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MODIFIED PERTURBATION THEORY FOR HYDROGEN ATOM IN SPACE WITH LORENTZ-COVARIANT DEFORMED ALGEBRA WITH MINIMAL LENGTH

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We study energy spectrum for the hydrogen atom problem in Dirac theory with Lorentz-covariant deformed algebra leading to minimal length. Developing perturbation theory free of divergencies we calculate the correction to any energy level in a simple case of deformation when one deformation parameter vanishes. Assuming that the effect of minimal length on energy spectrum cannot be seen experimentally we find the upper bound of minimal length.
Key words: deformed algebras, minimal length, Dirac equation, hydrogen atom.

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I. INTRODUCTION

String theory and quantum gravity independently suggest the existence of minimal length as a finite lower bound to the possible resolution of length [1–3]. Kempf *et al.* showed that minimal length can be introduced by modifying a canonical commutation relation [4–7]. The deformed commutation relation according to Kempf *et al.* in the D-dimensional space reads

$$\begin{aligned}
&[\hat{X}_i,\hat{P}_j]=i\hbar[(1+\beta\hat{P}^2)\delta_{ij}+\beta'\hat{P}_i\hat{P}_j],\\
&[\hat{X}_i,\hat{X}_j]=i\hbar\frac{2\beta-\beta'+(2\beta+\beta')\beta\hat{P}^2}{1+\beta\hat{P}^2}(\hat{P}_i\hat{X}_j-\hat{P}_j\hat{X}_i),\\
&[\hat{P}_i,\hat{P}_j]=0.
\end{aligned}\tag{1}$$

Here β and β' are two small nonnegative parameters.

But it should be noted that originally the deformed algebra leading to a quantized space-time was introduced by Snyder in the relativistic case [8]. In paper [9] $(D+1)$-dimensional two-parametric quantized space-time Lorentz-covariant deformed algebra was proposed as a generalization of Kempf's D-dimensional one.

$$\begin{aligned}
&[\hat{X}^\mu,\hat{P}^\nu]=-i\hbar[(1-\beta\hat{P}_\rho\hat{P}^\rho)g^{\mu\nu}-\beta'\hat{P}^\mu\hat{P}^\nu],\\
&[\hat{X}^\mu,\hat{X}^\nu]=i\hbar\frac{2\beta-\beta'-(2\beta+\beta')\beta\hat{P}_\rho\hat{P}^\rho}{1-\beta\hat{P}_\rho\hat{P}^\rho}\\
&\qquad\times(\hat{P}^\mu\hat{X}^\nu-\hat{P}^\nu\hat{X}^\mu),\\
&[\hat{P}^\mu,\hat{P}^\nu]=0,
\end{aligned}\tag{2}$$

with $g^{\mu\nu}=g_{\mu\nu}=\mathrm{diag}(1,-1,-1,\ldots,-1)$ being the metric tensor.

Algebra (2) contains the Snyder algebra as a special case ($D=3,\ \ \beta=0$). It is interesting to study the influence of the minimal length assumption on the properties of quantum systems. Such an interest is motivated by the perspective of checking the validity of such an assumption and a possibility to obtain the constraints on the deformation parameters. The hydrogen atom as a quantum-mechanical system with a highly accurate theoretical prediction and the most precise experimental data suits well for such an aim.

In the present paper we obtain a correction to the energy of some "problem" states of the hydrogen atom in Dirac theory in space with a special case of the deformation $\beta'=0$ of Lorentz-covariant commutation relation (2).

The paper is organized as follows. In Section II we calculate the energy correction of the ground state both for the hydrogen atom with Kempf's deformation and a relativistic hydrogen atom for Lorentz-covariant one ($\beta\neq 0,\beta'=0$) using conventional perturbation theory and an expansion of the ground state wave function over eigenfunctions of the distance operator. Considering the special case of deformation in Section III we obtain a correction to the energy for states with $j=\frac{1}{2}$ of Coulomb problem in Dirac theory using modified perturbation theory. Finally, in Section IV we conclude the obtained results.

II. CORRECTION TO THE GROUND STATE ENERGY

In this section we calculate a correction to the ground state energy of the hydrogen atom in Dirac theory with Lorentz-covariant deformed algebra with the minimal length using conventional perturbation theory and an expansion of the ground state wave function over eigenfunctions of the distance operator.

We study the Dirac equation in the (3+1)-dimensional case in the form of

$$\left[c\hat{\rho}_a(\hat{\sigma}_x\hat{P}^x+\hat{\sigma}_y\hat{P}^y+\hat{\sigma}_z\hat{P}^z)+mc^2\hat{\rho}_c-\frac{e^2}{R}\right]\psi=\hat{P}^0c\psi,\tag{3}$$

where operators of position $\hat{X}^\mu$ and momenta $\hat{P}^\mu$ satisfy the deformed commutation relation (2).

If we go to a stationary wave equation in the ordinary (undeformed) quantum mechanics, we will obtain an eigenequation for the Hamiltonian. The same eigenequation can be derived from the equality to zero of the deviation of the Hamiltonian. Such a coincidence is natural for undeformed quantum mechanics. For quantum mechanics with Lorentz-covariant deformed algebra (2) it is not the case. If $\beta' \neq 0$, then $[\hat{H}, \hat{P}_0] \neq 0$ and we have a problem with deriving the stationary equation. Therefore we consider a simple case of the deformation $\beta \neq 0, \beta' = 0$.

Algebras (1) and (2) are rather similar. We can apply our method to a less complicated problem of the hydrogen atom in space with Kempf's deformed commutation relation. Now we are going to obtain the correction to ground state energy of the hydrogen atom problem in space with deformed algebra (1) using the matrix approach. We choose the following representation for the algebra in the case $\beta \neq 0, \beta' = 0$:

$$\begin{aligned} \hat{X}_i &= \hat{x}_i + \frac{\beta}{2}(\hat{p}^2\hat{x}_i + \hat{x}_i\hat{p}^2); \\ \hat{P}_i &= \hat{p}_i, \end{aligned} \tag{4}$$

with $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. For the undeformed Heisenberg algebra the position representation may be taken: $\hat{x}_i = x_i, \quad \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$.

It can be shown that for s-states the distance operator has a rather simple form

$$\hat{R} = r + \frac{\beta}{2}\left(\hat{p}^2 r + r\hat{p}^2 + \frac{2\hbar^2}{r}\right), \tag{5}$$

with $r = \sqrt{x_i^2}$. One can easily find the eigenvalues and eigenfunctions of $\hat{R}$:

$$\begin{aligned} l_n &= (2n+3)\hbar\sqrt{\beta}, \\ \phi_n(r) &= \sqrt{\frac{2}{\hbar\sqrt{\beta}}}\frac{1}{\sqrt{(n+1)(n+2)}}\varrho e^{-\frac{\varrho}{2}}Q_n^2(\varrho). \end{aligned} \tag{6}$$

Here $\varrho = \frac{2r}{\hbar\sqrt{\beta}}$ and the normalization condition for $\phi_n(r)$ is

$$\int_0^\infty dr|\phi_n(r)|^2 = 1. \tag{7}$$

It should be noted that our results are in good agreement with the ones obtained in [10] where the eigenproblem for squared distance operator in momentum representation without any assumption on deformation parameters and orbital quantum number was considered.

Now let us calculate a correction to the ground state of the hydrogen atom. We expand the wave function of the ground state

$$\psi_{1s}(r) = \sqrt{\frac{4}{a^3}}re^{-\frac{r}{a}} \tag{8}$$

over the eigenfunctions of $\hat{R}$

$$\psi_{1s}(r) = \sum_{n=0}^{\infty} Y_n\phi_n(r) \tag{9}$$

with the expansion coefficients

$$Y_n = 8\sqrt{\frac{(n+1)(n+2)}{2}}\left(a\hbar\sqrt{\beta}\right)^{\frac{3}{2}}\frac{(\hbar\sqrt{\beta}-a)^n}{(\hbar\sqrt{\beta}+a)^{n+3}}. \tag{10}$$

We write the Hamiltonian of the problem under consideration in the following form

$$\hat{H} = \hat{H}_0 + \hat{V}_\beta. \tag{11}$$

Here $\hat{H}_0 = \frac{\hat{p}^2}{2m} - \frac{e^2}{r}$ is the Hamiltonian of the unperturbed problem, $\hat{V}_\beta = \frac{e^2}{r} - \frac{e^2}{\hat{R}}$ is the perturbation operator.

The mean value of the inverse distance operator is

$$\begin{aligned} \langle\psi_{1s}(r)|\hat{R}^{-1}|\psi_{1s}(r)\rangle &= \sum_{n=0}^{\infty}\frac{Y_n^2}{(2n+3)\hbar\sqrt{\beta}} \\ &= \frac{8\sqrt{\hbar\sqrt{\beta}}}{3(a+\hbar\sqrt{\beta})^3}\ {}_2F_1\left(-\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; -\frac{(a-\hbar\sqrt{\beta})^2}{4a\hbar\sqrt{\beta}}\right), \end{aligned}$$

with ${}_2F_1\left(a, b; c; z\right)$ being the hypergeometric function. Expanding the latter expression over small β we obtain

$$\begin{aligned} \langle\psi_{1s}(r)|\hat{R}^{-1}|\psi_{1s}(r)\rangle = \frac{1}{a} + \frac{\hbar^2\beta}{a^3}\left(4\ln\frac{\hbar\sqrt{\beta}}{a} + 3\right) \\ + O(\beta^2). \end{aligned} \tag{12}$$

And a correction to the energy of the ground state is

$$\begin{aligned} \Delta E_{1s} &= \langle\psi_{1s}(r)|e^2\hat{r}^{-1} - e^2\hat{R}^{-1}|\psi_{1s}(r)\rangle \\ &= -\frac{\hbar^2\beta e^2}{a^3}\left(4\ln\frac{\hbar\sqrt{\beta}}{a} + 3\right) + O(\beta^2). \end{aligned} \tag{13}$$

A similar result was obtained in [12] using the so-called shifted expansion

$$\begin{aligned} \Delta E_{1s} = -\frac{\hbar^2\beta e^2}{a^3}\left(4\ln\frac{\hbar\sqrt{\beta}}{a} + 2\ln 2 + 4\gamma - 1)\right) \\ + o(\beta) \end{aligned} \tag{14}$$

The difference is in the coefficient near the linear over β term and it insignificantly affects the estimation of minimal length.

The same procedure can be provided for the deformation algebra (2). We choose the representation

$$\begin{aligned} \hat{X}^\mu &= \hat{x}^\mu + \frac{\beta}{2}\left(\hat{x}^\mu(\hat{p}^2 - (\hat{p}^0)^2) + (\hat{p}^2 - (\hat{p}^0)^2)\hat{x}^\mu\right); \\ \hat{P}^\mu &= \hat{p}^\mu, \end{aligned} \tag{15}$$

with $\hat{x}^\mu = x^\mu, \quad \hat{p}^\mu = i\hbar g^{\mu\nu}\frac{\partial}{\partial x^\nu}$ and find out that for the states with zero orbital quantum number the distance operator equals

$$\hat{R} = (1 - \beta(\hat{p}^0)^2)r + \frac{\beta}{2}\left(\hat{p}^2 r + r\hat{p}^2 + \frac{2\hbar^2}{r}\right). \tag{16}$$

We find eigenfunctions and eigenvalues from the following equation

$$\hat{R}e^{-iEt/\hbar}\varphi_n(r) = \lambda_n e^{-iEt/\hbar}\varphi_n(r). \tag{17}$$

Thus, we obtain

$$\lambda_n = (2n+3)\hbar\sqrt{\theta}(1-\beta E^2/c^2), \tag{18}$$
$$\varphi_n(r) = \sqrt{\frac{2}{\hbar\sqrt{\theta}}}\frac{1}{\sqrt{(n+1)(n+2)}}\rho e^{-\frac{\rho}{2}}Q_n^2(\rho).$$

Here $Q_n^2(\rho)$ are the Laguerre polynomials and we denote $\theta = \frac{\beta}{1-\beta E^2/c^2}$ and $\rho = \frac{2r}{\hbar\sqrt{\theta}}$. For the ground state of a relativistic hydrogen atom we can separate the radial part of the wave function from the spherical and spinor parts. So the radial part of the wave function is

$$\psi_{01}(r) = \frac{1}{\sqrt{\Gamma(s+1)}}\left(\frac{2}{a}\right)^{\frac{s+1}{2}} r^{\frac{s}{2}}e^{-\frac{r}{a}} \tag{19}$$

with the normalization condition

$$\int_0^\infty dr|\psi_{01}(r)|^2 = 1 \tag{20}$$

and $s = 2\sqrt{1-\alpha^2}$, α being the fine structure constant. Expanding $\psi_{01}(r)$ over the eigenfunctions $\varphi_n(r)$

$$\psi_{01}(r) = \sum_{n=0}^{\infty}\Upsilon_n\varphi_n(r) \tag{21}$$

for the expansion coefficient we find

$$\Upsilon_n = 2^{\frac{s+2}{2}}\sqrt{(n+1)(n+2)}\frac{\Gamma\left(\frac{s}{2}+2\right)}{\sqrt{\Gamma(s+1)}}\frac{a^{\frac{3}{2}}\left(\hbar\sqrt{\theta}\right)^{\frac{s+1}{2}}}{\left(a+\hbar\sqrt{\theta}\right)^{\frac{s+4}{2}}}\ {}_2F_1\left(-n,\frac{s+4}{2};3;\frac{2a}{a+\hbar\sqrt{\theta}}\right). \tag{22}$$

The mean value of the inverse distance operator in this case equals

$$\langle\psi_{01}(r)|\hat{R}^{-1}|\psi_{01}(r)\rangle = \sum_{n=0}^{\infty}\frac{\Upsilon_n^2}{(2n+3)\hbar\sqrt{\theta}(1-\beta E^2/c^2)}. \tag{23}$$

We failed to summarize this sum exactly, but preserving the leading term over the deformation parameter β and the fine structure constant α we obtained

$$\langle\psi_{01}(r)|\hat{R}^{-1}|\psi_{01}(r)\rangle = \langle\psi_{01}(r)|r^{-1}|\psi_{01}(r)\rangle + \frac{\hbar^2\beta}{a^3}\left(4\ln\frac{\hbar\sqrt{\beta}}{a}+\frac{1}{\alpha^2}\right). \tag{24}$$

Thus, the correction to the ground state energy of the hydrogen atom in space with Lorentz-covariant deformed algebra is

$$\Delta E_{01} = \langle\psi_{01}(r)|e^2\hat{r}^{-1} - e^2\hat{R}^{-1}|\psi_{01}(r)\rangle = -\frac{\hbar^2e^2\beta}{a^3}\left(4\ln\frac{\hbar\sqrt{\beta}}{a}+\frac{1}{\alpha^2}\right). \tag{25}$$

III. MODIFIED PERTURBATION THEORY

In our previous paper [11] we chose the following representation for algebra (2)

$$\begin{cases}\hat{X}^\mu = \hat{x}^\mu - \frac{\beta}{2}\left[\hat{p}_\rho\hat{p}^\rho\hat{x}^\mu + \hat{x}^\mu\hat{p}_\rho\hat{p}^\rho)\right],\\ \hat{P}^\mu = \hat{p}^\mu,\\ \hat{x}^\mu = x^\mu,\\ \hat{p}^\mu = i\hbar g^{\mu\nu}\frac{\partial}{\partial x^\nu}\end{cases} \tag{26}$$

and obtained the expansion up to the first order over the deformation parameter for the inverse distance operator

$$\hat{R}^{-1} = \frac{1}{r} - \frac{\beta}{2}\left(\frac{1}{r}(\hat{p^2} - (\hat{p^0})^2 + (\hat{p^2} - (\hat{p^0})^2\frac{1}{r} + \frac{2\hbar^2}{r^3}\right) \tag{27}$$

as well as calculated a correction to any energy level besides some "problem states" in a simple case of deformation when one deformation parameter vanishes, namely $\beta' = 0, \beta \neq 0$. For the states with $k = \pm 1$ ($j = |k| - 1/2 = 1/2$) we obtain the divergent mean value of the operator in square brackets in (27). It means that such an expansion is not correct for the mentioned states, that is we have to preserve all the terms of the series.

In this section we consider a modified perturbation theory, which was proposed in [12] to avoid divergences of the correction to s-levels of the nonrelativistic hydrogen atom problem with Kempf's deformed commutation relations. As we see in the previous section the result of [12] slightly differs from the exact one.

In our case such a modification of the perturbation theory gives a possibility to obtain a correction for the energy spectrum of the problem under consideration in

the case of states with $k = \pm 1$. Mind that we work with a simple case of deformation $\beta \neq 0, \quad \beta' = 0$.

In the same representation (26) we rewrite $\hat{R}$ as follows

$$\hat{R} = \sqrt{r^2 + b^2 - \beta(r^2 \hat{p}_\nu \hat{p}^\nu + \hat{p}_\nu \hat{p}^\nu r^2 - \hbar^2 D + \overline{b}^2)}, \quad (28)$$

where $\beta\overline{b}^2 = b^2$. Similarly to the nonrelativistic case we provide a shifted expansion in the vicinity of a new point $r^2 + b^2$. We consider the introduced parameter b to be sufficiently small to discard terms of higher order in the series expansion, but not zero.

We present the result of the extraction of the square root (28) in the following form

$$\hat{R} = \sqrt{r^2 + b^2} + \hat{C}(\beta), \quad (29)$$

where

$$\hat{C}(\beta) = \beta \hat{C}_1 + \beta^2 \hat{C}_2 + \beta^3 \hat{C}_3 + \dots \quad (30)$$

If we equate squared right hand sides of (28) and (29) we obtain the equations for operators $\hat{C}_1, \hat{C}_2, \dots$. For $\hat{C}_1$ we have

$$\hat{C}_1 \sqrt{r^2 + b^2} + \sqrt{r^2 + b^2} \hat{C}_1 = -\beta(r^2 \hat{p}_\nu \hat{p}^\nu + \hat{p}_\nu \hat{p}^\nu r^2 - \hbar^2 D + \overline{b}^2). \quad (31)$$

and

$$\hat{C}_1 = \frac{1}{2}\left(\frac{r^2}{\sqrt{r^2 + b^2}} \hat{p}^2 + \hat{p}^2 \frac{r^2}{\sqrt{r^2 + b^2}} - \frac{2p_0^2 r^2}{\sqrt{r^2 + b^2}} + \frac{2\hbar^2 - \overline{b}^2}{\sqrt{r^2 + b^2}} + \frac{\hbar^2 b^4}{(r^2 + b^2)^{5/2}} \right). \quad (32)$$

Expanding the inverse distance R^{-1} in the series over the parameter of deformation up to the first order β we have:

$$\hat{R}^{-1} = \frac{1}{\sqrt{r^2 + b^2}} + \frac{\beta}{2}\left(\frac{r^2}{r^2 + b^2} \hat{p}_\nu \hat{p}^\nu \frac{1}{\sqrt{r^2 + b^2}} + \frac{1}{\sqrt{r^2 + b^2}} \hat{p}_\nu \hat{p}^\nu \frac{r^2}{r^2 + b^2} \right) - \frac{2\hbar^2\beta - b^2}{2(r^2 + b^2)^{3/2}} - \frac{\hbar^2 \beta b^4}{2(r^2 + b^2)^{7/2}}. \quad (33)$$

Now the perturbation operator reads

$$\hat{V}_\beta = \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 + b^2}} - \frac{e^2\beta}{2}\left(\frac{r^2}{r^2 + b^2} \hat{p}_\nu \hat{p}^\nu \frac{1}{\sqrt{r^2 + b^2}} + \frac{1}{\sqrt{r^2 + b^2}} \hat{p}_\nu \hat{p}^\nu \frac{r^2}{r^2 + b^2} \right) + \frac{e^2(2\hbar^2\beta - b^2)}{2(r^2 + b^2)^{3/2}} + \frac{e^2 \hbar^2 \beta b^4}{2(r^2 + b^2)^{7/2}}. \quad (34)$$

We calculate the correction to the ground state ($p = 0, k = 1$) caused by perturbation $\hat{V}_\beta$.

$$\Delta E_{01}^{(1)}(b) = -\frac{s\beta\hbar^2 e^2}{a^3 \alpha^2} + \frac{8\beta\hbar^2 e^2}{a^3} \frac{1}{s(s-1)(s-2)} - \frac{e^2\beta\hbar^2}{a^3} \frac{2}{s(s-1)} + \frac{e^2}{a} \frac{2^{1-s}\pi}{\sin\frac{\pi s}{2}\Gamma(\frac{s}{2})^2} \left(\frac{2}{a}\right)^s \left(\frac{2\beta\hbar^2 b^{s-2}}{s} + \frac{2-s}{s^2} b^s + \frac{2\beta\hbar^2 b^{s-2}(2-s)(s+6)}{12s} \right). \quad (35)$$

Here $s = 2\sqrt{1 - \alpha^2}$ and α is the fine structure constant.

Expanding (35) into a series over the small α we obtain

$$\Delta E_{01}^{(1)} = -\frac{e^2\hbar^2\beta}{a^3\alpha^2} - \frac{2e^2\hbar^2\beta}{a^3} \ln\frac{\hbar^2\beta}{a^2} + \frac{e^2\hbar^2\beta}{a^3}\left(-\frac{25}{12} - 4\gamma + q^2 - 2\ln q \right) + O(\alpha^0) \quad (36)$$

Here we denote $b = q\hbar\sqrt{\beta}$.

As we see the calculated correction (36) depends on the unknown parameter q. Of course, if we calculate the mean value of the exact perturbation operator $\hat{V}_\beta = e^2\left(r^{-1} - \hat{R}^{-1}\right)$, we would not obtain any dependance on q. But if we assume that q does not depend on the parameter β, the contribution to the energy correction of the third term can be neglected in comparison with the former two terms and we obtain (25). Thus, this means that the proposed method of shifted expansion gives a result sufficiently close to the correct one.

We also calculate a correction to any excited state with $k = \pm 1$ (see *Appendix A*). Our result can be presented as

$$\widetilde{\Delta E}^{(1)}_{pk} = \Delta E^{(1)}_{pk} + \Theta^{(1)}_{pk}. \tag{37}$$

Here

$$\Theta^{(1)}_{pk} = \frac{e^2}{48a}\left(\frac{2\sqrt{2}\hbar\sqrt{\beta}}{an^*}\right)^s \frac{sk(n^*+k)+2pk^2}{n^{*3}} \times \frac{\pi^{\frac{3}{2}}4^{-s}(s^3+4s^2-12sk^2-24)\Gamma(p+s)}{p!\Gamma(s/2+1)^3\Gamma(s/2+1/2)\sin(\pi/2(s+2))}$$

the expression for $\Delta E^{(1)}_{pk}$ is see in [11]. The quantum number p is connected with the main quantum number n by $p = n - k$ and the quantum number k is connected with the total angular momentum quantum number by $|k| = j + 1/2$ and $s = 2\sqrt{k^2-\alpha^2}$, $n^* = \sqrt{p^2+2ps+k^2}$. As we noticed the correction to the energy slightly depends on the parameter q, thus here we choose $q = \sqrt{2}$ to vanish the term proportional to $(r^2+b^2)^{-3/2}$ in (34).

To check the validity of our calculus let us discuss the obtained result for the correction to any excited state with k=-1 (or orbital quantum number $l = 1$). If we neglect the term connected with $\frac{(p^0)^2}{r}$ in the expression for this correction and take the nonrelativistic limit ($\alpha \to 0$) we obtain the well known result for a correction to energy levels with $l = 1$ of the hydrogen atom in space with deformed algebra (1). This fact ensures us that expressions (35) and (37) are correct.

Finally, we mention that the average $\langle \frac{(p^0)^2}{r} \rangle$ has a significantly larger contribution to the energy correction than other terms. This result can be predicted as far as p^0 contains the rest energy, which considerably ranks over the bound state energy.

We can make an estimation of the minimal length comparing our results with the experimental data of the frequency emitted during the transition $1s - 2s$. The accuracy of the measurement is of the order 100 Hz [13]. If we assume that the effect on the hydrogen atom energy spectrum caused by the deformation of commutation relations is less than the measurement accuracy, we find

$$\Delta X \leq 10^{-19} \text{ m}. \tag{38}$$

IV. CONCLUSION

We study the hydrogen atom problem in the space with Lorentz-covariant deformed algebra with the minimal length. In our previous paper [11] using ordinary perturbation theory we calculated the correction to the energy spectrum for all states besides some "problem states" with $|k| = 1$. The problem is connected with the terms proportional to $1/r^3$ and $\frac{1}{r}\hat{p}^2 + \hat{p}^2\frac{1}{r}$ contained in the perturbation operator, because of their divergent contribution to the energy correction. To overcome these divergencies, in the present paper we develop the modified perturbation theory by analogy with the nonrelativistic case. It gives us a possibility to calculate a correction to the energy spectrum for any eigenstate including the "problem states". Assuming that the effects of minimal length on the energy spectrum of the hydrogen atom cannot yet be seen experimentally we find that the upper bound of the minimal length is of the order 10^{-19} m.

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APPENDIX A: CALCULATION OF THE CORRECTIONS TO ANY ENERGY LEVEL OF THE HYDROGEN ATOM WITH $|\mathbf{k}| = 1$

Let us calculate the mean value $\left\langle \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2+b^2}} \right\rangle$ in the state with $|k| = 1$ preserving the leading terms over small b only

$$\left\langle \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2+b^2}} \right\rangle = \frac{4\hbar^2 C^2(p,k)}{a^2\alpha^2}\left(I_{FF}(p,k) + (n^*-k)^2 I_{GG}(p,k) - \frac{(2p+s)(n^*-k)}{n^*} I_{FG}(p,k)\right). \tag{A.1}$$

Here

$$\begin{aligned} I_{FF}(p,k) &= \int_0^\infty F_{pk}(x)\left(\frac{e^2}{x} - \frac{e^2}{\sqrt{x^2+\bar{b}^2}}\right)F_{pk}(x)dx, \\ I_{GG}(p,k) &= \int_0^\infty G_{pk}(x)\left(\frac{e^2}{x} - \frac{e^2}{\sqrt{x^2+\bar{b}^2}}\right)G_{pk}(x)dx, \\ I_{FG}(p,k) &= \int_0^\infty F_{pk}(x)\left(\frac{e^2}{x} - \frac{e^2}{\sqrt{x^2+\bar{b}^2}}\right)G_{pk}(x)dx, \end{aligned} \tag{A.2}$$

and $\bar{b} = \frac{2b}{an^*}$. The expressions for the functions $F_{pk}(x)$ and $G_{pk}(x)$ are:

$$F_{pk}(x) = x^{\frac{s}{2}} e^{-\frac{x}{2}} Q_p^s(x),$$
$$G_{pk}(x) = x^{\frac{s}{2}} e^{-\frac{x}{2}} Q_{p-1}^s(x) \tag{A.3}$$
$$C(p,k) = \sqrt{\frac{(n^*+k)}{p!\Gamma(p+s+1)} \frac{\alpha^2}{n^{*4}}}.$$

To recall the structure of the relativistic hydrogen atom wave function see [14].

To calculate integrals (A.2) let us introduce the function f of positive s and non-negative small $\bar{b}$ by the following integral

$$f(s,\bar{b}) = \int_0^\infty x^s e^{-x} \left(\frac{1}{x} - \frac{1}{\sqrt{x^2+\bar{b}^2}} \right) dx. \tag{A.4}$$

In the asymptotic of small b we find the following expression for the latter function

$$f(s,\bar{b}) \approx \begin{cases} \frac{1}{2}\bar{b}^2\Gamma(s-2) - \frac{1}{2\sqrt{\pi}}\bar{b}^s\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right), & \text{if } s \le 2 \\ \frac{1}{2}\bar{b}^2\Gamma(s-2), & \text{if } s > 2\,. \end{cases}$$

If we present the Laguerre polynomial as

$$Q_p^s(x) = \sum_{i=0}^{p} \frac{(-1)^i x^i}{i!} \frac{\Gamma(p+1)\Gamma(s+p+1)}{\Gamma(p-i+1)\Gamma(s+i+1)}, \tag{A.5}$$

we find that

$$I_{FF}(p,k) = \sum_{i=0}^{p}\sum_{j=0}^{p} \frac{(-1)^{i+j}\Gamma^2(s+p+1)\Gamma^2(p+1) f(s+i+j,\bar{b})}{i!j!\Gamma(s+i+1)\Gamma(s+j+1)\Gamma(p-i+1)\Gamma(p-j+1)}. \tag{A.6}$$

Next, substituting (A.5) in (A.6) we obtain

$$I_{FF}(p,k) = \frac{\bar{b}^2}{2}\sum_{i=0}^{p}\sum_{j=0}^{p} \frac{(-1)^{i+j}\Gamma^2(s+p+1)\Gamma^2(p+1)\Gamma(s+i+j-2)}{i!j!\Gamma(s+i+1)\Gamma(s+j+1)\Gamma(p-i+1)\Gamma(p-j+1)}$$
$$- \frac{\bar{b}^s}{2\sqrt{\pi}} \frac{\Gamma^2(s+p+1)}{\Gamma^2(s+1)} \Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right). \tag{A.7}$$

We have already dealt with the same sum as in the first term of the latter expression when we calculated the integral $\int_0^\infty F_{pk}(x)\frac{1}{x^3}F_{pk}(x)dx$ from $\left\langle \frac{e^2\hbar^2\beta}{r^3} \right\rangle$ with the only difference that in that case the parameter s was larger than 2. Analyzing this sum we arrive at the conclusion that nothing changes for the case when $s = 2\sqrt{1-\alpha^2}$. Therefore we write

$$I_{FF} = \frac{\bar{b}^2}{2}\Gamma(s+p+1)\Gamma(p+1)\frac{s^2+6ps+3s+6p^2+6p+2}{s(s^2-1)(s^2-4)} -$$
$$- \frac{\bar{b}^s}{2\sqrt{\pi}} \frac{\Gamma^2(s+p+1)}{\Gamma^2(s+1)} \Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \tag{A.8}$$

and similarly for I_{GG}, I_{FG}. Finally, we obtain

$$\left\langle \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2+b^2}} \right\rangle = \frac{8e^2\hbar^2\beta}{a^3n^3}\left(-\frac{1}{2s(s^2-1)} + \frac{3k^2\,(2p+s)\,(2n+k(2p+s))}{2n^2s(s^2-1)(s^2-4)} \right)$$
$$+\frac{e^2}{2\sqrt{\pi}a}\left(\frac{2\sqrt{2}\hbar\sqrt{\beta}}{an^*}\right)^s \frac{sk(n^*+k)+2pk^2}{n^{*3}} \frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma(s+p+1)\Gamma(p+s)}{p!\Gamma(s+1)^2}.$$

Of course, modified perturbation theory works also in the case Eigenfunctions and eigenvalues $|k| > 1$. Since in such circumstances the second term in the last expression can be omitted due to its higher infinitesimal order over the deformation parameter β.

Thus, the eigenstates with $|k| > 1$ obey

$$\left\langle \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2+b^2}} \right\rangle = \left\langle \frac{e^2\hbar^2\beta}{r^3} \right\rangle, \tag{A.9}$$

in linear approximation over a small β, as it has to be.

A similar calculation can be provided for other mean values. As a result we obtain (37).

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МОДИФІКОВАНА ТЕОРІЯ ЗБУРЕНЬ ДЛЯ АТОМА ВОДНЮ У ПРОСТОРІ З ЛОРЕНЦ-КОВАРІАНТНОЮ ДЕФОРМОВАНОЮ АЛҐЕБРОЮ З МІНІМАЛЬНОЮ ДОВЖИНОЮ

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Ми вивчаємо енергетичний спектр атома водню в теорії Дірака з Лоренц-коваріантною деформованою алґеброю, що веде до мінімальної довжини. Розвиваючи вільну від розбіжностей теорію збурень, ми обчислили поправки до всіх енергетичних рівнів у простому випадку деформації, коли один із параметрів деформації дорівнює нулеві. Припускаючи, що вплив мінімальної довжини на енергетичний спектр не можна спостерегти експериментально, ми знайшли верхню межу для мінімальної довжини.