Following the terminology used in the bibliography we shall call the Hamiltonian corresponding to the above expression as *perturbation of Schrödinger operator by the dynamics of* V; see [4][1].

The model considered in this paper belongs to this line of research. Specifying $\Omega = (0,\pi)\times\mathbb{R}$ and $\Sigma = \{(x_1,0) \in \Omega\,, 0 < x_1 < \pi\}$ we consider the formal Hamiltonian

$$H_{\alpha\beta} = -\Delta + \left(\beta\frac{d^2}{d^2x_1} - \alpha\right)\delta(x-\Sigma), \qquad \alpha\in\mathbb{R}, \quad \beta\in\mathbb{R}, \tag{1.2}$$

where Δ stands for the two dimensional Laplacian in $L^2(\Omega)$ with Dirichlet boundary conditions (D.b.c.) on the boundaries $\partial\Omega$ of Ω.

In [4] the authors list some applications of perturbation by dynamics. The problem addresses the two component system where V may play the role of the Hamiltonian of a subsystem living on Σ. For example, in [4] there is mentioned an acoustic model where Σ relates to a membrane (if it is surface) or string (if it is linear). We believe that the Hamiltonian considered in this paper can be applied to this kind of models.

The main results of this paper can be formulated as follows.

- Construction of the self-adjoint operator corresponding to (1.2) and its resolvent.
- Deriving an explicit formula for the eigenvalues of $H_{\alpha\beta}$ and the corresponding eigenfunctions. Moreover the analysis of the functions counting the number of discrete points of spectrum as well as embedded eigenvalues depending on α and β (Theorem III.3 and the discussion after it).
- Construction of the generalized eigenvectors.

Let us mention that a particular case of the model in question was considered in [2] (for more details see the discussion after Theorem III.3).

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[1] In fact, in [4] a larger class Hamiltonians is studied; this class can be determined as a perturbation of the self-adjoint operator by the dynamics of V

II. HAMILTONIAN OF THE SYSTEM AND ITS RESOLVENT

In the following we use the notations introduced in the previous section. To precise the domain of our Hamiltonian it is convenient to use the space $W^{2,m}(\Omega) = \{f \in L^2(\Omega) : \|f\|^2_{2,m} = \sum_{k=0}^m (\sum_{|\alpha|=k} \|D^\alpha f\|^2) < \infty\}$, where D^α is a (distributional) derivative of the order of α and $\|\cdot\|$ is the norm in $L^2(\Omega)$ [2]. Since we are interested in the Hamiltonian with Dirichlet boundary conditions at $\partial\Omega$ it is natural to consider the space $W_0^{2,m}(\Omega)$ being the closure of $C_0^\infty(\Omega)$ (infinitely differentiable functions compactly supported in Ω) in the norm $\|f\|^2_{2,m}$.

We study the quantum system governed by the Hamiltonian which can be symbolically written as (1.2). To give a meaning of a self-adjoint operator to the formal expression (1.2) we consider the quadratic form

$$\begin{aligned}\mathcal{E}_{\alpha\beta}(\psi,\phi) &= \int_{\mathbb{R}^{\not\in}} \overline{\nabla\psi}(x)\nabla\phi(x)\mathrm{d}x \qquad (2.1)\\ &+ \beta\int_{\mathbb{R}} \frac{d}{dx_1} I_\Sigma\overline{\psi}(x_1)\frac{d}{dx_1} I_\Sigma\phi(x_1)\mathrm{d}x_1\\ &- \alpha\int_{\mathbb{R}} I_\Sigma\overline{\psi}(x_1) I_\Sigma\phi(x_1)\mathrm{d}x_1\,, \quad D(\mathcal{E}_{\alpha\beta}) = W_0^{2,1}(\Omega),\end{aligned}$$

symbol I_Σ denotes here the embedding $W_0^{2,1}(\Omega) \mapsto L^2(\Sigma) \equiv L^2$ defined by means of the convolution $I_\Sigma\psi = \psi * \delta(\cdot - \Sigma)$.

Relying on the results of [3] one can, standardly, show that the operator associated with $\mathcal{E}_{\alpha\beta}$ is self-adjoint; this operator gives a mathematical meaning to the formal expression (2.1) and in the following will be denoted as $H_{\alpha\beta}$.

In fact, $H_{\alpha\beta}$ is just the Laplace operator with the appropriate boundary conditions on Σ. A straightforward calculation shows that $H_{\alpha\beta} = -\Delta$ and

$$\begin{aligned}&D(H_{\alpha\beta}) = \{W_0^{2,2}(\Omega\setminus\Sigma)\cap W_0^{2,1}(\Omega),\\ &\phi(x_1,0^+) = \phi(x_1,0^-) = \phi(x_1,0)\,,\\ &\partial_2\phi(x_1,0^+) - \partial_2\phi(x_1,0^-) = \left(\beta\frac{d^2}{dx_1^2} - \alpha\right)\phi(x_1,0)\}.\end{aligned}$$

The absolutely continuous spectrum of $H_{\alpha\beta}$ is the same as the spectrum of pure Laplacian in $L^2(\Omega)$ with D.b.c, i.e.

$$\sigma_{\mathrm{ac}}(H_{\alpha\beta}) = [1,\infty)\,.$$

Now our aim is to recover discrete spectrum as well as embedded eigenvalues of $H_{\alpha\beta}$. The total spectrum of $H_{\alpha\beta}$ will be, standardly, denoted $\sigma(H_{\alpha\beta})$.

Let $z \in \rho(H_{\alpha\beta}) := \mathbb{C}\backslash\sigma(\mathbb{H}_{\alpha\beta})$ and $R_{\alpha\beta}(z)$ stand for the resolvent of $H_{\alpha\beta}$, i.e. $R_{\alpha\beta}(z) = (H_{\alpha\beta} - z)^{-1}$. The poles of resolvent state eigenvalues of $H_{\alpha\beta}$, cf. [3]. Therefore, our first aim is to derive an explicit form of $R_{\alpha\beta}(z)$.

Let us note that the resolvent of $-\Delta$ acting in $L^2(\Omega)$ and satisfying D.b.c. on $\partial\Omega$ is an integral operator with the kernel taking the following form

$$G(z;x,x') = \sum_{n\in\mathbb{N}} G_n(z;x,x')$$

where $x = (x_1,x_2)\,,\ x' = (x_1',x_2')$ and

$$G_n(z;x,x') = \frac{i}{\pi}\frac{\mathrm{e}^{i\tau_n(z)|x_2-x_2'|}}{\tau_n(z)}\sin nx_1 \sin nx_1'\,,$$

where $\tau_n(z) = (z-n^2)^{1/2}$ and $\mathrm{Im}\,\tau_\mathrm{n}(\mathrm{z}) > 0$. (Of course, the above kernel depends on the coupling constants α and β; without any danger of confusion we omit here the appropriate indices). Furthermore, let us introduce special notations for the embeddings $\hat{G}_n : L^2(\Sigma) \mapsto L^2(\Omega)$ acting as $\hat{G}_n f = \int_\Omega G_n(z;\cdot,(x_1',0))f(x_1')\mathrm{dx}_1'$ and $\check{G}_n : L^2(\Omega) \mapsto L^2(\Sigma)$, $\check{G}_n\psi = I_\Sigma\int_\Omega G_n(z;\cdot,x')\psi(x')\mathrm{dx}' = (\int_\Omega \mathrm{G}(\mathrm{z};\cdot,\mathrm{x}')\psi(\mathrm{x}')\mathrm{dx}') * \delta(\cdot-\Sigma)$.

Then the resolvent $R_{\alpha\beta}(z)$ is again an integral operator with the kernel

$$\begin{aligned}&G_{\alpha\beta}(z;x,x') = G(z;x,x') \qquad (2.2)\\ &+ \sum_{n\in\mathbb{N}} \hat{G}_n(z;x,\cdot) * \Gamma_n(z;\cdot,\cdot)^{-1} * \check{G}_n(z;\cdot,x')\,,\end{aligned}$$

where $\Gamma(z;\cdot,\cdot)^{-1}$ is an operator acting in $L^2(\Sigma)$ and is defined as the inverse of

$$\begin{aligned}&\Gamma_n(z;x_1,x_1') = \Gamma_n(z)\sin nx_1 \sin nx_1',\\ &\Gamma_n(z) = \left(\frac{1}{\alpha-\beta n^2} - \frac{i}{2}\frac{1}{\tau_n(z)}\right).\end{aligned}$$

Note that Γ_n is not well defined for $\alpha/=n^2\beta$. For this case we assume that $\Gamma_n^{-1} = 0$. Since the above formula represents the standard Krein like resolvent we omit here the analysis in detail, cf. [5].

To proceed further let us note that the resolvent $R_{\alpha\beta}(z)$ has the second sheet analytical continuation; precisely all the components forming (2.2) can be analytically continued cutting the Riemann plane along the half line $[n^2,\infty)$ and taking $\mathrm{Im(z)}\,\tau_\mathrm{n} \le 0$.

III. SPECTRAL ANALYSIS OF $H_{\alpha\beta}$: DISCRETE AND EMBEDDED EIGENVALUES, GENERALIZED EIGENVECTORS

Now we are ready to search for the poles of $R_{\alpha\beta}(z)$ which are defined as roots of the equations

$$\Gamma_n(z) = 0\,, \quad \text{where } n\in\mathbb{N}\,. \qquad (3.1)$$

A straightforward calculation shows that for $\mathrm{Im}\,\tau_n(z)>0$ the above equation has a real solution *if* $\alpha > \beta n^2$. Given $n\in\mathbb{N}$ such that $\alpha > \beta n^2$ the solution of (3.1) is by

$$\epsilon_n \equiv \epsilon_n(\alpha,\beta) = -\frac{(\alpha-\beta n^2)^2}{4} + n^2\,. \qquad (3.2)$$

[2]More precisely $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2}$.

Remark III.1 *Analysis of* ϵ_n. Note that ϵ_n can be considered as a certain kind of generalization of (1.1). Particularly, if $\beta = 0$, then the eigenvalues are given by

$$\epsilon_n(\alpha, 0) = -\frac{\alpha^2}{4} + n^2\,. \tag{3.3}$$

The component n^2 pushing up the eigenvalue w.r.t. (1.1) is a natural consequence of the fact that our quantum system lives in the strip Ω and x_1 coordinate of the momentum is quantized. A direct analysis of (2.2) shows that the Hamiltonian $H_{\alpha\beta}$ admits decomposition $H_{\alpha\beta} = \oplus_{n=1}^{\infty} H_{\alpha\beta,n}$ and given n the halfline $[n^2, \infty)$ states the continuous spectrum of $H_{\alpha\beta,n}$. Consequently, the continuous spectrum of $H_{\alpha\beta}$ is given by $[1, \infty)$.

Remark III.2 *Discrete and embedded eigenvalues.* Note that, generally, ϵ_n can live in the continuous spectrum of $H_{\alpha\beta}$ as well as apart from it. The eigenvalues belonging to a discrete spectrum are determined by $\epsilon_n < 1$ which is equivalent to

$$2\sqrt{n^2-1} + \beta n^2 < \alpha\,,$$

and, analogously the embedded eigenvalues

$$2\sqrt{n^2-1} + \beta n^2 \geq \alpha > \beta n^2\,.$$

Let us note that for $n = 1$ the latter inequality leads to a contradiction, this means that the eigenvalue labelled by $n = 1$ belongs to the discrete spectrum.

For the latter purpose we introduce the following notations

$$N_{\mathrm{d}} = \sharp\{n \in \mathbb{N} \,:\, 2\sqrt{n^2-1} + \beta n^2 < \alpha\,\}. \tag{3.4}$$

and

$$N_{\mathrm{e}} = \sharp\{n \in \mathbb{N} \,:\, 2\sqrt{n^2-1} + \beta n^2 \geq \alpha\, > \beta n^2\}\,. \tag{3.5}$$

In view of the previous discussion, N_{d}, N_{e} count the number of discrete spectrum points and embedded eigenvalues, respectively.

Finally, note that the function

$$f_n(x) = \mathrm{e}^{-\sqrt{n^2-\epsilon_n}|x_2|} \sin nx_1 = \mathrm{e}^{-(\alpha-\beta n^2)|x_2|/2} \sin nx_1\,, \tag{3.6}$$

where $x = (x_1, x_2)$, states the eigenvector corresponding to ϵ_n. Indeed, a standard calculation shows that $-\Delta f_n = \epsilon_n f_n$ and, moreover, f_n satisfies boundary conditions on Σ defined in $D(H_{\alpha\beta})$.

The following theorem completes the above statements.

Theorem III.3 *Given* $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}$ *operator* $H_{\alpha\beta}$ *has* $\sharp\{n \in \mathbb{N} \,:\, \alpha > n^2\beta\}$ *eigenvalues of the following form*

$$\epsilon_n = -\frac{(\alpha - \beta n^2)^2}{4} + n^2\,.$$

The number of discrete spectrum points and embedded eigenvalues are given by N_{d} *and* N_{e}*, respectively. Moreover,* f_n *defined by (3.6) states the eigenfunction corresponding to* ϵ_n.

In particular, the above theorem implies

1. If $\alpha \in \mathbb{R}$ and $\beta < 0$ then $N_{\mathrm{d}} = \infty$ and $N_{\mathrm{e}} \in \mathbb{N}$ provided there exists $n \in \mathbb{N} \,:\, \alpha < 2\sqrt{n^2-1} + \beta n^2$ otherwise $N_{\mathrm{e}} = 0$.

2. If $\alpha \geq 0$ and $\beta > 0$ then $N_{\mathrm{d}} \in \mathbb{N}$ provided $\alpha > \beta$ otherwise $N_{\mathrm{d}} = 0$ and $N_{\mathrm{e}} \in \mathbb{N} \cup \{0\}$

3. If $\alpha \leq 0$ and $\beta \geq 0$ then $N_{\mathrm{d}} = N_{\mathrm{e}} = 0$.

4. If $\alpha > 0$ and $\beta = 0$ then $N_{\mathrm{d}} \in \mathbb{N}$ and $N_{\mathrm{e}} = \infty$.

Note that for $\alpha = 0$ and $\beta < 0$ formulae (3.2) and (3.6) were obtained in [2] [3]. Partially, result 1. was stated in [2] as well (namely, if $\alpha = 0$ and $\beta < 0$, then $N_{\mathrm{d}} = \infty$; the embedded eigenvalue problem was not discussed in [2]).

As the final step we derive generalized eigenfunctions of $H_{\alpha\beta}$. Mind that the generalized eigenfuctions are distributions (not belonging to $D(H_{\alpha\beta})$) corresponding to "continuum eigenfuctions". Again straightforward calculations show that

$$\psi_n(p;\sigma,x) = \mathrm{e}^{i\sigma\sqrt{p^2-n^2}x_2} \sin nx_1 - \frac{1}{2i\sqrt{p^2-n^2}} \Gamma_n(p^2)^{-1} \mathrm{e}^{i\sigma\sqrt{p^2-n^2}|x_2|} \sin nx_1,$$

where $p^2 \in [n^2, \infty)$, satisfy

$$-\Delta\psi_n(p;\sigma,x) = p^2\psi_n(p;\sigma,x)$$

and fulfill the boundary conditions described in the definition of $D(H_{\alpha\beta})$. This means that the $\psi_n(p;\sigma,x)$ state the generalized eigenfunctions of $H_{\alpha\beta}$; $\psi_n(p;+1,x)$ correspond to the incidence from the bottom and analogously $\psi_n(p;-1,x)$ corresponds to the incidence from the top.

A. Final remarks and open problems

As was shown in the previous section the Hamiltonian of our system admits embedded eigenvalues under certain conditions. If we introduce an additional 'small' perturbation to this system, then we can expect that the embedded eigenvalues are recovered from the continuous spectrum and move to the second sheet continuation of the resolvent stating resonances. We postpone a detailed analysis of resonances to a forthcoming paper.

[3] To be fully specific we have to say that the constant β in our paper corresponds to $-\alpha$ in [2]; see formula (89) of [2]

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ГАМІЛЬТОНІАН ІЗ ДЕЛЬТА-ПОДІБНОЮ ВЗАЄМОДІЄЮ: ДИНАМІЧНІ ЗБУРЕННЯ

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У статті проаналізовано гамільтоніан двовимірної системи зі взаємодією на множині ковимірності один. Збурення в цій моделі задає відповідний оператор. У роботі виведено формулу для власних значень і відповідних власних функцій. Також проаналізовано функції, які дають змогу підрахувати кількість дискретних і включених точок спектра залежно від констант зв'язку в моделі. Вивчено знайдені узагальнені власні функції.