THE THEORY OF SPATIALLY PERIODIC EQUILIBRIUM STATES IN THE QUASI-NEUTRAL SYSTEM OF CHARGES ABOVE THE SURFACE OF LIQUID DIELECTRIC

Yu. V. Slyusarenko, D. M. Lytvynenko
National Science Center Kharkiv Institute of Physics and Technology,
Akhiezer Institute for Theoretical Physics, 1, Akademichna St., Kharkiv, UA–61108, Ukraine
Kharkiv V. N. Karazin National University, High-Technology Institute,
31, Kurchatov Ave., Kharkiv, UA–61108, Ukraine,
e-mail: d_m_litvinenko@kipt.kharkov.ua
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The theory of quasi-neutral equilibrium state of charges above the surface of liquid dielectric is built. The basis of the approach is the variation principle, modified for the considered system, and the Thomas–Fermi model. In terms of this method we obtain the self-consistency equations relating the parameters of such system description — the electrostatic field potential, the distribution function of charges and the profile of liquid dielectric surface. The equations are used to study the phase transition of the system to states with spatially periodic structure. The parameters of the phase transition with the formation of dimple crystals are obtained.

Key words: electrons, non-degenerate gas, liquid dielectric, variation principle, perturbation theory, phase transition.

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I. INTRODUCTION

The research of spatially periodic states in the system of charged particles above the surface of liquid dielectric since its prediction [1] and appearance of the first works on its theoretical grounding [2–4] and experimental detection [5] already has a rather long history. Soon after the discovery of the phenomenon of the spatially periodic structures formation in the system of charges above helium surface it was named Wigner crystallization. This name is caused by the analogy between this phenomenon and the phenomenon of three-dimensional periodic structures formation in the system of electrons in metals predicted by Wigner [6]. It should be noted that one can come to the prediction of the phase transition associated with a spatially periodic structures formation in such systems also in a different way [7]. However, the spatially periodic structures above the helium surface have the advantage of simplicity of their experimental realization. It is known that Wigner three-dimensional spatially periodic structures still have no experimental proof. This circumstance causes one of the main reasons for the relevance of the research in the area of spatially periodic structures in the system of charged particles above the dielectric surface. In particular, it is assumed that the study of such systems can be useful in terms of predicting the effects associated with the Wigner crystallization in three-dimensional systems, as well as the conditions of their experimental observation. Naturally, the research of the mentioned systems makes the results interesting themselves, regardless to the problems of three-dimensional Wigner crystallization. At the present moment the works [8–10] and the references available in them seem to contain the most complete material to judge the development of the research of charge systems above the liquid helium surface.

It should be clarified that the existing works are devoted to the study of the effects associated with the existence of both two-dimensional Wigner crystals [5] and macroscopic dimple lattices [11]. The theoretical papers describing the corresponding experiments are based on the idea of the energy spectrum of the solitary electron above the dielectric surface. The essence of such a simple approach is that solitary electron above the flat dielectric surface together with its electrostatic image can be considered as an analogue of a hydrogen-like atom with the corresponding energy spectrum [12]. It is clear that in the case of a large number of charges, the description of the system under such consideration faces not only mathematical obstacles, but also the "philosophical" or methodological ones. Such obstacles can be avoided due to the description of these systems using a consistent microscopic theory instead of the phenomenological one. This theory initially bases on the consideration of the researched object as a quantum mechanical system of many particles [13,14] in external attracting electric field, because the attractive field produced by polarization of the dielectric by the charges is insufficient to hold them above the dielectric surface.

The approach to such theory building is proposed in [13]. It is based on the usage of the variation principle, modified for the considered systems, and the Thomas-Fermi model. The variation principle takes into account the possible existence of the external attracting electrostatic field. It allows obtaining of the self-consistency equations that give the relation between the parameters describing such system (the potential of electrostatic field, the distribution function of charges and the surface profile of liquid dielectric). As an application of the de-

veloped theory the authors study the phase transitions to spatially periodic states in a system of charged particles above the liquid dielectric surface. They obtain the solutions of self-consistency equations and the parameters of such phase transitions, including the periods of reciprocal lattices of spatially periodic structures of a special type. To test the developed theory and its partial results the experimental data [11] have been used and the qualitative agreement of theoretical and experimental data is obtained. It should be noted that the theory [13, 14] is based on the assumption of the conservation of the number of charges above the dielectric surface. In other words, the system is not assumed quasi-neutral. The term "quasi-neutrality" in such cases means the absence of an electric field far from the dielectric surface, and it is assumed that the system contains a number of charges, which compensates the external attracting electrostatic field [8–10].

Comparing with the "charged" system studied in [13, 14], the quasi-neutral system of charged particles above the liquid dielectric surface has a number of peculiarities in behavior [5, 10]. Some of them become apparent only in the description of the system in the microscopic approach. For this reason, the description of the space-periodic structures the case of quasi-neutral systems above the liquid dielectric surface in the presence of external pressing electrostatic field must be done separately. In its turn, such a description requires some modification of certain theory assumptions proposed in [13,14] for the "charged" systems. The subject of the present work is the solution to such problems.

II. SELF-CONSISTENCY EQUATIONS FOR THE SYSTEM OF CHARGES ABOVE THE LIQUID DIELECTRIC SURFACE

It was noted above that instead of the charged system in the quasi-neutral one the number of charges is determined by the external field. In other words, changing of the external clamping field causes the changing of the number of charges above dielectric surface. This difference between the systems does not affect the formulation of the variation principle for them, see [13]. For this reason there is no difference in self-consistency equations for charged and neutral systems obtained within the approach [13]. Therefore in this paper there is no need to build the variation principle of article [13]. We only write the self-consistency equations obtained there.

However, let us briefly recall, in terms of what physical characteristics the system is described and define its geometry. We consider a system of identical particles with charge Q, mass m, spin S_Q, momentum $\mathbf{p}$ and energy $\varepsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m}$. The charges are located in vacuum above the surface of liquid dielectric film having thickness d and dielectric constant ε and surface tension coefficient α. We assume that the liquid dielectric film is located on a flat, solid dielectric substrate with a dielectric constant $\varepsilon_d \gg \varepsilon$. The surface profile of the liquid dielectric film is described by a function $\xi(\boldsymbol{\rho}) \equiv \xi(x, y)$,

where $\boldsymbol{\rho} \equiv \{x, y\}$ is a radius vector in the plane $z = 0$ of Cartesian coordinate system $\{z, x, y\}$. We assume the interface regions "1"–"3" in the direction of coordinates unlimited $\boldsymbol{\rho} \equiv \{x, y\}$. To avoid any questions concerned with "repulsion" of the same charged particles along $\boldsymbol{\rho}$, we assume that the system is located in a vessel with walls at $\rho \to \infty$. These walls prohibit charges to leave the system along the unperturbed flat liquid dielectric surface.
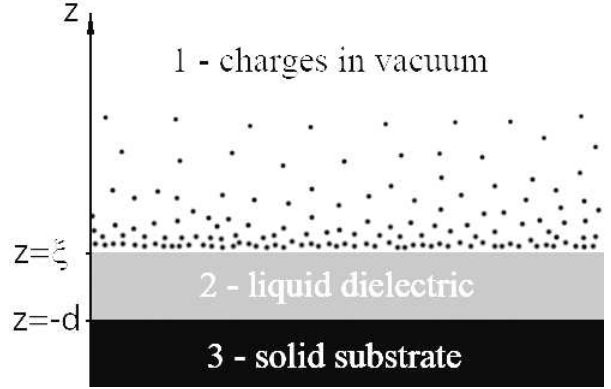

Fig. 1. Schematic structure of the studied system

Let the particles be acted by external attracting electric field E directed along the z axis. We also assume the existence of a potential barrier prohibiting the penetration of charges inside the liquid dielectric film. All physical quantities related to the region $z > \xi(\boldsymbol{\rho})$ are marked by index "1", the physical quantities related to the liquid dielectric film ($\xi(\boldsymbol{\rho}) > z > -d$) - by index "2", and the physical quantities related to the dielectric solid substrate ($z < -d$) - by index "3" (see Fig. 1).

Let us introduce the parameters describing the system. In the area "1" the system is completely described by distribution function of particles $f_{\mathbf{p}}(\mathbf{r})$, electric field potential $\varphi_1^{(i)}(\mathbf{r})$ created by the system of charges, external attractive electrostatic field potential $\varphi_1^{(e)}(\mathbf{r})$ and the surface profile $\xi(\boldsymbol{\rho})$ of liquid dielectric. In the area "2" the system is described by the surface profile $\xi(\boldsymbol{\rho})$ of liquid dielectric and the total electric field potential. The "total" potential means the sum of the external electric field potential in liquid dielectric and the field potential induced by charges from the area "1". The area "3" is characterized by total electric field potential in the solid substrate.

To obtain the self-consistent equations for the equilibrium values of basic parameters describing the system $f_{\mathbf{p}}(\mathbf{r})$, $\xi(\boldsymbol{\rho})$ and $\varphi_1^{(i)}(\mathbf{r})$ it is necessary to solve the problem of determining of the maximum of the system entropy S

$$S = -\frac{g}{(2\pi\hbar)^3} \int d\mathbf{r}\, d\mathbf{p} \left(\bar{f} \ln \bar{f} + (1 - \bar{f}) \ln (1 - \bar{f})\right),$$

(1)

$$\bar{f} = \frac{(2\pi\hbar)^3}{g} f_{\mathbf{p}}(\mathbf{r}), \quad g = 2S_Q + 1$$

under the following conditions. Firstly, in the given (constant) external attracting field the total number of particles in the system $N = \int d\mathbf{r} \, d\mathbf{p} \, f_{\mathbf{p}}(\mathbf{r})$, the total energy of the system

$$
\begin{aligned}
E_t &= \int\limits_{V_1} d\mathbf{r} \, d\mathbf{p} \, f_{\mathbf{p}} \varepsilon_{\mathbf{p}} + \frac{Q}{2} \int\limits_{V_1} d\mathbf{r} \, \varphi_1^{(i)} n \\
&+ Q \int\limits_{V_1} d\mathbf{r} \, \varphi_1^{(e)} n + \int\limits_{V_1} d\mathbf{r} \, \frac{\left(\nabla \varphi_1^{(e)}\right)^2}{8\pi} \\
&+ \frac{\varepsilon}{8\pi} \int\limits_{V_2} d\mathbf{r} \, (\nabla \varphi_2)^2 + \frac{\varepsilon_d}{8\pi} \int\limits_{V_3} d\mathbf{r} \, (\nabla \varphi_3)^2 \qquad (2) \\
&+ \frac{\alpha}{2} \int dS \left\{ (\nabla_{\boldsymbol{\rho}} \xi(\boldsymbol{\rho}))^2 + \kappa^2 \xi^2(\boldsymbol{\rho}) \right\}, \\
& dS = d^2\rho \sqrt{1 + (\nabla_{\boldsymbol{\rho}} \xi(\boldsymbol{\rho}))^2}, \\
& \nabla_{\boldsymbol{\rho}} \equiv \partial / \partial \boldsymbol{\rho}, \quad \varphi_j = \varphi_j^{(i)} + \varphi_j^{(e)}
\end{aligned}
$$

and its total momentum $\mathbf{P} = \int d\mathbf{r} \, d\mathbf{p} f_{\mathbf{p}}(\mathbf{r}) \, \mathbf{p}$ must remain constant. If the system as a whole is at rest, its total momentum must be zero. Secondly, in the absence of charges above liquid dielectric film the its surface profile cannot be transformable. And thirdly, in all three regions of the system the Poisson equations must take place. Let us also note that in Eq. (2) V_j, $j = 1, 2, 3$ are the volumes of "1","2" and "3" regions, correspondingly. Besides that the definition of particle density is used

$$
n(\mathbf{r}) = \int d\mathbf{p} \, f_{\mathbf{p}}(\mathbf{r}). \qquad (3)
$$

The problem on determining the conditional maximum of entropy can be reduced to the problem on unconditional minimum of thermodynamical potential $\tilde{\Omega}$ determining (see [13] for details):

$$
\begin{aligned}
\tilde{\Omega} &= -S + Y_0 E + Y_i P_i + Y_4 N \\
&+ \int d\rho \, \lambda_\xi(\rho) \, \xi(\rho)|_{N=0} \qquad (4) \\
&+ \int d\mathbf{r} \, \lambda(\mathbf{r}) \left\{ \Delta\varphi(\mathbf{r}) + 4\pi Q n(\mathbf{r}) \right\},
\end{aligned}
$$

where $Y_0, Y_i, Y_4, \lambda(\mathbf{r}), \lambda_\xi(\rho)$ are the corresponding Lagrange multipliers to the above conditions.

The solution of such a variation problem in [13] results in the following equation

$$
\begin{aligned}
&\frac{gT}{(2\pi\hbar)^3} \int d\mathbf{p} \ln \left(1 - \frac{(2\pi\hbar)^3}{g} f_{\mathbf{p}}(\mathbf{r}) \right) \Bigg|_{z=\xi(\boldsymbol{\rho})} = \frac{\varepsilon}{8\pi} \left((\nabla\varphi_2(\mathbf{r}))^2 - \left(\nabla\varphi_2^{(e)}(\mathbf{r})\right)^2 \right)_{z=\xi(\boldsymbol{\rho})} \\
&+ \alpha \left\{ \kappa^2 \xi(\boldsymbol{\rho}) \sqrt{1 + (\nabla_{\boldsymbol{\rho}} \xi(\boldsymbol{\rho}))^2} - \nabla_{\boldsymbol{\rho}} \left[\frac{\nabla_{\boldsymbol{\rho}} \xi(\boldsymbol{\rho}) \left(1 + \frac{\kappa^2}{2} \xi^2(\boldsymbol{\rho}) + \frac{3}{2} (\nabla_{\boldsymbol{\rho}} \xi(\boldsymbol{\rho}))^2 \right)}{\sqrt{1 + (\nabla_{\boldsymbol{\rho}} \xi(\boldsymbol{\rho}))^2}} \right] \right\},
\end{aligned}
$$

$$(5)$$

where the distribution function of charges $f_{\mathbf{p}}(\mathbf{r})$ is given by

$$
f_{\mathbf{p}}(\mathbf{r}) = \frac{\theta(z - \xi(\boldsymbol{\rho})) \, g}{(2\pi\hbar)^3 \left(1 + e^{\frac{\varepsilon_{\mathbf{p}} - \mu + Q\varphi_1}{T}} \right)}. \qquad (6)
$$

$\theta(z)$ is the Heaviside step function, and the constant κ in (5) is defined by the expression

$$
\kappa^2 = \frac{\rho}{\alpha} (g + f), \qquad (7)
$$

where g is gravity acceleration, α is the surface tension of liquid dielectric, ρ is its density, and $f \sim d^{-4}$ is van der Waals constant, which in the case of a massive liquid dielectric ($d \to \infty$) is negligible compared to g. In the case of a thin dielectric film the gravity force acting on atoms of liquid dielectric becomes negligibly small as compared to van der Waals forces (see [8, 10] and references therein). E.g., such situation takes place for liquid helium films thinner then $d \sim 10^{-4}$ cm.

Eqs. (5), (6) together with the equations for the electric field potentials, both external and induced by charges in all three regions of the system,

$$
\begin{aligned}
\Delta\varphi_1^{(i)}(\mathbf{r}) + 4\pi Q \int d\mathbf{p} f_{\mathbf{p}}(\mathbf{r}) &= 0, \\
\Delta\varphi_2^{(i)}(\mathbf{r}) = 0, \quad \Delta\varphi_3^{(i)}(\mathbf{r}) &= 0, \qquad (8) \\
\Delta\varphi_j^{(e)}(\mathbf{r}) = 0, \quad j = 1, 2, 3,
\end{aligned}
$$

form a set of self-consistent equations. However, this system must be supplemented by the boundary conditions for the electric fields and their potentials at the interfaces $z = \xi(\boldsymbol{\rho})$ and $z = -d$. For the sake of convenience we shall write the boundary conditions for the fields during the description of the phase transition with the formation of the system of spatially periodic structures.

III. SCENARIO OF THE PHASE TRANSITION RESULTING IN SPATIALLY PERIODIC STRUCTURES AND THE BOUNDARY CONDITIONS FOR THE ELECTRIC FIELDS

The scenario of the phase transition resulting in the transformation of the surface of liquid dielectric film is assumed as follows (see, e.g., [8,10,13]). As already mentioned, the external electric field attracting charges to the flat surface of liquid dielectric film causes its subsidence within the area of this field acting. Moreover, the bottom of this deflection remains flat. Therefore, the deformation of liquid dielectric surface, leaving the bottom deflection flat, can be characterized by a single parameter $\bar{\xi}$ (depth of subsidence). If the flat surface of undeformed dielectric is described by plane, then the value of $\bar{\xi}$ should be negative, $\bar{\xi} < 0$. With further increase of the external electric field the depth increases and the surface of bottom deformation remains flat up to a certain critical value of the total electric field E_c on the dielectric surface,

$$E_c = \left| \frac{\partial \varphi_1(z, \boldsymbol{\rho})}{\partial z} \right|_{z = \bar{\xi}}. \tag{9}$$

Naturally, in this case the inequality $|\bar{\xi}| < d$ takes place, if the liquid dielectric is a film with thickness d located on a solid substrate.

With further growth of the clamping electric field the surface profile of the formed deflection bottom the can be deformed and acquire a periodic structure. Hence, the phase transition to a spatially periodic structures in this system occurs on a background of a flat structure of liquid dielectric.

It should be noted that the control parameter for this phase transition can be not only the external electric field, but also the temperature. As mentioned above, under the conditions of electric neutrality the charge density above liquid dielectric surface is determined by the external field. Moreover, it will be seen further that in the transition point these two parameters (external electric field and temperature) are related by an equation describing a certain curve.

According to the above scenario of the phase transition the surface profile of liquid dielectric in a phase with a lower symmetry, may be represented as follows [13,14]:

$$\xi(\boldsymbol{\rho}) = \bar{\xi} + \tilde{\xi}(\boldsymbol{\rho}), \tag{10}$$

where $\tilde{\xi}(\boldsymbol{\rho})$ is the spatially inhomogeneous surface profile forming as the result of the phase transition on the background of the flat bottom surface $z = \bar{\xi}$ above a homogeneous deformation. Thus, the surface profile $\tilde{\xi}(\boldsymbol{\rho})$ is the order parameter of the considered phase transition. In the symmetric phase this quantity has zero value, in an asymmetric one it describes the spatially periodic structure of the surface. So, near the critical point from the asymmetric phase, the inequality

$$|\bar{\xi}| \gg \left| \tilde{\xi}(\boldsymbol{\rho}) \right| \tag{11}$$

takes place. Let us recall that in the theory of phase transitions term "asymmetric phase" means the phase formed as a result of phase transition, and this phase has a lower symmetry than the initial one. In this case the initial phase is called symmetric. Let us also note that in the case of Eq. (11) taking place in the neighborhood the phase transition point, and the zero value of the order parameter at the point, the second order phase transition occurs [15].

To describe the phase transitions associated with the transformation of liquid dielectric surface and formation of spatially periodic structures in the researched system we must obtain the following quantities: $\bar{\xi}$, $\tilde{\xi}(\boldsymbol{\rho})$ and the distributions of charges and fields in the system as the result of phase transition. For this purpose we use Eqs. (5)–(8) supplemented by the boundary conditions for the characteristics of the electric field at the interfaces of the three regions. In the case of an external constant and uniform electric field acting on the system, the potentials φ_1, φ_2 and φ_3 in all three areas can be given as

The boundary conditions for the potentials φ_j on the boundaries $z = \xi(\boldsymbol{\rho})$ and $z = -d$ can be written as:

$$\varphi_1(z, \boldsymbol{\rho}) \,|_{z=\xi} = \varphi_2(z, \boldsymbol{\rho}) \,|_{z=\xi} \, ,$$
$$\varphi_2(z, \boldsymbol{\rho}) \,|_{z=-d} = \varphi_3(z, \boldsymbol{\rho}) \,|_{z=-d} \, ,$$
$$((\mathbf{n}(\boldsymbol{\rho}) \cdot \nabla) \{\varepsilon \varphi_2(z, \boldsymbol{\rho}) - \varphi_1(z, \boldsymbol{\rho})\})_{z=\xi} = 0,$$
$$\left\{ \varepsilon \frac{\partial \varphi_2(z, \boldsymbol{\rho})}{\partial z} - \varepsilon_d \frac{\partial \varphi_3(z, \boldsymbol{\rho})}{\partial z} \right\}_{z=-d} = 0,$$
$$\varphi_1^{(e)}(z, \boldsymbol{\rho}) \,|_{z=\xi} = \varphi_2^{(e)}(z, \boldsymbol{\rho}) \,|_{z=\xi} \, ,$$
$$\varphi_2^{(e)}(z, \boldsymbol{\rho}) \,|_{z=-d} = \varphi_3^{(e)}(z, \boldsymbol{\rho}) \,|_{z=-d} \, ,$$
$$\left((\mathbf{n}(\boldsymbol{\rho}) \cdot \nabla) \, \varphi_1^{(e)}(z, \boldsymbol{\rho}) \right)_{z=\xi}$$

$$= \varepsilon \left((\mathbf{n}(\boldsymbol{\rho}) \cdot \nabla) \, \varphi_2^{(e)}(z, \boldsymbol{\rho}) \right)_{z=\xi},$$
$$\left\{ \varepsilon \frac{\partial \varphi_2^{(e)}(z, \boldsymbol{\rho})}{\partial z} - \varepsilon_d \frac{\partial \varphi_3^{(e)}(z, \boldsymbol{\rho})}{\partial z} \right\}_{z=-d} = 0, \tag{12}$$

where $\mathbf{n}(\boldsymbol{\rho})$ is the normal to the surface with profile $\xi(\boldsymbol{\rho})$ at the point $\boldsymbol{\rho}$.

$$\mathbf{n}(\boldsymbol{\rho}) = \sigma \left\{ -\frac{\partial \xi}{\partial x}, -\frac{\partial \xi}{\partial y}, 1 \right\}$$
$$\sigma = \left(1 + (\nabla \xi)^2 \right)^{-1/2}. \tag{13}$$

These boundary conditions correspond to the case when the surface charges on the boundaries are absent. Eq. (12) must be also supplemented by the limitation conditions of the fields at infinity

$$\left| \frac{\partial \varphi_1^{(e)}}{\partial z} \right|_{z \to +\infty} < +\infty, \quad \left| \frac{\partial \varphi_3^{(e)}}{\partial z} \right|_{z \to -\infty} < +\infty,$$
$$\left| \frac{\partial \varphi_1}{\partial z} \right|_{z \to +\infty} < +\infty, \quad \left| \frac{\partial \varphi_3}{\partial z} \right|_{z \to -\infty} < +\infty. \tag{14}$$

Further on we shall consider the surface profile which is not very different from the flat one, and show how Eqs. (5)–(8) change in this case. It is shown in [13] that if the surface profile slowly changes along the coordinate, we have

$$|\partial \xi(\boldsymbol{\rho})/\partial x| \ll 1, \quad |\partial \xi(\boldsymbol{\rho})/\partial y| \ll 1. \qquad (15)$$

When Eqs. (11)–(15) take place, we can expect the distribution of charges and fields in the system to be a little different from the distributions taking place in the case of a flat dielectric surface $z = \bar{\xi}$. Then, the potentials of the external $\varphi_j^{(e)}$, $j = 1, 2, 3$ and the total φ_j fields can be given as

$$\varphi_j(z, \boldsymbol{\rho}) = \bar{\varphi}_j(z) + \tilde{\varphi}_j(z, \boldsymbol{\rho}),$$
$$\varphi_j^{(e)}(z, \boldsymbol{\rho}) = \bar{\varphi}_j^{(e)}(z) + \tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho}), \qquad (16)$$

where $\bar{\varphi}_j(z)$ and $\bar{\varphi}_j^{(e)}(z)$ are the potentials of total and external electric fields respectively, in all of the described above regions of the system (but not on the boundaries) in the case of flat liquid dielectric surface $z = \bar{\xi}$. Potentials $\tilde{\varphi}_j(z, \boldsymbol{\rho})$ and $\tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho})$ describe small potential distortions in all these regions due to the surface inhomogeneity with profile $\tilde{\xi}(\boldsymbol{\rho})$. As the distortions of the potential by the undulated surface are assumed to be week, the following inequalities take place

$$|\bar{\varphi}_j(z)| \gg |\tilde{\varphi}_j(z, \boldsymbol{\rho})|,$$
$$\left|\bar{\varphi}_j^{(e)}(z)\right| \gg \left|\tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho})\right|. \qquad (17)$$

We shall further assume that the initially flat surface profile and then deformed as a result of the phase transition $\tilde{\xi}(\boldsymbol{\rho})$

$$\tilde{\xi}(\boldsymbol{\rho}) = \sum_{\mathbf{q} \neq 0} \xi_{\mathbf{q}} e^{i\mathbf{q}\rho},$$
$$\xi_{\mathbf{q}} = \frac{1}{(2\pi)^2} \int d\boldsymbol{\rho}\, \xi(\boldsymbol{\rho}) e^{-i\mathbf{q}\rho} \qquad (18)$$

is spatially periodic. In the case of $\tilde{\xi}(\boldsymbol{\rho})$ periodicity (see Eq. (18)), the Eq. (10) leads to:

$$\bar{\xi} \equiv \langle \xi(\boldsymbol{\rho}) \rangle, \quad \tilde{\xi}(\boldsymbol{\rho}) = \xi(\boldsymbol{\rho}) - \langle \xi(\boldsymbol{\rho}) \rangle, \qquad (19)$$

where $\langle ... \rangle$ is averaging over the period.

The periodic structure of $\tilde{\xi}(\mathbf{q})$ allows searching the potentials $\tilde{\varphi}_j(z, \boldsymbol{\rho})$ (see Eq. (16)) in the form:

$$\tilde{\varphi}_j(z, \boldsymbol{\rho}) = \sum_{\mathbf{q} \neq 0} \tilde{\varphi}_{j\mathbf{q}}(z) e^{i\mathbf{q}\rho},$$
$$\tilde{\varphi}_{j\mathbf{q}}(z) = \frac{1}{(2\pi)^2} \int d\boldsymbol{\rho}\, \tilde{\varphi}_j(z, \boldsymbol{\rho}) e^{-i\mathbf{q}\rho},$$
$$\tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho}) = \sum_{\mathbf{q} \neq 0} \tilde{\varphi}_{j\mathbf{q}}^{(e)}(z) e^{i\mathbf{q}\rho}, \qquad (20)$$
$$\tilde{\varphi}_{j\mathbf{q}}^{(e)}(z) = \frac{1}{(2\pi)^2} \int d\boldsymbol{\rho}\, \tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho}) e^{-i\mathbf{q}\rho}.$$

Taking into account Eqs. (16), (20), we easily see that

$$\bar{\varphi}_j(z) \equiv \langle \varphi_j(z, \boldsymbol{\rho}) \rangle, \quad \langle \tilde{\varphi}_j(z, \boldsymbol{\rho}) \rangle = 0,$$
$$\bar{\varphi}_j^{(e)}(z) \equiv \left\langle \varphi_j^{(e)}(z, \boldsymbol{\rho}) \right\rangle, \quad \left\langle \tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho}) \right\rangle = 0.$$

To describe the phase transition on the scenario described in the beginning of this section, it is necessary to determine the order parameter $\tilde{\xi}(\boldsymbol{\rho})$. Considering the phase transition as a second order one, we are able to obtain the order parameter $\tilde{\xi}(\boldsymbol{\rho})$ as a function of the control parameters T, E, n_s near the critical values T_c, E_c, n_{sc} using the perturbation theory in the small parameters $\tilde{\xi}(\boldsymbol{\rho})$, $\tilde{\varphi}_j(z, \boldsymbol{\rho})$ and $\tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho})$.

Taking into account Eqs. (15)–(18) after substituting Eqs. (15), (16), into Eqs. (5)–(8) and keeping the terms linear in $\tilde{\xi}(\boldsymbol{\rho})$, $\tilde{\varphi}_j(z, \boldsymbol{\rho})$, $\tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho})$ and $T - T_c$, $E - E_c$, $n - n_{sc}$, we obtain the equations describing the spatial structure of the liquid dielectric surface and the distribution of charges and fields in the asymmetric phase near the critical surface (see the note above). Let us write the equations describing the system above the surface of the liquid dielectric film $z = \bar{\xi}$, i.e., in the region "1". These equations are the main approximation of the described perturbation theory. Subsequently, the charges above the liquid dielectric surface are considered to be electrons, and therefore in the corresponding formulae we put the charge of an electron $Q = -e$ in the place of Q. Due to the periodicity of the small quantities $\tilde{\xi}(\boldsymbol{\rho})$, $\tilde{\varphi}_j(z, \boldsymbol{\rho})$ and $\tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho})$, see Eqs. (18), (20), the main approximation is obtained by averaging over the period of the self-consistent equations Eqs. (5)–(8). The components representing the averaged values of terms quadratic in $\tilde{\xi}(\boldsymbol{\rho})$, $\tilde{\varphi}_j(z, \boldsymbol{\rho})$ and $\tilde{\varphi}_j^{(e)}(z, \boldsymbol{\rho})$ are small in comparison with the main approximation, therefore they can be omitted. Then the Poisson equation in the first region in the main approximation has the form

$$\frac{\partial^2}{\partial z^2} \bar{\varphi}_1(z) = 4\pi e n(z) \theta(z - \bar{\xi}),$$
$$n(z) = \int d^3 p\, f_{\mathbf{p}}(z), \qquad (21)$$
$$f_{\mathbf{p}}(z) = \frac{g}{(2\pi\hbar)^3} \left(1 + e^{\frac{\varepsilon_{\mathbf{p}} - (e\bar{\varphi}_1(z) + \mu)}{T}}\right)^{-1}.$$

The main order approximation of Eq. (5) in the mentioned parameters gives the equation to determine $\bar{\xi}$:

$$\frac{gT}{(2\pi\hbar)^3} \int d\mathbf{p} \ln\left(1 - \frac{(2\pi\hbar)^3}{g} f_{\mathbf{p}}(z)\right)\bigg|_{z=\bar{\xi}}$$
$$= \frac{\varepsilon}{8\pi} \left(\left(\frac{\partial \bar{\varphi}_2(z)}{\partial z}\right)^2 - \left(\frac{\partial \bar{\varphi}_2^{(e)}(z)}{\partial z}\right)^2\right)_{z=\bar{\xi}}$$
$$+ \alpha \kappa^2 \bar{\xi}. \qquad (22)$$

Due to the absence of charges in regions "2" and "3", the equations for the potentials $\bar{\varphi}_2(z)$ and $\bar{\varphi}_3(z)$ have the form:

$$\frac{\partial^2}{\partial z^2} \bar{\varphi}_2(z) = 0, \quad \frac{\partial^2}{\partial z^2} \bar{\varphi}_3(z) = 0. \qquad (23)$$

Let us remind that the external field potentials $\bar{\varphi}_j^{(e)}(z)$ in all three regions are described by the same equations (Laplace equations):

$$\frac{\partial^2}{\partial z^2}\bar{\varphi}_j^{(e)}(z) = 0. \qquad (24)$$

To make the system of Eqs. (21)–(24) self-contained, the same averaging procedure of period is done with the boundary conditions Eq. (12). As a result we obtain the relation between the total potentials $\bar{\varphi}_j(z)$ and the potentials of external electric field $\bar{\varphi}_j^{(e)}(z)$ on the borders of three regions by the boundary conditions:

$$\bar{\varphi}_1|_{z=\bar{\xi}} = \bar{\varphi}_2|_{z=\bar{\xi}}, \quad \bar{\varphi}_2|_{z=-d} = \bar{\varphi}_3|_{z=-d},$$
$$\varepsilon\,\bar{\varphi}_2'|_{z=\bar{\xi}} = \bar{\varphi}_1'|_{z=\bar{\xi}}, \quad \varepsilon_d\,\bar{\varphi}_3'|_{z=-d} = \varepsilon\,\bar{\varphi}_2'|_{z=-d},$$
$$\bar{\varphi}_1^{(e)}\Big|_{z=\bar{\xi}} = \bar{\varphi}_2^{(e)}\Big|_{z=\bar{\xi}}, \quad \bar{\varphi}_2^{(e)}\Big|_{z=-d} = \bar{\varphi}_3^{(e)}\Big|_{z=-d}, \qquad (25)$$
$$\varepsilon\,\bar{\varphi}_2^{(e)'}\Big|_{z=\bar{\xi}} = \bar{\varphi}_1^{(e)'}\Big|_{z=\bar{\xi}},$$
$$\bar{\varphi}_3^{(e)'}\Big|_{z=-d} = \frac{\varepsilon}{\varepsilon_d}\bar{\varphi}_2^{(e)'}\Big|_{z=-d}.$$

It is easy to see that the solution of the problem on the phase transition description starts from solving the equations of the main approximation (21)–(24) with boundary conditions Eq. (25). This procedure provides obtaining the distribution of charges and fields in the system in case of flat surface of liquid dielectric, which is given by the equation $z = \bar{\xi}$. The value of $\bar{\xi}$ obtained from Eq. (22) determines the subsidence level of the flat surface due to the influence of charges (electrons) on it.

To obtain the critical parameters of the considered phase transition the higher orders of perturbation theory are involved. Below we shall formulate the system of self-consistent Eqs. (5)–(8) in the first order of this theory. For the further simplicity of calculations, we assume the resulting periodic structure to be one-dimensional with the period along the x axis equal to a, so $q = q_x = \frac{2\pi}{a}$. In this case further on instead of the vector $\mathbf{q}$, directed along x axis, we write its corresponding projection q. Let us try the quantities $\tilde{\xi}_q$, $\tilde{\varphi}_{jq}(z)$ and $\tilde{\varphi}_{jq}^{(e)}(z)$ in the following form

$$\tilde{\xi}_q(z) = \sum_{l=1}^{\infty}\tilde{\xi}_q^{(l)}, \quad \tilde{\varphi}_{jq}(z) = \sum_{l=1}^{\infty}\tilde{\varphi}_{jq}^{(l)}(z),$$
$$\tilde{\varphi}_{jq}^{(e)}(z) = \sum_{l=1}^{\infty}\tilde{\varphi}_{jq}^{(e)(l)}(z), \qquad (26)$$

where

$$\tilde{\xi}_q^{(1)} = \tilde{\xi}_{q_0}^{(1)}\left(\Delta(q-q_0) + \Delta(q+q_0)\right),$$
$$\tilde{\xi}_q^{(2)} = \tilde{\xi}_{2q_0}^{(2)}\left(\Delta(q-2q_0) + \Delta(q+2q_0)\right),$$
$$\tilde{\varphi}_{jq}^{(1)}(z) = \tilde{\varphi}_{jq_0}^{(1)}(z)\left(\Delta(q-q_0) + \Delta(q+q_0)\right),$$
$$\tilde{\varphi}_{jq}^{(2)} = \tilde{\varphi}_{j2q_0}^{(2)}\left(\Delta(q-2q_0) + \Delta(q+2q_0)\right), \qquad (27)$$
$$\tilde{\varphi}_{jq}^{(e)(1)}(z) = \tilde{\varphi}_{jq_0}^{(e)(1)}(z)$$
$$\times\left(\Delta(q-q_0) + \Delta(q+q_0)\right),$$
$$\tilde{\varphi}_{jq}^{(e)(2)} = \tilde{\varphi}_{j2q_0}^{(e)(2)}\left(\Delta(q-2q_0) + \Delta(q+2q_0)\right).$$

In Eq. (27) is the Kronecker symbol

$$\Delta(q) = \begin{cases} 0, & q \neq 0 \\ 1, & q = 0 \end{cases}.$$

In Eq. (26) we assume the appearing periodic structure to be one-dimensional with a period along x axis equal to a, so

$$q = q_x = 2\pi/a.$$

We also assume that $\tilde{\varphi}_{jq}(z) = \tilde{\varphi}_{j-q}(z)$ and $\tilde{\xi}_q = \tilde{\xi}_{-q}$, thereby considering the real values of these quantities, so

$$\tilde{\xi}(x) = 2\sum_{l=1}^{+\infty}\tilde{\xi}^{(l)}\cos lq_0 x,$$
$$\tilde{\varphi}_j(x,z) = 2\sum_{l=1}^{+\infty}\tilde{\varphi}_j^{(l)}(z)\cos lq_0 x,$$
$$\tilde{\varphi}_j^{(e)}(x,z) = 2\sum_{l=1}^{+\infty}\tilde{\varphi}_j^{(e)(l)}(z)\cos lq_0 x.$$

Then, the linear approximation for Eqs. (5), (8) in small values of the first harmonics of $\tilde{\xi}_q$ and $\tilde{\varphi}_{jq}(z)$ has the following form

$$\frac{\partial^2\tilde{\varphi}_1^{(1)}}{\partial z^2} - q_0^2\tilde{\varphi}_1^{(1)} = 4\pi e^2\frac{\partial n}{\partial\mu}\tilde{\varphi}_1^{(1)},$$
$$\frac{\partial^2\tilde{\varphi}_2^{(1)}}{\partial z^2} - q_0^2\tilde{\varphi}_2^{(1)} = 0, \quad \frac{\partial^2\tilde{\varphi}_3^{(1)}}{\partial z^2} - q_0^2\tilde{\varphi}_3^{(1)} = 0,$$

$$\frac{\varepsilon}{4\pi}\left(\bar{\varphi}_2^{(e)'}\frac{\partial\tilde{\varphi}_2^{(e)(1)}}{\partial z} - \bar{\varphi}_2'\frac{\partial\tilde{\varphi}_2^{(1)}}{\partial z}\right)_{z=\bar{\xi}}$$
$$-\left(en\left(\tilde{\varphi}_1^{(1)} + \bar{\varphi}_1'\tilde{\xi}^{(1)}\right)\right)_{z=\bar{\xi}} \qquad (28)$$
$$= \alpha\tilde{\xi}^{(1)}\left(\kappa^2 + q^2\left(1 + \frac{\kappa^2\bar{\xi}^2}{2}\right)\right).$$

And the same approximation for the boundary conditions Eq. (12) can be expressed as:

$$\left((\bar{\varphi}_1' - \bar{\varphi}_2')\tilde{\xi}^{(1)} + \tilde{\varphi}_1^{(1)} - \tilde{\varphi}_2^{(1)}\right)_{z=\bar{\xi}} = 0,$$
$$\left(\bar{\varphi}_1''\tilde{\xi}^{(1)} + \frac{\partial\tilde{\varphi}_1^{(1)}}{\partial z} - \varepsilon\frac{\partial\tilde{\varphi}_2^{(1)}}{\partial z}\right)_{z=\bar{\xi}} = 0,$$
$$\left(\tilde{\varphi}_2^{(1)} - \tilde{\varphi}_3^{(1)}\right)_{z=-d} = 0, \qquad (29)$$
$$\left(\varepsilon\frac{\partial\tilde{\varphi}_2^{(1)}}{\partial z} - \varepsilon_d\frac{\partial\tilde{\varphi}_3^{(1)}}{\partial z}\right)_{z=-d} = 0.$$

Similarly, the first approximation of the considered perturbation theory for Eq. (5) for the external potential,

supplemented by the appropriate boundary conditions has the form

$$\frac{\partial^2 \tilde{\varphi}_j^{(1)}}{\partial z^2} - q_0^2 \tilde{\varphi}_j^{(1)} = 0, \quad j = 1, 2, 3,$$

$$\left(\left(\bar{\varphi}_1^{(e)'} - \bar{\varphi}_2^{(e)'} \right) \tilde{\xi}^{(1)} + \tilde{\varphi}_1^{(e)(1)} - \tilde{\varphi}_2^{(e)(1)} \right)_{z=\bar{\xi}} = 0,$$

$$\left(\tilde{\varphi}_2^{(e)(1)} - \tilde{\varphi}_3^{(e)(1)} \right)_{z=-d} = 0,$$

$$\left(\frac{\partial \tilde{\varphi}_1^{(e)(1)}}{\partial z} - \varepsilon \frac{\partial \tilde{\varphi}_2^{(e)(1)}}{\partial z} \right)_{z=\bar{\xi}} = 0, \qquad (30)$$

$$\left(\varepsilon \frac{\partial \tilde{\varphi}_2^{(e)(1)}}{\partial z} - \varepsilon_d \frac{\partial \tilde{\varphi}_3^{(e)(1)}}{\partial z} \right)_{z=-d} = 0.$$

Eqs. (28)–(30) allow solving the above problem of describing the phase transition associated with the formation of spatially periodic structures above the liquid dielectric film surface in the studied system. We also note that the values of T and E, included in the coefficients multiplied by small deviations $\tilde{\xi}^{(1)}, \tilde{\varphi}_j^{(e)(1)}, \tilde{\varphi}_j^{(1)}$ are related by the already mentioned critical surface. The linear approximation of the discussed equations does not contain the terms proportional to $T - T_c$ and $E - E_c$ because they have a higher order of smallness. This fact causes obtaining the higher orders of perturbation theory to calculate the dependence of the order parameter $\tilde{\xi}^{(1)}$ on the control parameters T, E near the critical surface. The following sections are devoted to solving this problem, and to solving the system of Eqs. (21)–(25) and Eqs. (28)–(30).

It should be particularly emphasized that in the present paper the system of charges is not considered to be localized in any plane as, for instance, in [1–5, 8–10]. These papers are concerned with the two-dimensional triangular crystal structures formed by the electrons above liquid helium surface. The exceptional cases are those where the so-called "dimple" crystals are described. As is seen from the above problem this work describes a three-dimensional system of charges by the distribution function depending on the coordinates of the half-space above the liquid dielectric surface. So as we shall see further, the present work considers the spatially periodic structure along the directions parallel to the plane (x, y), caused by the spatial periodicity of the surface profile of the liquid dielectric film.

IV. DISTRIBUTION OF CHARGES AND FIELDS IN THE ELECTRICALLY NEUTRAL SYSTEM ABOVE A FLAT LIQUID DIELECTRIC SURFACE

Eqs. (21)–(24), can be solved using the procedure proposed in [14] (see also [13]), where a similar problem is considered for the case of the charge distribution above the flat boundary of a massive solid dielectric. To solve the first equation in Eq. (21) it is convenient to rewrite it as follows:

$$\frac{\partial^2 \bar{\varphi}_1(z)}{\partial z^2} = 4\pi e \nu \int\limits_0^\infty \frac{d\varepsilon \varepsilon^{1/2}}{1 + e^{(\varepsilon - \psi)/T}}, \qquad (31)$$

where we introduce the following designations

$$\psi(z) \equiv \mu + e\bar{\varphi}_1(z), \quad \nu = \frac{\sqrt{2}}{\pi^2 a_0^{3/2} e^3}, \qquad (32)$$

and take into account that the electron spin is $1/2$ and $a_0 \equiv \frac{\hbar^2}{me^2}$. The quantity ψ is usually named as electrochemical potential and a_0 as Bohr radius.

The order of Eq. (31) can be lowered, see [13]. As a result we obtain the equation

$$\frac{\partial \bar{\varphi}_1}{\partial z} = -\left\{ \frac{16\pi}{3} \nu \int\limits_0^\infty \frac{d\varepsilon \varepsilon^{3/2}}{1 + e^{(\varepsilon - \psi)/T}} + C_1 \right\}^{1/2}, \qquad (33)$$

where C_1 is an arbitrary integration constant. We choose the sign of the square root in this equation from the following considerations: the force acting on the negative charges at $z > \bar{\xi}$ must press these charges to the dielectric surface. Eq. (33) is greatly simplified, if we take into account the quasi-neutrality of the system. In fact, the condition of quasi-neutrality of the system means that there are no particles at infinity, so (see Eq. (21)) we have

$$\lim_{z \to \infty} \frac{1}{1 + e^{(\varepsilon - \psi)/T}} = 0,$$

and taking into account Eq. (33) leads us to

$$\lim_{z \to \infty} \frac{\partial \varphi_1(z)}{\partial z} = -\sqrt{C_1}.$$

In the electrically neutral case the electric field $E(z) = -\frac{\partial \varphi_1(z)}{\partial z}$ must be zero at infinity, so $C_1 = 0$ and after using the definitions from (32) Eq. (33) has the form:

$$\frac{\partial \psi}{\partial z} = -\left(\frac{16\pi e^2 \nu}{3} \int\limits_0^\infty \frac{d\varepsilon \varepsilon^{3/2}}{1 + e^{(\varepsilon - \psi)/T}} \right)^{1/2}, \qquad (34)$$

In [14] we show that in the most general case, the electron gas can be degenerate near the dielectric surface and it is non-degenerate at a considerable distance from it. The condition of electron gas degeneracy depends on temperature, particle density and the external clamping field. If these parameters satisfy the non-degeneracy condition of the electron gas near the dielectric surface, then this gas is non-degenerate in the entire space above the dielectric. In the present paper we consider exactly this situation. In this case, the distribution function of electrons must be close to the Boltzmann distribution function (see [15]):

$$\frac{1}{1 + e^{(\varepsilon_{\mathbf{p}} - \psi)/T}} \approx e^{(\varepsilon_{\mathbf{p}} - \psi)/T}, \qquad (35)$$

whereby the expression for the gas density (see Eq. (21)) takes the form (see [13, 14] [7, 8]):

$$n(z) \equiv \nu \int\limits_0^\infty \frac{d\varepsilon \varepsilon^{1/2}}{1 + e^{(\varepsilon - \psi(z))/T}} \approx \frac{\sqrt{\pi}}{2} \nu T^{\frac{3}{2}} e^{\frac{\psi}{T}}. \qquad (36)$$

The left side of Eq. (36) determines the density of the number of charges above the flat dielectric surface and it is obtained from Eq. (21) and from the requirement

$$\int d\mathbf{x} \sum_{\mathbf{p}} f_{\mathbf{p}}(\mathbf{x}) = \int d\boldsymbol{\rho} \int\limits_0^\infty dz \int \frac{d\mathbf{p} f_{\mathbf{p}}(z)}{(2\pi\hbar)^3}$$
$$= S \int\limits_0^\infty dz \, n(z) = N, \qquad (37)$$

where N is the total number of particles in the system, S is the area of flat dielectric surface. In fact, Eq. (37) is approximate, because a certain part of charges is associated with a spatially periodic structure of liquid dielectric surface. However, according to Eqs. (15)–(18) we think that the number of such particles is small comparing to the total number of charges above the surface of liquid dielectric film.

Using Eqs. (35), (36), we easily make sure that in the case of the non-degenerate electron gas above the dielectric surface, Eq. (34) has the analytical solution:

$$e^{\frac{\psi}{T}} = \frac{4T^2 B^{-1}}{\left(z - \bar{\xi} + C_2\right)^2}, \; B \equiv 4\pi^{\frac{3}{2}} T^{\frac{5}{2}} e^2 \nu. \qquad (38)$$

To obtain the integration constant C_2 in Eq. (38), we use Eq. (37). Substituting Eq. (36) in it and taking into account Eq. (38), we obtain the following expression:

$$n_s = \int\limits_\xi^\infty dz n(z) = \frac{T}{2\pi e^2 C_2}, \qquad (39)$$

where $n_s \equiv \frac{N}{S}$ is the number of volume charges per area unit of the flat dielectric surface. The solution of Eqs. (38), (39) makes it possible to obtain the dependence of the electric field $E_1(z) = -\frac{\partial \bar{\varphi}_1(z)}{\partial z}$ on the coordinate z above the liquid dielectric surface $z = \bar{\xi}$.

$$E_1(z) = \frac{4\pi e n_s}{1 + \frac{z - \bar{\xi}}{2z_0}}, \; z_0 = \frac{T}{4\pi e^2 n_s}, \qquad (40)$$

and the electron density $n(z)$:

$$n(z) = \frac{n_s}{2z_0} \frac{1}{\left(1 + \left(z - \bar{\xi}\right)/2z_0\right)^2}. \qquad (41)$$

This result is in agreement with the expression for the charge density obtained in [16].

Eq. (41) allows giving the physical meaning of z_0. To do this we build the dependence of quantity $\Delta(z) = \frac{1}{n_s} \int\limits_{\bar{\xi}}^z n(x) dx$ on the distance from the surface z. This

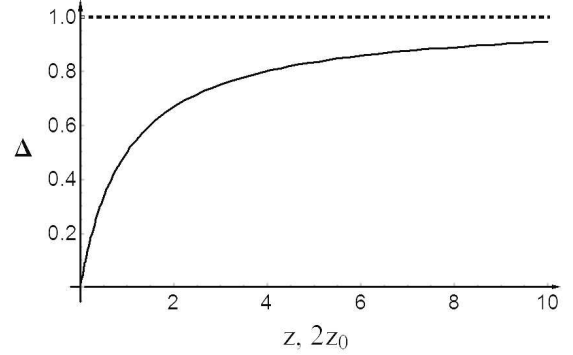quantity gives the part of charges below z from the total number of particles above the dielectric surface.

Fig. 2.

Eq. (41) shows that $\Delta\left(\bar{\xi} + 2z_0\right) = 0.5$. This value means that a half of all particles of the system is located below distance $2z_0$ from the surface. Noticing that $\Delta\left(\bar{\xi} + 20z_0\right) \approx 0.9$ and $\Delta\left(\bar{\xi} + 200z_0\right) \approx 0.99$, we conclude that at the distance $\bar{\xi} + 20z_0$ from the dielectric surface and moreover $\bar{\xi} + 200z_0$ the charges are nearly absent with the accuracy of 10 and 1 percent respectively. In the region of the electric field and temperature satisfying the non degeneracy condition Eq. (35) of the gas, the value of $2z_0$ is nearly 10^{-7} cm, i.e., 10 Å.

Eq. (40) shows that in the case of the electrically neutral system of charges in an external field E the number of particles n_s per area unit of the dielectric surface is determined by the value of this external clamping field. In fact, Eq. (40) shows that in the case $n_s = 0$, the electric field in region "1" is also equal to zero, $E_1(z) = 0$. $E_1(z)$ is a superposition of the external electric field E and the "internal" field $E_i(z)$ generated by the charges above the dielectric surface:

$$E_1(z) = E + E_i(z). \qquad (42)$$

But in the case of absence of charges, the electric field created by them is zero $E_i(z) = 0$, so

$$E_1(z) = E, \quad n_s = 0. \qquad (43)$$

Eq. (42) shows that the equality $E_1(z) = 0$ at $n_s = 0$ takes place only if $E = 0$. In other words, Eqs. (40), (42), (43) show that the number of particles per area unit of the dielectric surface is zero only when the external clamping field is absent. Thus, the clamping field in the electrically neutral case is the only factor holding the particles above the dielectric surface (see also [8–10]). So, we see that in this system the total field $E_i(z)$ created by all factors (field of the charges, the field of dielectric polarized by charges and the external field) minus the external field E

$$E_i(z) = \frac{4\pi e n_s}{1 + \left(z - \bar{\xi}\right)/2z_0} - E,$$

must be zero on the dielectric surface $E_i\left(\bar{\xi}\right) = 4\pi e n_s - E = 0$, otherwise, if the external field E is turned off, the charges will be still located above the dielectric surface. So, in this electro-neutral case

$$E = 4\pi e n_s. \qquad (44)$$

Let us note that Eq. (44) has exactly the same form as the field inside a flat capacitor produced by oppositely charged flat infinite plates having a surface charge density $\sigma = e n_s$. In this sense, Eq. (44) is in agreement with the corresponding expressions in [8–10] (see also [13,14]).

According to Eqs. (21), (23)–(25), (40), (44) the potentials of the total electric field and the external electric field in liquid dielectric film and solid substrate are given by the expressions:

$$\bar{\varphi}_1(z) = \varphi_0 - 2E z_0 \ln\left(1 + \left(z - \bar{\xi}\right)/2z_0\right),$$
$$\bar{\varphi}_2(z) = -\frac{E}{\varepsilon}\left(z - \bar{\xi}\right) + \varphi_0,$$
$$\bar{\varphi}_3(z) = -\frac{E}{\varepsilon_d}\left(z + d\right) + \frac{E}{\varepsilon}\left(d + \bar{\xi}\right) + \varphi_0,$$
$$\bar{\varphi}_1^{(e)}(z) = -E\left(z - \bar{\xi}\right) + \varphi_0^{(e)}, \qquad (45)$$
$$\bar{\varphi}_2^{(e)}(z) = -\frac{E}{\varepsilon}\left(z - \bar{\xi}\right) + \varphi_0^{(e)},$$
$$\bar{\varphi}_3^{(e)}(z) = -\frac{E}{\varepsilon_d}\left(z + d\right) + \frac{E}{\varepsilon}\left(d + \bar{\xi}\right) + \varphi_0^{(e)},$$

where the following notations are made $\bar{\varphi}_{1\xi} = \bar{\varphi}_{2\xi} \equiv \varphi_0$, $\bar{\varphi}_{1\xi}^{(e)} = \bar{\varphi}_{2\xi}^{(e)} \equiv \varphi_0^{(e)}$.

Basing on Eqs. (40), (44), (45) we define the subsidence of the surface in terms of the problem parameters, starting from Eq. (22) and using Eq. (35) we have:

$$\bar{\xi} = -\frac{E^2}{8\pi\alpha\kappa^2}. \qquad (46)$$

As it should be in the absence of charges according to Eq. (46), the value of $\bar{\xi}$ is zero. Eq. (46) allows imposing a natural restriction on the value of the clamping electric field, and therefore the maximum possible surface density of electrons. Indeed, in the case of equilibrium state of the system described by the self-consistent Eqs. (5)–(8), the subsidence depth of the liquid dielectric film surface must be substantially smaller in absolute value than the thickness of the film:

$$\left|\bar{\xi}\right| \ll d. \qquad (47)$$

This condition allows defining the maximum value clamping field, which is significantly higher than the values of the fields under our consideration

$$E \ll E^m, \quad E^m = 2\kappa\sqrt{2\pi\alpha d}. \qquad (48)$$

For example, for a liquid helium film having thickness $d = 0.1\,\mathrm{cm}$, we have $E^m \approx 5.4 \cdot 10^3\,\mathrm{V/cm}$, or in terms of the surface density of electrons (see Eq. (44)) $n_s^m \approx 2.85 \cdot 10^9\,\mathrm{cm}^{-2}$. This value is in good agreement with the value of $n_s^{cr} \approx 2 \cdot 10^9\,\mathrm{cm}^{-2}$, above which electrons start to fall into helium, forming the so-called bubbles [17]. In other words, the inequality Eq. (44) takes place, if $n_s \leq 10^9\,\mathrm{cm}^{-2}$.

In terms of the quantities introduced above the non-degeneracy criterion for the electron gas above the surface of liquid dielectric film has the following form (see Eqs. (35), (38)):

$$2^{3/2} n_s^2 a_0^4 \left(\frac{\pi e^2}{T a_0}\right)^{5/2} \ll 1. \qquad (49)$$

This inequality breaks in the case of low temperatures and strong external clamping fields (in this connection see [14]).

So, in the case of electrical neutrality, Eqs. (40)–(49) are the solution to the problem of obtaining the distribution of non-degenerate electron gas and fields in a system of charges above the flat surface of liquid dielectric in external clamping field. The next section is devoted to considering the possibility of formation of spatially-periodic states resulting from the phase transition in this system.

V. CRITICAL PARAMETERS OF THE PHASE TRANSITION IN THE SYSTEM TO A STATE WITH SPATIALLY PERIODIC SURFACE PROFILE OF THE LIQUID DIELECTRIC

The starting point for the study of the critical parameters of such phase transition are Eqs. (28) and the boundary conditions Eq. (29). Taking into account Eqs. (36) and (41) the first equation in Eq. (21) can be written as:

$$\frac{\partial^2 \tilde{\varphi}_1^{(1)}}{\partial z^2} = \left(q_0^2 + \frac{1}{2z_0^2}\left(1 + \frac{z - \bar{\xi}}{2z_0}\right)^{-2}\right)\tilde{\varphi}_1^{(1)}, \qquad (50)$$

where due to Eqs. (32), (36) we use the expression $\frac{\partial n}{\partial \mu} = \frac{\partial n}{\partial \psi} = \frac{n}{T}$. Making the following substitution

$$\tilde{\varphi}_1^{(1)}(z) = y^{\frac{1}{2}}\eta(y), \ y = 2q_0 z_0 + q_0\left(z - \bar{\xi}\right), \qquad (51)$$

Eq. (50) turns into the following form:

$$y^2 \eta'' + y \eta' - \eta\left(y^2 + \left(\frac{3}{2}\right)^2\right) = 0. \qquad (52)$$

This is a modified Bessel equation and its solution is given by the expression (see [18] [10])

$$\eta(y) = C_1 I_{\frac{3}{2}}(y) + C_2 K_{\frac{3}{2}}(y), \qquad (53)$$

where $I_\nu(y)$ and $K_\nu(y)$ are the modified Bessel functions of the first and second kind, respectively

$$I_\nu(y) = \sum_{k=0}^{\infty} \frac{\left(\frac{y}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)},$$
$$K_\nu(y) = \frac{\pi}{2}\frac{I_{-\nu}(y) - I_\nu(y)}{\sin \pi\nu}. \qquad (54)$$

If $\nu = 3/2$, they can be expressed in terms of elementary functions:

$$K_{\frac{3}{2}}(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \left(1 + \frac{1}{y}\right),$$
$$I_{\frac{3}{2}}(y) = \sqrt{\frac{2}{\pi y}} \left(chy - \frac{shy}{y}\right). \tag{55}$$

It is easily seen that the function $I_{\frac{3}{2}}(y)$ does not satisfy the limitation condition Eq. (11) for the electric field in region "1" at $z \to +\infty$. For this reason we put $C_1 = 0$ in Eq. (54) and the potential $\tilde{\varphi}_1^{(1)}(z)$ has the following expression:

$$\tilde{\varphi}_1^{(1)}(z) = C_2^{(1)} e^{-y(z)} \left(1 + \frac{1}{y(z)}\right). \tag{56}$$

In the general case the solutions of the second and the third equations in Eq. (27) are:

$$\tilde{\varphi}_2^{(1)}(z) = C_1^{(2)} e^{q_0 z} + C_2^{(2)} e^{-q_0 z},$$
$$\tilde{\varphi}_3^{(1)}(z) = C_1^{(3)} e^{q_0 z} + C_2^{(3)} e^{-q_0 z}. \tag{57}$$

Taking into account the finiteness of the electric fields at $z \to -\infty$ the constant $C_2^{(3)}$ in Eq. (57) must be set equal to zero, $C_2^{(3)} \equiv 0$. The constants $C_2^{(1)}$, $C_1^{(2)}$, $C_2^{(2)}$, $C_1^{(3)}$ are obtained from the boundary conditions Eq. (29). After doing a number of cumbersome but simple calculations, these constants are expressed through the value of $\tilde{\xi}^{(1)}$. According to Eqs. (18), (26) and (27) this value represents the Fourier transform of the profile of a spatially periodic surface of liquid dielectric film on a solid substrate:

$$\tilde{\varphi}_1^{(1)}(z) = E\tilde{\xi}^{(1)} e^{q_0(\bar{\xi}-z)}$$
$$\times \left(1 + \frac{1}{y_0 + q_0(z - \xi)}\right) G(q_0)$$
$$\tilde{\varphi}_2^{(1)}(z) = \tilde{\xi}^{(1)} \left(e^{q(z-\bar{\xi})} - C e^{q(\bar{\xi}-z)}\right) EF(q), \tag{58}$$
$$\tilde{\varphi}_3^{(1)}(z) = \tilde{\xi}^{(1)} e^{q(z-\bar{\xi})} EF(q)(1-\delta),$$

where for the purpose of simplicity the following notations are introduced:

$$F(q_0) = \left(1 + y_0 + (\varepsilon^{-1} - 1)(1 + y_0 + y_0^2)\right)$$
$$\times \frac{1}{(1-C)(1 + y_0 + y_0^2) + y_0 \varepsilon (1+C)(1+y_0)},$$
$$G(q_0) = y_0 \left(1 - C + (\varepsilon - 1)(C+1) y_0\right)$$
$$\times \frac{1}{(1-C)(1 + y_0 + y_0^2) + y_0 \varepsilon (1+C)(1+y_0)}, \tag{59}$$
$$C = \delta e^{-2q(d+\bar{\xi})}, \quad \delta \equiv \frac{\varepsilon_d - \varepsilon}{\varepsilon_d + \varepsilon},$$
$$y_0 \equiv y(z = \xi) = 2q_0 z_0.$$

The solution of Eq. (28) for the potentials $\tilde{\varphi}_j^{(e)(1)}$, $j = 1, 2, 3$ is given by

$$\tilde{\varphi}_1^{(e)}(z) = C_2^{(e1)} e^{-q_0 z},$$
$$\tilde{\varphi}_2^{(e)(1)}(z) = C_1^{(e2)} e^{q_0 z} + C_2^{(e2)} e^{-q_0 z}, \tag{60}$$
$$\tilde{\varphi}_3^{(e)(1)}(z) = C_1^{(e3)} e^{q_0 z}.$$

The form of these solutions is chosen to satisfy the condition of the external fields finiteness at $z \to \pm\infty$. The constants in Eq. (60) are obtained from the linear approximation of the corresponding boundary conditions Eq. (29). So, the potentials $\tilde{\varphi}_j^{(e)(1)}$ can be written as follows:

$$\tilde{\varphi}_1^{(e)(1)}(z) = (\varepsilon + 1) CE\tilde{\xi}^{(1)} F^{(e)}(q) e^{q(\bar{\xi}-z)},$$
$$\tilde{\varphi}_2^{(e)(1)}(z) = \tilde{\xi}^{(1)} EF^{(e)}(q_0)$$
$$\times \left(C e^{q_0(\bar{\xi}-z)} - e^{q_0(z-\bar{\xi})}\right), \tag{61}$$
$$\tilde{\varphi}_3^{(e)(1)}(z) = (1-\delta)\tilde{\xi}^{(1)} EF^{(e)}(q_0) e^{q_0(z-\bar{\xi})},$$

where

$$F^{(e)}(q) = \frac{(1 - \varepsilon^{-1})}{(\varepsilon(1+C) + 1 - C)}.$$

Taking into account Eq. (19), we substitute Eqs. (58), (61) into the last equation in Eq. (28) and obtain the following expression:

$$\Phi(q_0) \tilde{\xi}^{(1)} = 0, \tag{62}$$

where the function $\Phi(q_0)$ has the form:

$$\Phi(\mathbf{q}_0) \equiv \frac{E^2}{8\pi z_0} \left((1 + y_0^-) G(q_0) - 1\right.$$
$$\left. - y_0(1 + C)\left(F(q_0) + F^{(e)}(q_0)\right)\right) \tag{63}$$
$$+ \alpha(\kappa^2 + q_0^2 \beta), \quad \beta = \left(1 + \frac{\kappa^2 \bar{\xi}^2}{2}\right).$$

Eq. (62) has two solutions: $\tilde{\xi}^{(1)} = 0$ and $\Phi(q_0) = 0$. The first one is trivial and means the absence of the spatially periodic structure on the liquid dielectric surface. In other words, the liquid dielectric surface stays flat. The existence of a phase transition to the state with a spatially periodic surface profile of the liquid dielectric assumes that $\tilde{\xi}^{(1)} \neq 0$, (see Eqs. (15), (18), (19)). Thus, we must choose the second solution:

$$\Phi(q_0) = 0. \tag{64}$$

This equation determines only the norm of the vector $\mathbf{q}_0$ (see Eq. (59)) as a function of the physical parameters of the problem - the temperature T_c, the external clamping field E_c or the density of electron number n_{sc} (see Eq. (44)), the density of the dielectric ρ and its surface tension coefficient α, the dielectric constants of dielectric ε solid substrate ε_d. By fixing q_0 (for example, for the known values of the lattice period), we obtain a curve relating the regions of critical parameters T_c and E_c.

Let us analyze the obtained Eq. (64). Preliminary numeric estimates show that q_0 is actually independent on T_c. Indeed, Eq. (64) contains T_c by means of y_0 (see Eqs. (59), (40)). Further on it is shown that this value is small in the considered region of temperatures and clamping fields. This fact means that according to the above scenario of phase transition, if the clamping field has a fixed value, the described phase transition occurs at the temperatures lower than those that satisfy the condition of gas non-degeneracy (see Eq. (49)). Hence, the results obtained in the present paper are inapplicable for the description of the temperature phase transition under a fixed value of clamping field.

It should be noted that in the case of small periods of the described periodic structures in comparison with the thickness of liquid dielectric film and especially compared with the linear dimensions of the platform where this effect can be observed:

$$q_0 d \gg 1, \qquad (65)$$

we can put $C = 0$ (see Eq. (59)). In the region of E_c and T_c satisfying Eq. (49) (that is the region of charge gas non-degeneracy), and for low enough values of T_c for dielectric (He II in this case) to stay liquid, where the inequality takes place

$$q z_0 \ll 1, \qquad (66)$$

Eq. (64) tends to

$$\kappa^2 + q_0^2 \beta = \frac{q_0 E_c^2}{4\pi\alpha} \frac{\varepsilon + 3}{\varepsilon + 1}. \qquad (67)$$

It is seen that Eq. (67) does not depend on temperature, because of the condition Eq. (66). It should be noted that the approximations (65), (66) are made only to illustrate the obtained results, and the absence of temperature dependence is not their consequence. The numerical evaluation shows practically the absence of the temperature dependence of Eq. (64). This fact corresponds to the made assumption.

Eq. (67) has two roots:

$$q_0 = \frac{E_c^2}{8\pi\alpha\beta} \frac{\varepsilon + 3}{\varepsilon + 1}$$
$$\times \left(1 \pm \sqrt{1 - \beta \left(\frac{8\pi\alpha\kappa}{E_c^2} \left(\frac{\varepsilon + 1}{\varepsilon + 3} \right) \right)^2} \right). \qquad (68)$$

For the described states it is natural to choose the root with the sign "+", because it is logical to assume that with the increasing of the clamping field (but taking into account Eqs. (48), (49)), and hence the number of particles, the distance between them is reduced. Fig. 3 shows the "+" solution of Eq. (68) by a solid line and the numeric solution of Eq. (64) by dots.
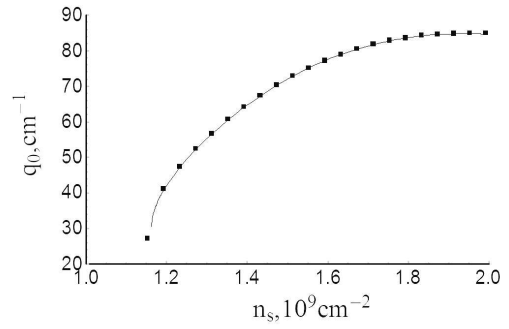

Fig. 3.

These solutions show that the considered periodic structures in the system of non degenerate gas of charges above liquid dielectric surface take place starting from the value of the clamping field corresponding to $n_s \approx 1.153 \cdot 10^9 \, \mathrm{cm}^{-2}$ (see Eq. (44)). According to Eq. (27) the lattice constant is equal to $a = 2\pi/q_0 \approx 0.23 \, \mathrm{cm}$ at this point. With further growth of n_s the corresponding lattice constant decreases tending to the value of $0.074 \, \mathrm{cm}$. Further growth of n_s practically holds the period unchanged. By reaching the critical value of the clamping field [Eq. (48)] the considered periodic structure collapses.

Let us also note the following circumstance. The first harmonic of the Fourier transform of gas density deviation from the equilibrium value provided by the periodic structure forming on the dielectric surface has the form

$$\tilde{n}^{(1)} \approx \frac{\partial n}{\partial \bar{\varphi}_1} \tilde{\varphi}_1^{(1)} + \frac{\partial n}{\partial \bar{\xi}} \tilde{\xi}^{(1)}. \qquad (69)$$

According to Eqs. (36), (58), (65), (66) we have

$$\tilde{n}^{(1)} \approx -2q_0 n\left(\bar{\xi}\right) \tilde{\xi}^{(1)}. \qquad (70)$$

As the deviations of density and surface profile have the opposite sign, we can conclude that increasing of density occurs above the deepening of dielectric surface profile or so-called "dimple". Correspondingly, the rarefaction of charges density occurs above the surface hump (see Fig. 4).
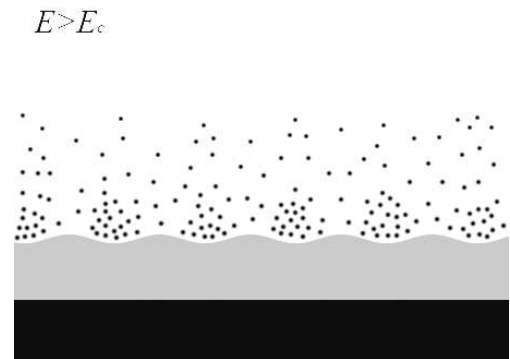
$E > E_c$

Fig. 4. Schematic structure of the asymmetric phase

Let us obtain the expression for the order parameter $\tilde{\xi}^{(1)}$ near the critical value of control parameter E_c. It is obvious from this section that the linear approximation of the considered perturbation theory is not sufficient for this purpose. Noticing that

$$\left(\tilde{\xi}\right)_{\mathbf{q}} = \tilde{\xi}_q,$$
$$\left(\tilde{\xi}^2\right)_{\mathbf{q}} = \sum_{q',q''} \tilde{\xi}_{q'}\tilde{\xi}_{q''}\Delta\left(q' + q'' - q\right), \tag{71}$$

$$\left(\tilde{\xi}^3\right)_{\mathbf{q}} = \sum_{q',q'',q'''} \tilde{\xi}_{q'}\tilde{\xi}_{q''}\tilde{\xi}_{q'''}\Delta\left(q' + q'' + q''' - q\right), \tag{72}$$

and that in the main nonvanishing approximation of the following expression takes place

$$\left(\tilde{\xi}^2\right)_{q_0} \approx 2\tilde{\xi}^{(1)}\tilde{\xi}^{(2)}, \quad \left(\tilde{\xi}^3\right)_{q_0} \approx 3\left(\tilde{\xi}^{(1)}\right)^3,$$
$$\left(\tilde{\xi}\tilde{\varphi}_j\right)_{q_0} \approx \tilde{\varphi}_j^{(1)}\tilde{\xi}^{(2)} + \tilde{\xi}^{(1)}\tilde{\varphi}_j^{(2)}, \tag{73}$$

we write Eq. (5) in the third order of the perturbation theory for small parameters $\tilde{\xi}(\boldsymbol{\rho})$, $\tilde{\varphi}_j(z,\boldsymbol{\rho})$, $\tilde{\varphi}_j^{(e)}(z,\boldsymbol{\rho})$, and $E - E_c$. Taking into account the above note about the absence of the temperature phase transition in non-degenerate gas of particles in further calculations we put $T = T_c$. According to Eqs. (18), (20), (26), (27), and the fourth equation in Eq. (28) we obtain the equation for the Fourier transforms $\tilde{\xi}_q$, $\tilde{\varphi}_{jq}(z)$ and $\tilde{\varphi}_{jq}^{(e)}(z)$ at $q = q_0$

$$\Bigg(ne\left(\left(\bar{\varphi}_1''\tilde{\xi}^{(1)} + \frac{\partial\tilde{\varphi}_1^{(1)}}{\partial z}\right)\tilde{\xi}^{(2)} + \frac{\partial\tilde{\varphi}_1^{(2)}}{\partial z}\tilde{\xi}^{(1)} + \frac{\bar{\varphi}_1'''}{2}\left(\tilde{\xi}^{(1)}\right)^3 + \frac{3}{2}\frac{\partial^2\tilde{\varphi}_1^{(1)}}{\partial z^2}\left(\tilde{\xi}^{(1)}\right)^2\right)$$

$$\frac{3e^2 n}{2T}\left(\tilde{\varphi}_1^{(1)} + \bar{\varphi}_1'\tilde{\xi}^{(1)}\right)\left(\tilde{\xi}^{(1)}\left(\bar{\varphi}_1''\tilde{\xi}^{(1)} + 2\frac{\partial\tilde{\varphi}_1^{(1)}}{\partial z}\right) + \frac{e}{T}\left(\tilde{\varphi}_1^{(1)} + \bar{\varphi}_1'\tilde{\xi}^{(1)}\right)^2\right)$$

$$+\frac{ne^2}{T}\left(\tilde{\varphi}_1^{(1)} + \bar{\varphi}_1'\tilde{\xi}^{(1)}\right)\left(\tilde{\varphi}_1^{(2)} + \bar{\varphi}_1'\tilde{\xi}^{(2)}\right) + e\left(\tilde{\varphi}_1^{(1)} + \bar{\varphi}_1'\tilde{\xi}^{(1)}\right)\frac{\partial n}{\partial E}\left(E - E_c\right)\Bigg)_{z=\bar{\xi}}$$

$$+\frac{\varepsilon}{4\pi}\left(\bar{\varphi}_2' q_0^2\left(\tilde{\varphi}_2^{(1)}\tilde{\xi}^{(2)} + 4\tilde{\varphi}_2^{(2)}\tilde{\xi}^{(1)}\right) + q_0^2\left(4\tilde{\varphi}_2^{(1)}\tilde{\varphi}_2^{(2)} + \frac{\partial\tilde{\varphi}_2^{(1)}}{\partial z}\tilde{\xi}^{(1)}\left(\frac{3\bar{\varphi}_2'}{2}\tilde{\xi}^{(1)} + 8\tilde{\varphi}_2^{(1)}\right)\right)\right) \tag{74}$$

$$+2\frac{\partial\tilde{\varphi}_2^{(1)}}{\partial z}\frac{\partial\tilde{\varphi}_2^{(2)}}{\partial z} - q_0^2\left(4\tilde{\varphi}_2^{(e)(1)}\tilde{\varphi}_2^{(e)(2)} + \frac{\partial\tilde{\varphi}_2^{(e)(1)}}{\partial z}\tilde{\xi}^{(1)}\left(\frac{3\bar{\varphi}_2^{(e)'}}{2}\tilde{\xi}^{(1)} + 8\tilde{\varphi}_2^{(e)(1)}\right)\right)$$

$$-\bar{\varphi}_2^{(e)'}q_0^2\left(4\tilde{\varphi}_2^{(e)(2)}\tilde{\xi}^{(1)} + \tilde{\varphi}_2^{(e)(1)}\tilde{\xi}^{(2)}\right) - 2\frac{\partial\tilde{\varphi}_2^{(e)(1)}}{\partial z}\frac{\partial\tilde{\varphi}_2^{(e)(2)}}{\partial z}\Bigg)_{z=\bar{\xi}}$$

$$-\kappa^2\alpha q_0^2\tilde{\xi}^{(1)}\left(\bar{\xi}\tilde{\xi}^{(2)} - \left(\tilde{\xi}^{(1)}\right)^2\left(4 + 3q_0^2\kappa^{-2}\left(1 - \frac{\kappa^2\bar{\xi}^2}{4}\right)\right)\right)\Bigg) = 0.$$

It is easily seen that obtaining $\tilde{\xi}^{(1)}$ requires writing the equations giving the relation between the second harmonics $\tilde{\varphi}_j^{(2)}$, $\tilde{\varphi}_j^{(e)(2)}$, $\tilde{\xi}^{(2)}$ and the first harmonic $\tilde{\xi}^{(1)}$. That is why we apply the analogous procedure to Eqs. (5), (8) at $q = 2q_0$:

$$\frac{\partial^2\tilde{\varphi}_1^{(2)}}{\partial z^2} - 4q_0^2\tilde{\varphi}_1^{(2)} = \frac{4\pi e^2 n}{T}\left(\tilde{\varphi}_1^{(2)} + \frac{e}{2T}\left(\tilde{\varphi}_1^{(1)}\right)^2\right), \quad \frac{\partial^2\tilde{\varphi}_2^{(2)}}{\partial z^2} - 4q_0^2\tilde{\varphi}_2^{(2)} = 0, \quad \frac{\partial^2\tilde{\varphi}_3^{(2)}}{\partial z^2} - 4q_0^2\tilde{\varphi}_3^{(2)} = 0,$$

$$\Bigg(en\left(\tilde{\varphi}_1^{(2)} + \bar{\varphi}_1'\tilde{\xi}^{(2)} + \left(\frac{\bar{\varphi}_1''}{2}\tilde{\xi}^{(1)} + \frac{\partial\tilde{\varphi}_1^{(1)}}{\partial z}\right)\tilde{\xi}^{(1)} + \frac{e}{2T}\left(\tilde{\varphi}_1^{(1)} + \bar{\varphi}_1'\tilde{\xi}^{(1)}\right)^2\right)\Bigg)_{z=\bar{\xi}}$$

$$+\frac{\varepsilon}{4\pi}\left(\bar{\varphi}_2'\left(\frac{\partial\tilde{\varphi}_2^{(2)}}{\partial z} + q_0^2\tilde{\varphi}_2^{(1)}\tilde{\xi}^{(1)}\right) - \bar{\varphi}_2^{(e)'}\left(\frac{\partial\tilde{\varphi}_2^{(e)(2)}}{\partial z} + q_0^2\tilde{\varphi}_2^{(e)(1)}\tilde{\xi}^{(1)}\right)\right.$$

$$\left.+\left(\frac{\partial\tilde{\varphi}_2^{(1)}}{\partial z}\right)^2 - \left(\frac{\partial\tilde{\varphi}_2^{(e)(1)}}{\partial z}\right)^2 + q_0^2\left(\left(\tilde{\varphi}_2^{(e)(1)}\right)^2 - \left(\tilde{\varphi}_2^{(1)}\right)^2\right)\right)_{z=\bar{\xi}} \tag{75}$$

$$+\alpha\left(\tilde{\xi}^{(2)}\left(\kappa^2 + 4q^2\left(1 + \frac{\kappa^2\bar{\xi}^2}{2}\right)\right)\right) + \bar{\xi}q^2\kappa^2\frac{5}{2}\left(\tilde{\xi}^{(1)}\right)^2\Bigg) = 0.$$

Taking into account Eq. (58) the general solution of the first equation in Eq. (75) is obtained by the method of arbitrary constants variation. Then after simple calculations we obtain

$$\tilde{\varphi}_1^{(2)}(z) = C_1 K_{\frac{3}{2}}(2y(z))$$
$$+ C_2 I_{\frac{3}{2}}(y)(2y(z)) + \frac{E}{z_0}\left(\tilde{\xi}^{(1)}\right)^2 G_1(y(z)), \qquad (76)$$

where the following notation is introduced (see also Eq. (59)):

$$G_1(x) = e^{2(y_0 - x)}\frac{(1+x)}{x^2}\frac{G(q_0)^2}{8}. \qquad (77)$$

The second and third equations in Eq. (75) have the following solutions:

$$\tilde{\varphi}_2^{(2)} = B_1 e^{2q_0(\bar{\xi}-z)} + B_2 e^{2q_0(z-\bar{\xi})},$$
$$\tilde{\varphi}_3^{(2)} = C_3 e^{2q_0(z-\bar{\xi})} + C_4 e^{2q_0(\bar{\xi}-z)}. \qquad (78)$$

Taking into account Eqs. (51), (76) and the constraints (14) we put $C_2 = 0$ and $C_4 = 0$ in the obtained solutions (76), (78).

To obtain the constants C_2, B_1, B_2, C_3 we place Eqs. (76), (78) into the Fourier transform of Eq. (12) at $q = 2q_0$. As a result we obtain

$$\tilde{\varphi}_1^{(2)}(z) = E e^{-2q_0(z-\bar{\xi})}\left(1 + \frac{1}{2y(z)}\right)\left(G(2q_0)\tilde{\xi}^{(2)} + z_0^{-1}\left(\tilde{\xi}^{(1)}\right)^2 G^{(2)}\right) + \frac{E}{z_0}\left(\tilde{\xi}^{(1)}\right)^2 G_1(y(z)),$$

$$\tilde{\varphi}_2^{(2)}(z) = E\left(e^{2q_0(z-\bar{\xi})} - \delta^{-1}C^2 e^{2q_0(\bar{\xi}-z)}\right)\left(F(2q_0)\tilde{\xi}^{(2)} + z_0^{-1}\left(\tilde{\xi}^{(1)}\right)^2 F^{(2)}\right), \qquad (79)$$

$$\tilde{\varphi}_3^{(2)}(z) = E e^{2q_0(z-\bar{\xi})}(1-\delta)\left(F(2q_0)\tilde{\xi}^{(2)} + z_0^{-1}\left(\tilde{\xi}^{(1)}\right)^2 F^{(2)}\right),$$

where the following notations are introduced

$$F^{(2)} = \frac{H_2\left(1 + 2y_0 + 4y_0^2\right) - H_1 2y_0\left(1 + 2y_0\right)}{\left(1 - C^2/\delta\right)\left(1 + 2y_0 + 4y_0^2\right) + 2y_0\varepsilon\left(1 + C^2/\delta\right)\left(1 + 2y_0\right)},$$

$$G^{(2)} = -\frac{4y_0^2\left(H_1\left(1 - C^2/\delta\right) + H_2\varepsilon\left(1 + C^2/\delta\right)\right)}{\left(1 - C^2/\delta\right)\left(1 + 2y_0 + 4y_0^2\right) + 2y_0\varepsilon\left(1 + C^2/\delta\right)\left(1 + 2y_0\right)},$$

$$H_1 = \frac{1}{4y_0}\left(1 - 2y_0 G_1'(y_0) - 2y_0^2\left(\left(1 + y_0^{-1}\right)\left(1 + y_0^{-2}\right)G(q_0) - \varepsilon(1-C)F(q_0)\right)\right), \qquad (80)$$

$$H_2 = \frac{1}{4}\left(1 + 4G_1(y_0) - 2y_0\left(\left(1 + y_0^{-1} + y_0^{-2}\right)G(q_0) + (1+C)F(q_0)\right)\right).$$

To obtain the values of $\tilde{\varphi}_j^{(e)(2)}$ we write the set of equations (28) and boundary conditions (30) for the Fourier transforms of the external field potential at $q = 2q_0$ and taking into account that set (75) contains only $\tilde{\varphi}_2^{(e)(2)}$ from all of the external potentials $\tilde{\varphi}_j^{(e)(2)}$, we present only the solution for this value

$$\tilde{\varphi}_2^{(e)(2)} = E\left(\frac{C^2}{\delta}e^{2q(\bar{\xi}-z)} - e^{2q(z-\bar{\xi})}\right)$$

$$\times\left(F^{(e)}(2q)\tilde{\xi}^{(2)} + F_2^{(e)}\left(\tilde{\xi}^{(1)}\right)^2 z_0^{-1}\right),$$
$$F_2^{(e)} = F^{(e)}(q)F^{(e)}(2q)\frac{y_0}{2}\frac{1 + C + \varepsilon(1-C)}{1 - \varepsilon^{-1}}. \qquad (81)$$

Substituting Eqs. (58), (61), (79), (81) into the last equation from Eq. (75) and making the approximations

Eqs. (65), (66), we have:

$$\tilde{\xi}^{(2)} = \frac{\gamma}{z_0}\left(\tilde{\xi}^{(1)}\right)^2,$$
$$\gamma = \frac{\bar{\xi}}{8z_0\left(1 + 2q^2\left(2\kappa^{-2} + \bar{\xi}^2\right)\right)}. \qquad (82)$$

Eq. (82) shows that all the Fourier transforms of the second harmonics $\tilde{\varphi}_j^{(2)}$, $\tilde{\varphi}_j^{(e)(2)}$ and $\tilde{\xi}^{(2)}$ are proportional to the squared Fourier transform of the first harmonic $\tilde{\xi}^{(1)}$, which is the order parameter of the considered problem. To obtain its dependence on the control parameters of the phase transition - temperature T and the external clamping field E we substitute Eqs. (58), (61), (79), (81) and (82) into Eq. (74). After some lengthy calculations the following expression is obtained

$$\tilde{\xi}^{(1)}\Psi\left(\tilde{\xi}^{(1)}, T, E\right) = 0, \qquad (83)$$

where after taking into account approximations

Eqs. (65), (66) the following notations are introduced

$$\Psi\left(\tilde{\xi}^{(1)}, T, E\right) = E_c e y_0 \frac{\partial n_{\bar{\xi}}}{\partial E}\left(E - E_c\right)$$

$$-E_c^2\left(\tilde{\xi}^{(1)}\right)^2\left(32\pi z_{0c}^3\right)^{-1} U\left(T_c, E_c, q_0\right),$$

$$U\left(T_c, E_c, q_0\right) = (1 - 2\gamma) \tag{84}$$

$$+y_{0c}^2\left(\gamma + \frac{z_{0c}}{|\bar{\xi}|}\left(4 + 3q_0^2\left(\frac{1}{\kappa^2} - \frac{\bar{\xi}^2}{4}\right)\right)\right).$$

Eq. (83) has two solutions. The first solution $\tilde{\xi}^{(1)} = 0$ is trivial, because in this case there is no phase transition. In the case of non-zero deformation of liquid dielectric surface profile, formed as a result of the phase transition, we have the following solution of Eq. (83)

$$\Psi\left(\tilde{\xi}^{(1)}, T, E\right) = 0 \tag{85}$$

giving the relation between the first harmonic of the Fourier transform of the dielectric surface profile perturbations and the parameter of the phase transition E. Taking into account Eq. (84), (40), (41) we can rewrite Eq. (85) in a simpler form

$$\tilde{\xi}^{(1)} = 4z_{0c}\left(\frac{q_0 z_{0c}}{U\left(T_c, E_c, q_0\right)}\right)^{1/2}\sqrt{\frac{E}{E_c} - 1}. \tag{86}$$

The last expression determines the periodic structure of liquid dielectric near E_c and it is obtained assuming Eqs. (65), (66) taking place, or

$$d^{-1} \ll q_0 \ll z_0^{-1}. \tag{87}$$

Eqs. (86) and (82) also contain the parameters κ and $\bar{\xi}^{-1}$, having the dimension of inverse length. In order to estimate them, we use Eqs. (7), (46) and the concrete values of clamping field E and temperature T.

Let us consider the macroscopic case, when the periods of arising periodic structures a are comparable with the thickness d of liquid dielectric (see e.g. [11])

$$d > a. \tag{88}$$

In this case, the formulas obtained in the approximation Eq. (65) are still applicable, because the condition of Eq. (89) allows putting $C = 0$ with high accuracy. Considering the liquid dielectric to be helium having thickness $d = 1\,\mathrm{cm}$ and external clamping field capable to hold up to $n_s = 1.4 \cdot 10^9\,\mathrm{cm}^{-2}$ of electrons above the helium surface unit, we have the following relation between these parameters

$$d^{-1} < q_0, \kappa, \bar{\xi}^{-1} \ll z_0^{-1}. \tag{89}$$

The value of z_0 is calculated for the temperature region, where the non-degeneracy condition Eq. (49) is satisfied. Using the Eq. (89) we simplify Eq. (86)

$$\tilde{\xi}^{(1)} = \chi\sqrt{\frac{E}{E_c} - 1},$$

$$\chi = 8z_{0c}^2\left(\frac{q_0}{|\bar{\xi}|}\left(1 + 2q_0^2\left(\frac{2}{\kappa^2} + \bar{\xi}^2\right)\right)\right)^{1/2}. \tag{90}$$

In the framework of the developed perturbation theory Eqs. (79), (82), (86), (90) allow obtaining the expressions for the amplitudes of the first and the second harmonics of all electrostatic potentials of the system. However, we do not give the expressions for the second harmonics because of their bulky form and low informativity in the context of this paper tasks.

VI. CONCLUSION

Thus, in this paper we propose a theory of equilibrium states of a system of identical charges above the ideal liquid dielectric surface in external constant clamping electric field. The system is assumed quasi-neutral, that is the number of charges generating their own electric field above the dielectric surface is exactly as large as it is necessary to compensate an external electric field far from the surface. In other words, the number of trapped charges above the liquid dielectric surface depends only on the external attractive field. It is shown that in this case, that the compensatory electric field generated by charges is equivalent to the neutralizing field produced inside a flat capacitor produced by oppositely charged flat infinite plates having a surface charge density $\sigma = en_s$ (n_s is the number of charges above the area unit of the dielectric surface). In terms of the built theory we obtain the system of self-consistent equations for the parameters describing the system — the potential of the electrostatic field, the distribution function of charges above the dielectric surface and the surface profile of the dielectric. The proposed theory is applied for the description of phase transitions in such the system to the states with spatially-periodic structures in the case when the gas of charges is far from degeneracy state. By solving self-consistent equations near the transition point we obtain the equation of the critical curve separating the symmetric and asymmetric phases. In this case this the equation giving the relation between the temperature of the system and the electrostatic clamping field. For the particular cases we obtain the equations for the reciprocal lattice vectors of spatially periodic structures as a function of temperature and external clamping field. In some special cases we define the reciprocal lattice vectors of the spatially-periodic structures as functions of only the external clamping field due to their weak dependence on temperature.

It should be reminded again that in this paper we consider the system of charges not localized in any plane, as do the authors of, e.g., [5, 8–10], where the electron crystals are considered as a two-dimensional triangular lattice. Naturally, except for the works considering the dimple crystals. As in this paper we consider a three-dimensional system of charges characterized by the distribution function depending on the coordinates of the half-space above liquid dielectric surface, so the two-dimensional crystal structures have automatically no possibility to occur. In this sense the spatially periodic structures described in the previous sections characterizing the surface profile as well as the charge and field

distribution are more similar to the characteristics of the structures in systems with dimple crystals researched in the experiment. In this connection there arises the question: are the experimentally observed spatially periodic structures two-dimensional in fact?

The answer to this question may be as follows. In this paper we describe the phase transition in the studied system to a state with a spatially periodic structures near the phase curve in the small order parameter of the perturbation theory. The equation for the reciprocal lattice vector is obtained from the linear approximation in the small order parameter $\tilde{\xi}(\mathbf{q}_0)$, see Eqs. (10), (11), (19). According to Eqs. (58), (61), this value determines the other amplitudes of the described periodic structure. According to Eq. (90) the value of $\tilde{\xi}(\mathbf{q}_0)$ tends to zero proportionally to the square root $\sqrt{E - E_c}$ when the values of clamping field are close to E_c. In other words, if the lattice constant of the periodic structures is large, then the inhomogeneities of the surface profile of liquid dielectric can be "invisible". Perhaps, that is why the spatially periodic structures observed in the experiments are associated with the formation of two-dimensional crystals formed by electrons above the liquid helium surface.

In conclusion, we consider the agreement between of the results obtained in this paper and the available experimental data. At first glance, the experimental data in [5] is obtained under the conditions adequate for our reviews. This paper describes the experiment of a phase transition from the electron system to an electronic two-dimensional crystal on the surface of liquid helium under the specific values of the control parameters - temperature and clamping electric field. The measurements are made in the case when the field produced by electrons is compensated by the external clamping field of the capacitor in accordance with the ratio of Eq. (44) type.

So, the field on the upper capacitor plate is zero (the bottom plate is immersed below the helium surface at $d = 0.1$ cm depth). Under such conditions the electron gas is considered to obey the Boltzmann distribution. This assumption is made by the authors basing on the estimates of the Fermi energy of two-dimensional electron gas, according to the well-known expression:

$$\varepsilon_F = \pi m^{-1} \hbar^2 n_s \approx 0.03 \, \text{K}.$$

The Fermi energy is one order less than the temperature of the system in the experiment, $T = 0.42$ K. This fact seems to allow a correct comparison between the theoretical and experimental results. However, these results are substantially different. Moreover, under the conditions of the experiments [5] inequality (49) breaks. In this paper it is formulated as evidence of non-degenerate of the gas of charges above the dielectric surface. But this fact contradicts the above condition of paper [5].

However, such a disagreement of the estimates is, apparently, caused by the way of considering the system of charges above the liquid dielectric surface — as a two-dimensional or as a three-dimensional one. In fact, in the three-dimensional case, the criterion of electron gas degeneracy can be the value of ratio between the temperature of the system and the Fermi energy. However, in the three-dimensional case the calculation of the Fermi energy is a more complex problem than in the two-dimensional case. It requires solving the problem on obtaining the distribution of charges and fields in the system of a completely degenerate electron gas. We are working on this problem now, and in some cases it can be solved even analytically. The non-degeneracy condition Eq. (49) is obtained in the high temperature approach, and therefore it can be obtained in terms of the used approximations and serves as a criterion of their validity.

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YU. V. SLYUSARENKO, D. M. LYTVYNENKO

ТЕОРІЯ ПРОСТОРОВО-ПЕРІОДИЧНИХ РІВНОВАЖНИХ СТАНІВ У КВАЗІНЕЙТРАЛЬНІЙ СИСТЕМІ ЗАРЯДІВ НАД ПОВЕРХНЕЮ РІДКОГО ДІЕЛЕКТРИКА

Ю. В. Слюсаренко, Д. М. Литвиненко

Національний науковий центр Харківський фізико-технічний інститут,
Інститут теоретичної фізики ім. А.І. Ахієзера, вул. Академічна, 1, Харків, 61108, Україна
Харківський національний університет ім. В.Н.Каразіна, Інститут високих технологій
пр. Курчатова, 31, Харків, 61108, Україна, e-mail: d_m_litvinenko@kipt.kharkov.ua

Запропоновано мікроскопічну теорію квазінейтральних рівноважних станів зарядів над поверхнею рідкого діелектрика на основі уявлень про систему багатьох тотожних частинок. В основу підходу покладено варіаційний принцип, модифікований для розглянутих систем, і модель Томаса–Фермі. У межах розвиненої теорії виписано рівняння самоузгодження, що зв'язують параметри опису такої системи — потенціал статичного електричного поля, функцію розподілу зарядів і профіль поверхні рідкого діелектрика. Рівняння використано для дослідження фазових перетворень системи до просторово-періодичних станів. Визначено параметри фазового перетворення системи до просторово-періодичних станів типу лункових кристалів.