JOURNAL OF PHYSICAL STUDIES
v. 26, No. 2 (2022) 2001(20 p.)

RELATIVISTIC SYMMETRIES OF THE DEFORMED DIRAC EQUATION THROUGH THE IMPROVED HULTHÉN PLUS A CLASS OF YUKAWA POTENTIAL INCLUDING A COULOMB-LIKE TENSOR INTERACTION IN DEFORMATION QUANTUM MECHANICS

Abdelmadjid Maireche
Laboratory of Physics and Material Chemistry, Physics department,
Sciences Faculty, University of M'sila,
BP 239 Chebilia-M'sila, Algeria,
e-mail: abdelmadjid.maireche@univ-msila.dz

(Received 20 December 2021; in final form 24 May 2022; accepted 24 May 2022; published online 24 June 2022)

We approximately solve the deformed Dirac equation for a new improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction (IHCYPCTI for short) in the context of extended relativistic quantum mechanics ERQM symmetries including an improved Coulomb-like tensor interaction with an arbitrary spin-orbit coupling quantum number k. Within the framework of the spin and pseudospin (p-spin) symmetry, we obtain the global new energy eigenvalue which equals the energy eigenvalue in usual relativistic QM plus the corrected energy induced by three infinitesimal additive parts of the Hamiltonian corresponding to the spin-orbit interaction, the new modified Zeeman and the rotational Fermi term by using the parametric of Bopp's shift method and standard perturbation theory with an approximation to the centrifugal term. The new values that we got appeared sensitive to the quantum numbers $(j, k, l, \widetilde{l}, s, \widetilde{s}, m, \widetilde{m})$, the mixed potential depths (A, B), the range of the potential δ, and noncommutativity parameters (Θ, σ, χ). The mixed potential which in some particular cases gives solutions of different potentials: the improved Hulthén, the improved Yukawa potential and the improved Coulomb-like problem along with their bound state energies are obtained.

Key words: Dirac equation, Hulthén plus a class of Yukawa potential, noncommutative geometry, Bopp's shift method and star products.

DOI: https://doi.org/10.30970/jps.26.2001

I. INTRODUCTION

The relativistic Dirac equation DE is an effective tool that researchers use to study various high-energy physical systems that have spin-1/2 especially in nuclear, atomic physics, and hadronic physics. This equation plays a significant role in various fields of physics. It expands to include very important physical symmetries under various potentials. The solutions of these potentials in 3D and multidimensional space have attracted attention since the beginning of the study of relativistic quantum mechanics RQM. The Hulthén and inversely quadratic Yukawa potentials are the most important potentials in the field of strong quantum physics. The first potential is considered a short-range potential, extensively used to study the bound states of the interaction systems such as atomic, nuclear and particle physics, while the second potential studies the strong interactions between nucleons. We have noticed many works that deal with the study of the two potentials in relativistic and nonrelativistic regimes. Hamzavi and Rajabi obtained the approximate analytical bound states of the DE for scalar-vector-tensor Hulthén potentials in the presence of spin and pseudospin (p-spin) symmetries (s(p)pin-sy) with any arbitrary spin-orbit coupling number ASOCN-k, using the Pekeris approximation and the generalized parametric Nikiforov-Uvarov method (NUM), and obtained both energy eigenvalues and corresponding wave functions in their closed forms [1]. In 2013, Aydoğdu *et al.* obtained solutions of the relativistic DE for the scalar and vector Hulthén potentials with the Yukawa-type tensor potential using the NUM for s(p)pin-sy with the Yukawa-type tensor potential for an ASOCN-k, and they deduced the energy eigenvalue equations and corresponding upper and lower spinor wave functions in both s(p)pin-sy cases [2]. Ikhdair *et al.* solved the DE approximately for the attractive scalar and repulsive vector Hulthén potentials including a Coulomb-like tensor potential with ASOCN-j and obtained the analytic energy spectrum and the corresponding two-component upper and lower spinors of the two Dirac particles through the NUM within the framework of the s(p)pin-sy concept [3]. Ikhdair and Falaye obtained the approximate relativistic bound state of a spin-1/2 particle in the field of the Yukawa potential and a Coulomb-like tensor interaction with ASOCN k under the s(p)pin-sy using the asymptotic iteration method and obtained energy eigenvalues and the corresponding wave functions [4]. Ikhdair and Hamzavi solved approximately the DE for a generalized inversely quadratic Yukawa potential IQYP including a Coulomb-like tensor interaction with ASOCN-k within the framework of the s(p)pin-sy and obtained the energy eigenvalue equation and the corresponding eigenfunctions, in a closed form with the help of the parametric NUM [5]. Ikot *et al.* solved the DE for the energy-dependent Yukawa potential including a tensor interaction term within the framework of the s(p)pin-sy limits with ASOCN-k and obtained explicitly the energy eigenvalues and the corresponding wave function using the NUM [6].

Recently, the phenomenon of combining different potentials has appeared, as it has become a source of interesting research inspiration for researchers. The real goal is to search for more applications in various physical and chemical fields alike. We will confine ourselves to mentioning three models of paramount importance. Magu *et al.* solved the DE for Manning–Rosen plus a class of Yukawa potential including a Coulomb-like tensor potential with ASOCN-k within the framework of the spin and p-spin symmetry and obtained the energy eigenvalue equation and the corresponding eigenfunctions in a closed form by using the NUM [7]. Ikot *et al.* presented the DE for the Mobius square-Yukawa potentials including the tensor interaction term within the framework of s(p)pin-sy limit with ASOCN-k and obtained the energy eigenvalues and the corresponding wave functions using the supersymmetry method [8]. Ahmadov *et al.* examined the bound state solutions of the DE under the s(p)pin-sy for Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction, they employed an improved scheme to deal with the centrifugal (pseudo centrifugal) term and obtained the relativistic energy eigenvalues and associated Dirac spinor components of wave functions using the NUM and SUSYQM methods [9]. In our current research, we will focus on the latter to shed more light on it. We will process it in a space that includes wider symmetries in order to discover more applications and powers. This space is known to researchers as noncommutative quantum mechanics or extended quantum mechanics EQM and deformation quantum mechanics DQM. Why do we resort to the option of conducting the study in this expanded space? Considering the successes achieved by adopting the principles of quantum mechanics known in the literature, there are still many problems without solutions such as the quantum gravity, string theory and the divergence problem of the standard model. There are promising indications that it is possible to find solutions to these problems [10–19]. In addition to the well-known postulate $[\widehat{x}_\mu \overset{*}{,} \widehat{p}_\nu] \neq 0$, physicists have extended the symmetries of quantum mechanics QM to include more new postulates such as $[\widehat{x}_\mu \overset{*}{,} \widehat{x}_\nu] \neq 0$ and $[\widehat{p}_\mu \overset{*}{,} \widehat{p}_\nu] \neq 0$. The idea of DQM is old and dates to the early years of QM, where the idea was proposed by Snyder [20] in 1947 and its geometric analysis was introduced by Connes in 1991 and 1994 [21, 22]. Seiberg and Witten extended earlier ideas about the appearance of NC geometry in string theory with a nonzero B-field and obtained a new version of gauge fields on noncommutative gauge theory[23]. In the context of some deformed canonical commutation relations leading to isotropic nonzero minimal uncertainties in the position coordinates, Quesne and Tkachuk (2005) solved exactly the DE for the Dirac oscillator using supersymmetric quantum mechanical and shape-invariance methods to derive both the energy spectrum and wavefunctions in the momentum representation [24]. In the next year, the same authors generalized the D-dimensional (β, β')-two-parameter deformed algebra with minimal length, which was introduced by Kempf to a Lorentz-covariant algebra in a $(D+1)$-dimensional quantized spacetime that reproduces the Snyder algebra case for $D = 3$ and $\beta = 0$ [25]. There were multiple contributions to the study of the Hulthén potential or Yukawa potential singly or combined with another potential in relativistic and nonrelativistic regimes in the symmetries of deformation quantum mechanics [26–31], but as regards the combination of Hulthén with IQYP in the symmetries of deformed Dirac theory DDT, no researcher has yet addressed it, to the best of my knowledge. I hope that this study will help to discover more investigations in the sub-atomic scales and gain more scientific knowledge of elementary particles in the field of Nano-scales. The research reported in the present paper was motivated by the fact that the study of the improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction (IHCYPCTI, in short) in the DDT symmetries has not been reported in the available literature. In this work, the vector and scalar IHCYPCTI model ($V_{\text{hcy}}(\widehat{r})$, $S_{\text{hcy}}(\widehat{r})$) to be employed is defined as:

$$V_{\text{hcy}}(\widehat{r}) = V_{\text{hcy}}(r) - \frac{1}{2r}\frac{V_{\text{hcy}}(r)}{\partial r}\begin{cases}\mathbf{L}\Theta + O\left(\Theta^2\right) = V_{\text{ts}}^{\text{s}}(r) \text{ for spin-sy,}\\ \widetilde{\mathbf{L}}\Theta + O\left(\Theta^2\right) = V_{\text{hcy}}^{\text{p}}(r) \text{ for pspin-sy,}\end{cases} \tag{1.1}$$

and

$$S_{\text{hcy}}(\widehat{r}) = S_{\text{hcy}}(r) - \frac{1}{2r}\frac{S_{\text{hcy}}(r)}{\partial r}\begin{cases}\mathbf{L}\Theta + O\left(\Theta^2\right) = S_{\text{hcy}}^{\text{s}}(r) \text{ for spin-sy,}\\ \widetilde{\mathbf{L}}\Theta + O\left(\Theta^2\right) = S_{\text{ts}}^{\text{p}}(r) \text{ for pspin-sy,}\end{cases} \tag{1.2}$$

where $(V_{\text{hcy}}(r), S_{\text{hcy}}(r))$ are the vector and scalar potentials [9] according to the view of RQM known in the literature:

$$\begin{cases} V_{\rm hcy}\left(r\right) = -\dfrac{Ze^{2}\delta\exp\left(-\delta r\right)}{1-\exp\left(-\delta r\right)} - \dfrac{Ae^{-\delta r}}{r} - \dfrac{Be^{-2\delta r}}{r^{2}}, \\ S_{\rm hcy}\left(r\right) = -\dfrac{Z_{\rm s}e^{2}\delta\exp\left(-\delta r\right)}{1-\exp\left(-\delta r\right)} - \dfrac{A_{\rm s}e^{-\delta r}}{r} - \dfrac{B_{\rm s}e^{-2\delta r}}{r^{2}}, \end{cases} \tag{2}$$

where $A/A_{\rm s}$ and $B/B_{\rm s}$ are the depths of the studied potential, δ is the screening parameter, ($\widehat{r}$ and r) is the distance between the two particles in deformation of the Dirac theory symmetries and QM symmetries, respectively. The two couplings $\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\Theta$ are the scalar product of the usual components of the angular momentum operators ($\mathbf{L}\left(L_x, L_y, L_z\right)$ or $\widetilde{\mathbf{L}}\left(\widetilde{L}_x, \widetilde{L}_y, \widetilde{L}_z\right)$) and the modified noncommutativity vector $\Theta\left(\theta_{12}, \theta_{23}, \theta_{13}\right)/2$ which presents the noncommutativity elements parameter. The modified algebraic structure of covariant canonical commutation relations MASCCCRs, Lie structure and quantum plane in the DDT in the representations of Schrödinger, Heisenberg, and interactions pictures, as follows (we have used the natural units $\hbar = c = 1$) [32–42]:

$$\left[x_{\mu}^{\rm (S,H,I)},\ p_{\nu}^{\rm (S,H,I)}\right] = i\hbar\delta_{\mu\nu} \Longrightarrow \left[\widehat{x}_{\mu}^{\rm (S,H,I)} \overset{*}{,}\ \widehat{p}_{\nu}^{\rm (S,H,I)}\right] = i\hbar_{\rm eff}\delta_{\mu\nu} \tag{3}$$

and

$$\left[x_{\mu}^{\rm (S,H,I)}, x_{\nu}^{\rm (S,H,I)}\right] = 0 \Longrightarrow \left[\widehat{x}_{\mu}^{\rm (S,H,I)} \overset{*}{,} \widehat{x}_{\nu}^{\rm (S,H,I)}\right] = \begin{cases} i\theta_{\mu\nu} \text{ with } \theta_{\mu\nu}\in IC, \text{ Canonical\_structure}, \\ if_{\mu\nu}^{\alpha}\widehat{x}_{\alpha}^{\rm (S,H,I)} \text{ with } f_{\mu\nu}^{\alpha}\in IC, \text{ Lie\_structure}, \\ iC_{\mu\nu}^{\alpha\beta}\widehat{x}_{\alpha}^{\rm (S,H,I)}\widehat{x}_{\beta}^{\rm (S,H,I)} \text{with } C_{\mu\nu}^{\alpha\beta}\in IC, \text{ Quantum\_plane} \end{cases} \tag{4}$$

with $\widehat{x}_{\mu}^{\rm (S,H,I)} = \left(\widehat{x}_{\mu}^{\rm S}, \widehat{x}_{\mu}^{\rm H}, \widehat{x}_{nc\mu}^{\rm I}\right)$ and $\widehat{p}_{\mu}^{\rm (S,H,I)} = \left(\widehat{p}_{\mu}^{\rm S}, \widehat{p}_{\mu}^{\rm H}, \widehat{p}_{\mu}^{\rm I}\right)$ are the generalized coordinates and the corresponding generalizing coordinates in the DDT symmetries while IC denotes the complex number field. The uncertsainty relations will be changed to the following formula in the new symmetries as follows:

$$\left|\Delta x_{\mu}^{\rm (S,H,I)}\Delta p_{\nu}^{\rm (S,H,I)}\right| \geqslant \hbar\delta_{\mu\nu}/2$$
$$\Longrightarrow \begin{cases} \left|\Delta\widehat{x}_{\mu}^{\rm (S,H,I)}\Delta\widehat{p}_{\nu}^{\rm (S,H,I)}\right| \geqslant \hbar_{\rm eff}\delta_{\mu\nu}/2, \\ \left|\Delta\widehat{x}_{\mu}^{\rm (S,H,I)}\Delta\widehat{p}_{\nu}^{\rm (S,H,I)}\right| \geqslant \left|\theta_{\mu\nu}\right|/2. \end{cases} \tag{5}$$

It is important to note that Eqs. (3) and (4) are covariant equations (the same behavior of $\widehat{x}_{\mu}^{\rm (S,H,I)}$) under the Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame. We have extended the MASCCCRs to include the Heisenberg and interaction pictures in DDT. Here $\hbar_{\rm eff} \cong \hbar$ is the effective Planck constant, $\theta_{\mu\nu} = \epsilon_{\mu\nu}\theta$ (θ is the non-commutative parameter, and $\epsilon_{\mu\nu}$is just an antisymmetric number, for example $\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = -\epsilon_{32} = -\epsilon_{21} = -\epsilon_{31} = 1$, $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0$) which is an infinitesimal parameter if compared to the energy values and elements of antisymmetric (3×3) real matrices, and $\delta_{\mu\nu}$ is the Kronecker symbol. The symbol $*$ denotes the Weyl–Moyal star product, which is generalized between two ordinary functions $h(x)g(x)$ to the new deformed form $\hat{h}(\widehat{x})\hat{g}(\widehat{x})$ which is expressed with the Weyl–Moyal star product $h(x)*g(x)$ in the symmetries of deformation of the Dirac theory symmetries as follows [43–53]:

$$h(x)*g(x) = \begin{cases} \exp\left(i\epsilon^{\mu\nu}\theta\partial_{\mu}^{x}\partial_{\nu}^{x}\right)(hg)(x), \\ \text{canonical\_structure}, \\ \exp\left(\frac{i}{2}x_{nc\mu}^{\rm (S,H,I)}g_{k}\left(i\partial_{\mu}^{x}, i\partial_{\nu}^{x}\right)\right)(hg)(x), \\ \text{Lie\_structure}, \\ iq^{G\left(u,v,\partial_{\mu}^{u},\partial_{\nu}^{v}\right)}h\left(u,v\right)g\left(u',v'\right)\rfloor_{u'\to u}^{v'\to v}, \\ \text{quantum\_plane} \end{cases} \tag{6}$$

with

$$g_{\alpha}\left(k,p\right) = -k_{\mu}p_{\nu}f_{k}^{\nu\nu} + \frac{1}{6}k_{\mu}p_{\nu}\left(p_{\alpha}-k_{\alpha}\right)f_{l}^{\nu\nu}f_{m}^{l\alpha} + \dots\ .$$

In the current paper, we apply the modified algebraic structure of covariant canonical commutation relations MASCCCRs in the DDT, which allows us to rewrite to the following simple form at the first order of noncommutativity parameter $\epsilon^{\mu\nu}\theta$ as follows [49–53]:

$$(h * g)(x) \approx (hg)(x) - \frac{i\epsilon^{\mu\nu}\theta}{2}\partial_\mu^x h \partial_\nu^x g\rfloor_{x^\mu = x^\nu} + O\left(\theta^2\right). \quad (7)$$

The indices $(\mu, \nu = 1, 2, 3)$ and $O\left(\theta^2\right)$ stand for the second and higher-order terms of the NC parameter. Physically, the second term in the last equation presents the effects of space-space noncommutativity. The purpose of this paper is to investigate the (k, l)-states solution of the deformed Dirac equation within Bopp's shift and standard perturbation theory methods to generate an accurate new energy spectrum in the deformation of the Dirac theory symmetries and the deformation of the Schrödinger theory symmetries. Our current work is structured into five sections. The first section includes the scope and purpose of our investigation, while the remaining parts of the paper are structured as follows. A review of the DE with Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction is presented in Sect. II. Section III is devoted to studying the deformed Dirac equation by applying the usual Bopp's shift method and the like Greene and Aldrich approximation for the centrifugal term to obtain the effective potential of the improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction in DDT symmetries. Furthermore, via standard perturbation theory, we find the expectation values of some radial terms to calculate the corrected relativistic energy generated by the effect of the perturbed effective potential of the improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction, we derive the global corrected energy with the improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction in the deformed Dirac symmetries. We will also treat some important special cases, including the study of nonrelativistic cases as a nonrelativistic limit. The last section is devoted to a summary and conclusion.

II. REVISED DE UNDER HCYP INCLUDING A COULOMB-LIKE TENSOR INTERACTION

This section is devoted to a brief review of a physical system that interacted with the Hulthén plus a class of Yukawa potential HCYP including a Coulomb-like tensor interaction; this system can be described by the following equation:

$$\begin{pmatrix} \widehat{\alpha}\mathbf{p} + \widehat{\beta}\left(M + S_{\text{hcy}}(r)\right) - i\widehat{\beta}\widehat{r}U(r) \\ -\left(E_{nk} - V_{\text{hcy}}(r)\right) \end{pmatrix} \Psi_{nk}(r, \Omega) = 0. \quad (8)$$

The vector potential $V_{\text{hcy}}(r)$ due to the four-vector linear momentum operator A^μ $(V_{\text{hcy}}(r),\ \mathbf{A} = \mathbf{0})$ and the space-time scalar potential $S_{\text{hcy}}(r)$ due to the mass, E_{nk} represents the relativistic eigenvalues, (n, k) represent the principal and spin-orbit coupling terms, respectively. $U(r)$ is the tensor interaction, $\widehat{\alpha}_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$, $\widehat{\beta} = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}$ and σ_i are the usual Pauli matrices. Since the Hulthén plus a class of Yukawa potential have spherical symmetry, allowing the solutions of the known form $\Psi_{nk}(r, \Omega) = \frac{1}{r}\begin{pmatrix} F_{nk}(r) Y_{jm}^l(\Omega) & iG_{nk}(r) Y_{jm}^{\widetilde{l}}(\Omega) \end{pmatrix}$, $F_{nk}(r)$ and $G_{nk}(r)$ that represent the upper and lower components of the Dirac spinors $\Psi_{nk}(r, \Omega)$ while $Y_{jm}^l(\Omega)$ and $Y_{jm}^{\widetilde{l}}(\Omega)$ are the spin and pseudospin spherical harmonics and m is the projection on the z-axis. The upper and lower components $F_{nk}(r)$ and $G_{nk}(r)$ satisfy the following second-order differential equations:

$$\left[\frac{d^2}{dr^2} - k(k+1)r^{-2} + U_{\text{eff}}^{\text{ct}}(r) - \left(M + E_{nk} - \Delta_{\text{hcy}}(r)\right)\left(M - E_{nk} + \Sigma_{\text{hcy}}(r)\right) + \frac{\frac{d\Delta_{\text{hcy}}(r)}{dr}\left(\frac{d}{dr} + \frac{k}{r} - U(r)\right)}{M + E_{nk} - \Delta_{\text{hcy}}(r)}\right] F_{nk}(r) = 0, \quad (9)$$

$$\left[\frac{d^2}{dr^2} - k(k-1)r^{-2} + U_{\text{eff}}^{\text{ct}}(r) - \left(M + E_{nk} - \Delta_{\text{hcy}}(r)\right)\left(M - E_{nk} + \Sigma_{\text{hcy}}(r)\right) + \frac{\frac{d\Sigma_{\text{hcy}}(r)}{dr}\left(\frac{d}{dr} - \frac{k}{r} + U(r)\right)}{M + E_{nk} + \Sigma_{\text{hcy}}(r)}\right] G_{nk}(r) = 0. \quad (10)$$

Here $U_{\text{eff}}^{\text{ct}}(r) = \frac{2kU(r)}{r} - \frac{dU(r)}{dr} - U^2(r)$ and $U(r) = -\frac{H}{r}$, $H = \frac{Z_a Z_b}{4\pi\epsilon_0}$, $\Sigma_{\text{hcy}}(r)$ and $\Delta_{\text{hcy}}(r)$ are determined by:

$$\begin{cases} \Sigma_{\text{hcy}}(r) = V_{\text{hcy}}(r) + S_{\text{hcy}}(r) = -\frac{Ze^2\delta\exp(-\delta r)}{1 - \exp(-\alpha\delta r)} - \frac{Ae^{-\delta r}}{r} - \frac{Be^{-2\delta r}}{r^2} \\ \text{and } \frac{d\Delta_{\text{hcy}}(r)}{dr} = 0 \ \ (\Delta = C_{\text{sp}}) \text{ for spin sy limit,} \\ \\ \Delta_{\text{hcy}}(r) = V_{\text{hcy}}(r) - S_{\text{hcy}}(r) = -\frac{Ze^2\delta\exp(-\delta r)}{1 - \exp(-\alpha\delta r)} - \frac{Ae^{-\delta r}}{r} - \frac{Be^{-2\delta r}}{r^2} \\ \text{and } \frac{d\Sigma_{\text{hcy}}(r)}{dr} = 0 \ \ (\Sigma = C_{\text{ps}}) \text{ for p-spin sy limit.} \end{cases} \quad (11.1)$$

We obtain the following second-order Schrödinger-like equation in RQM symmetries, respectively:

$$\left[\frac{d^2}{dr^2} - k\,(k+1)\,r^{-2} + U_{\text{eff}}^{\text{ct}}\,(r) - \left(M + E_{nk}^{\text{sp}} - C_{\text{sp}}\right)\left(M - E_{nk}^{\text{sp}} + \Sigma_{\text{hcy}}\,(r)\right)\right] F_{nk}\,(r) = 0, \tag{11.2}$$

$$\left[\frac{d^2}{dr^2} - k\,(k-1)\,r^{-2} + U_{\text{eff}}^{\text{ct}}\,(r) - \left(M + E_{nk}^{\text{ps}} - \Delta_{\text{hcy}}\,(r)\right)\left(M - E_{nk}^{\text{ps}} + C_{\text{ps}}\right)\right] G_{nk}\,(r) = 0, \tag{11.3}$$

with $k\,(k-1)$ and $k\,(k+1)$ equal to $\widetilde{l}\left(\widetilde{l}-1\right)$ and $l\,(l+1)$, respectively. The authors of Refs. [9] use the NU and SUSYQM methods and the Greene–Aldrich approximation for the centrifugal term to obtain the expressions for the wave function as hypergeometric polynomials ${}_2F_1\left(-n, 2\beta^{nk} + 2\zeta^{nk} + n; 1 + 2\beta^{nk}; s\right)$ in RQM symmetries as follows:

$$F_{nk}\,(s) = D_{nk}^{\text{sp}} s^{\beta^{nk}} \,(1-s)^{\zeta^{nk}} \;{}_2F_1\left(-n, 2\beta^{nk} + 2\zeta^{nk} + n; 1 + 2\beta^{nk}; s\right), \tag{12}$$

$$G_{nk}\,(s) = \widetilde{D}_{nk}^{\text{ps}} s^{\widetilde{\beta}^{nk}} \,(1-s)^{\widetilde{\zeta}^{nk}} \;{}_2F_1\left(-n, 2\widetilde{\beta}^{nk} + 2\widetilde{\zeta}^{nk} + n; 1 + 2\widetilde{\beta}^{nk}; s\right) \tag{13}$$

with $D_{nk}^{\text{sp}} = \frac{\Gamma\left(n+2\beta^{\text{nk}-}+1\right)}{n!\Gamma\left(2\beta^{\text{nk}-}+1\right)} C_{nk}^{\text{sp}}$, $\widetilde{D}_{nk}^{\text{ps}} = \frac{\Gamma\left(n+2\widetilde{\beta}^{\text{nk}-}+1\right)}{n!\Gamma\left(2\widetilde{\beta}^{\text{nk}-}+1\right)} \widetilde{C}_{nk}^{\text{ps}}$, $s = \exp\left(-2\delta r\right)$ and:

$$\begin{cases} 4\beta^{nk2} = \left(M^2 - E_{nk}^{\text{sp}2} - C_{\text{sp}}\left(M - E_{nk}^{\text{sp}}\right)\right)\delta^{-2}, \\ 4\;\widetilde{\beta}^{nk2} = \left(M^2 - E_{nk}^{ps2} + C_{\text{ps}}\left(M + E_{nk}^{\text{ps}}\right)\right)\delta^{-2}, \\ 2\zeta^{nk} = \sqrt{1/4 - B\left(M + E_{nk}^{\text{sp}} - C_{\text{sp}}\right) + (k+H)(k+H+1)}, \\ 2\widetilde{\zeta}^{nk} = \sqrt{1/4 + B\left(M - E_{nk}^{\text{ps}} + C_{\text{ps}}\right) + (k+H)(k+H-1)}, \end{cases} \tag{14}$$

where C_{nk}^{sp} and $\tilde{C}_{nk}^{\text{ps}}$ are the normalization constants:

$$\begin{cases} C_{nk}^{\text{sp}} = \sqrt{\dfrac{2\delta n!\left(n + \zeta^{nk} + \beta^{\text{nk}-}\right)\Gamma\left(2\beta^{\text{nk}-}+1\right)\Gamma\left(n + 2\beta^{\text{nk}-} + 2\zeta^{nk}\right)}{\left(n+\zeta^{nk}\right)\Gamma\left(2\beta^{\text{nk}-}\right)\Gamma\left(n+2\beta^{\text{nk}-}+1\right)\Gamma\left(n+2\zeta^{nk}\right)}}, \\ \widetilde{C}_{nk}^{\text{sp}} = \sqrt{\dfrac{2\delta n!\left(n + \widetilde{\zeta}^{nk} + \widetilde{\beta}^{\text{nk}-}\right)\Gamma\left(2\widetilde{\beta}^{\text{nk}-}+1\right)\Gamma\left(n + 2\widetilde{\beta}^{\text{nk}-} + 2\widetilde{\zeta}^{nk}\right)}{\left(n+\widetilde{\zeta}^{nk}\right)\Gamma\left(2\widetilde{\beta}^{\text{nk}-}\right)\Gamma\left(n+2\widetilde{\beta}^{\text{nk}-}+1\right)\Gamma\left(n+2\widetilde{\zeta}^{nk}\right)}}. \end{cases} \tag{15}$$

For the spin symmetry and the p-spin symmetry, the equations of energy are given by:

$$M^2 - E_{nk}^{\text{sp}2} - C_{\text{sp}}\left(M - E_{nk}^{\text{sp}}\right) = -\left[\frac{\alpha^2 + (k+H)(k+H+1)}{n + 1/2 + 2\zeta^{nk}} - \frac{2\,(2n+1)\,\zeta^{nk}}{n + 1/2 + 2\zeta^{nk}}\right]^2, \tag{16}$$

$$M^2 - E_{nk}^{ps2} + C_{\text{ps}}\left(M + E_{nk}^{\text{ps}}\right) = -\left[\frac{\alpha^2 + (k+H)(k+H-1)}{n + 1/2 + 2\widetilde{\zeta}^{nk}} - \frac{2\,(2n+1)\,\widetilde{\zeta}^{nk}}{n + 1/2 + 2\widetilde{\zeta}^{nk}}\right]^2 \tag{17}$$

with $\alpha^2 = \frac{\left(2\delta Ze^2 + 2A\delta\right)\left(M + E_{nk}^{\text{sp}} - C_{\text{sp}}\right)}{4\delta^2}$ and $\widetilde{\alpha}^2 = -\frac{\left(2\delta Ze^2 + 2A\delta\right)\left(M - E_{nk}^{\text{ps}} + C_{\text{ps}}\right)}{4\delta^2}$.

III. THE NEW SOLUTIONS OF DDE UNDER IHCYPCTI IN THE DDT SYMMETRIES

A. Review of Bopp's shift method

Let us begin this subsection by finding the DDE in the symmetries of extended RQM under IHCYPCTI. Our objective is achieved by applying the new principles which we described in the introduction, Eqs. (4) and (7), summarized in new relationships MASCCCRs, and the notion of the Weyl–Moyal star product. These data allow us to rewrite the usual radial Dirac equations in Eq. (8) in the DDT symmetries as follows:

$$\left(\widehat{\alpha}\mathbf{p}+\widehat{\beta}\left(M+S_{\mathrm{hcy}}(r)\right)-i\widehat{\beta}\widehat{r}U(r)-\left(E_{nk}-V_{\mathrm{hcy}}(r)\right)\right)*\Psi_{nk}(r,\Omega)=0. \tag{18}$$

Thus, the upper and lower components $F_{nk}(r)$ and $G_{nk}(r)$ satisfy the following second-order differential equations in the DDT symmetries:

$$\left(\frac{d^2}{dr^2}-k(k+1)r^{-2}+U_{\mathrm{eff}}^{\mathrm{ct}}(r)-\left(M+E_{nk}^{\mathrm{sp}}-C_{\mathrm{sp}}\right)\left(M-E_{nk}^{\mathrm{sp}}+\Sigma_{\mathrm{hcy}}(r)\right)\right)*F_{nk}(r)=0 \tag{19}$$

and

$$\left(\frac{d^2}{dr^2}-k(k-1)r^{-2}+U_{\mathrm{eff}}^{\mathrm{ct}}(r)-\left(M+E_{nk}^{\mathrm{ps}}-\Delta_{\mathrm{hcy}}(r)\right)\left(M-E_{nk}^{\mathrm{ps}}+C_{\mathrm{ps}}\right)\right)*G_{nk}(r)=0. \tag{20}$$

Among the possible paths to finding solutions to Eqs. (19) and (20) is the application of the Connes method [21, 22], or the Seiberg and Witten map [23, 54]. Specialists know that the star product can be translated into an ordinary product known in the literature using what is called Bopp's shift method. Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules $(x,p)\rightarrow\left(\widehat{x}=x-\frac{i}{2}\partial_{\mathrm{p}},\,\widehat{p}=p+\frac{i}{2}\partial_x\right)$ instead of the ordinary correspondence $(x,\,p)\rightarrow\left(\widehat{x}=x,\,\widehat{p}=p+\frac{i}{2}\partial_x\right)$, respectively. This procedure is known as Bopp's shifts and this quantization procedure is known as Bopp's quantization [55–58]. This method has achieved considerable success in recent years. For illustrations of its application in treating the nonrelativistic deformed Schrödinger equation NR-DSE over a significant number of typical potentials, see references [27, 51, 59–64]. The success of this method was not limited to the DSE, but extended to the study of various relativistic physics problems, for example the deformed Klein–Gordon equation DKGE (see references [28–32, 65–73]), for the DDE (see references [26, 50, 52, 53, 74]) and the deformed Duffin–Kemmer–Petiau equation DDKPE [75, 76]. Thus, Bopp's shift method BSM is based on reducing second-order linear differential equations of the DSE, DKG, DDE and DDKPE with the Weyl–Moyal star product to second-order linear differential equations of SE, KGE, DE, and DKPE without the Weyl–Moyal star product with simultaneous translation in the space-space. It is worth motioning that BSM permutes to reduce the above equations to the simplest form:

$$\left(\frac{d^2}{dr^2}-k(k+1)\widehat{r}^{-2}+U_{\mathrm{eff}}^{\mathrm{ct}}(\widehat{r})\right.$$
$$\left.-\left(M+E_{nk}^{\mathrm{sp}}-C_{\mathrm{sp}}\right)\left(M-E_{nk}^{\mathrm{sp}}+\Sigma_{\mathrm{hcy}}(\widehat{r})\right)\right)F_{nk}(r)=0 \tag{21}$$

and

$$\left(\frac{d^2}{dr^2}-k(k-1)\widehat{r}^{-2}+U_{\mathrm{eff}}^{\mathrm{ct}}(\widehat{r})\right.$$
$$\left.-\left(M+E_{nk}^{\mathrm{ps}}-\Delta_{\mathrm{hcy}}(\widehat{r})\right)\left(M-E_{nk}^{\mathrm{ps}}+C_{\mathrm{ps}}\right)\right)G_{nk}(r)=0. \tag{22}$$

The MASCCCRs with the notion of the Weyl–Moyal star product in Eqs. (4) become new MASCCCRs with an ordinary product known in literature as follows (see, e. g., [55–58]):

$$\begin{cases}\left[\widehat{x}_\mu^{(\mathrm{S,H,I})},\,\widehat{p}_\nu^{(\mathrm{S,H,I})}\right]=i\hbar_{\mathrm{eff}}\delta_{\mu\nu},\\ \left[\widehat{x}_\mu^{(\mathrm{S,H,I})},\,\widehat{x}_\nu^{(\mathrm{S,H,I})}\right]=i\theta_{\mu\nu}.\end{cases} \tag{23}$$

The generalized positions and momentum coordinates $\widehat{x}_\mu^{(\mathrm{S,H,I})}$ and $\widehat{p}_\mu^{(\mathrm{S,H,I})}$ in the symmetries of extended RQM are defined as [63–66]:

$$\begin{cases}\widehat{x}_\mu^{(\mathrm{S,H,I})}=x_\mu^{(\mathrm{S,H,I})}-\sum\limits_{\nu=1}^{3}\frac{i\theta_{\mu\nu}}{2}p_\nu^{(\mathrm{S,H,I})},\\ \widehat{p}_\mu^{(\mathrm{S,H,I})}=p_\mu^{(\mathrm{S,H,I})},\end{cases} \tag{24}$$

here $x_\mu^{(\mathrm{S,H,I})} = \left(x_\mu^{\mathrm{S}}, x_\mu^{\mathrm{H}}, x_\mu^{\mathrm{I}}\right)$ and $p_\mu^{(\mathrm{S,H,I})} = \left(p_\mu^{\mathrm{S}}, p_\mu^{\mathrm{H}}, p_\mu^{\mathrm{I}}\right)$ are corresponding coordinates in the RQM symmetries. This allows us to find the operator $\widehat{r}^2$ equal ($r^2 - \mathbf{L\Theta}$, $r^2 - \widetilde{\mathbf{L}}\mathbf{\Theta}$) for spin symmetry and p-spin, respectively [26, 50, 52, 53, 74] while the new operators $V_{\mathrm{hcy}}(\widehat{r})$, $U_{\mathrm{eff}}^{\mathrm{ct}}(\widehat{r})$, $k(k+1)\widehat{r}^{-2}$and $k(k-1)\widehat{r}^{-2}$ in the DDT symmetries, are expressed as:

$$\begin{cases} V_{\mathrm{hcy}}^{\mathrm{s}}(\widehat{r}) = V(r) - \frac{\partial V_{\mathrm{hcy}}(r)}{\partial r}\frac{\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right), \\ V_{\mathrm{hcy}}^{\mathrm{p}}(\widehat{r}) = V(r) - \frac{\partial V_{\mathrm{hcy}}(r)}{\partial r}\frac{\widetilde{\mathbf{L}}\mathbf{\Theta}}{2r} + O\left(\Theta^2\right), \\ U_{\mathrm{eff}}^{\mathrm{ct\text{-}s}}(\widehat{r}) = U_{\mathrm{eff}}^{\mathrm{ct}}(r) - \frac{\partial U_{\mathrm{eff}}^{\mathrm{ct}}(r)}{\partial r}\frac{\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right), \\ U_{\mathrm{eff}}^{\mathrm{ct\text{-}p}}(\widehat{r}) = U_{\mathrm{eff}}^{\mathrm{ct}}(r) - \frac{\partial U_{\mathrm{eff}}^{\mathrm{ct}}(r)}{\partial r}\frac{\widetilde{\mathbf{L}}\mathbf{\Theta}}{2r} + O\left(\Theta^2\right), \\ k(k+1)\widehat{r}^{-2} = k(k+1)r^{-2} + k(k+1)r^{-4}\mathbf{L}\Theta + O\left(\Theta^2\right), \\ k(k-1)\widehat{r}^{-2} = k(k-1)r^{-2} + k(k-1)r^{-4}\widetilde{\mathbf{L}}\Theta + O\left(\Theta^2\right). \end{cases} \tag{25}$$

Substituting Eqs. (25) into Eqs. (21) and (22), we obtain the following two Shrödinger-like equations:

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + U_{\mathrm{eff}}^{\mathrm{ct\text{-}s}}(r) - \left(M + E_{nk}^{\mathrm{sp}} - C_{\mathrm{sp}}\right)\left(M - E_{nk}^{\mathrm{sp}} + \Sigma_{\mathrm{hcy}}(r)\right) - \Sigma_{\mathrm{hcy}}^{\mathrm{pert}}(r)\right] F_{nk}(r) = 0 \tag{26}$$

and

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + U_{\mathrm{eff}}^{\mathrm{ct\text{-}p}}(r) - \left(M + E_{nk}^{\mathrm{ps}} - \Delta_{\mathrm{hcy}}(r)\right)\left(M - E_{nk}^{\mathrm{ps}} + C_{\mathrm{ps}}\right) - \Delta_{\mathrm{hcy}}^{\mathrm{pert}}(r)\right] G_{nk}(r) = 0 \tag{27}$$

with

$$\Sigma_{\mathrm{hcy}}^{\mathrm{pert}}(r) = -\frac{\partial U_{\mathrm{eff}}^{\mathrm{ct\text{-}s}}(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} + \frac{k(k+1)}{r^4}\mathbf{L}\Theta - \left(M + E_{nk}^{\mathrm{sp}} - C_{\mathrm{sp}}\right)\frac{\partial V(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} \tag{28}$$

and

$$\Delta_{\mathrm{hcy}}^{\mathrm{pert}}(r) = -\frac{\partial U_{\mathrm{eff}}^{\mathrm{ct\text{-}p}}(r)}{\partial r}\frac{\widetilde{\mathbf{L}}\Theta}{2r} + \frac{k(k-1)}{r^4}\widetilde{\mathbf{L}}\Theta - \left(M - E_{nk}^{\mathrm{ps}} + C_{\mathrm{ps}}\right)\frac{\partial V(r)}{\partial r}\frac{\widetilde{\mathbf{L}}\Theta}{2r}. \tag{29}$$

By comparing Eqs. (11.1) and (11.2) and Eqs. (26) and (27), we observe two additive potentials $\Sigma_{\mathrm{hcy}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{hcy}}^{\mathrm{pert}}(r)$. Moreover, these terms are proportional to the infinitesimal noncommutativity parameter Θ. From a physical point of view, this means that these two spontaneously generated terms $\Sigma_{\mathrm{hcy}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{hcy}}^{\mathrm{pert}}(r)$ as a result of the topological properties of deformation space–space can be considered very small compared to the fundamental terms $\Sigma_{\mathrm{hcy}}(r)$ and $\Delta_{\mathrm{hcy}}(r)$, respectively. A direct calculation gives $\frac{\partial V_{\mathrm{hcy}}(r)}{\partial r}$ and $\frac{\partial U_{\mathrm{eff}}^{\mathrm{yu}}(r)}{\partial r}$ as follows:

$$\frac{\partial V_{\mathrm{hcy}}(r)}{\partial r} = \frac{4Ze^2\delta^2\exp(-2\delta r)}{(1-\exp(-2\delta r))} + \frac{Ze^2\delta^2\exp(-4\delta r)}{(1-\exp(-2\delta r))^2} - \frac{2A\delta e^{-2\delta r}}{r} + \frac{Ae^{-2\delta r}}{r^2} + \frac{4B\delta e^{-4\delta r}}{r^2} + \frac{2Be^{-4\delta r}}{r^3}, \tag{30}$$

$$\frac{\partial U_{\mathrm{eff}}^{\mathrm{yu}}(r)}{\partial r} = \frac{H}{r^2}. \tag{31}$$

It should be noted that for convenience, we substitute $\delta \to 2\delta$ in the Hulthén potential. Substituting Eq. (30) and (31) into Eqs. (28) and (29), we obtain spontaneously generated terms $\Sigma_{\mathrm{hcy}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{hcy}}^{\mathrm{pert}}(r)$ as follows:

$$\Sigma_{\mathrm{hcy}}^{\mathrm{pert}}(r) = -\left(M + E_{nk}^{\mathrm{sp}} - C_{\mathrm{sp}}\right)\begin{pmatrix} \frac{2Ze^2\delta^2\exp(-2\delta r)}{r(1-\exp(-2\delta r))} + \frac{2Ze^2\delta^2\exp(-4\delta r)}{r(1-\exp(-2\delta r))^2} \\ -\frac{A\delta e^{-2\delta r}}{r^2} + \frac{Ae^{-2\delta r}}{2r^3} + \frac{B\delta e^{-4\delta r}}{r^3} + \frac{Be^{-4\delta r}}{r^4} \end{pmatrix}\mathbf{L}\Theta + \left(\frac{k(k+1)}{r^4} - \frac{H}{2r^3}\right)\mathbf{L}\Theta + O\left(\Theta^2\right) \tag{32}$$

and

$$\Delta_{\rm hcy}^{\rm pert}\left(r\right) = -\left(M - E_{nk}^{\rm ps} + C_{\rm ps}\right)\begin{pmatrix} \frac{2Ze^{2}\delta^{2}\exp(-2\delta r)}{r(1-\exp(-2\delta r))} + \frac{2Ze^{2}\delta^{2}\exp(-4\delta r)}{r(1-\exp(-2\delta r))^{2}} \\ -\frac{A\delta e^{-2\delta r}}{r^{2}} + \frac{Ae^{-2\delta r}}{2r^{3}} + \frac{B\delta e^{-4\delta r}}{r^{3}} + \frac{Be^{-4\delta r}}{r^{4}} \end{pmatrix}\widetilde{\mathbf{L}}\Theta + \left(\frac{k\left(k-1\right)}{r^{4}} - \frac{H}{2r^{3}}\right)\widetilde{\mathbf{L}}\Theta + O\left(\Theta^{2}\right). \tag{33}$$

For spin symmetry, we first consider Eq. (26), which contains the improved Hulthén plus a class of Yukawa and also an improved Coulomb-like tensor potential in the deformation of Dirac theory symmetries. It can be solved exactly only for $k = 0$ and $k = -1$ in the absence of tensor interaction $H = 0$, since the two centrifugal terms (proportional to $k\left(k+1\right)r^{-2}$ and $k\left(k+1\right)r^{-4}$) vanished. In the case of arbitrary k, an appropriate approximation needs to be employed on the centrifugal terms. We apply the following improved approximation which was applied by Greene and Aldrich [77]:

$$\frac{1}{r^{2}} \approx \frac{4\delta^{2}e^{-2\delta r}}{\left(1-e^{-2\delta r}\right)^{2}} = \frac{4\delta^{2}s}{\left(1-s\right)^{2}} \Longrightarrow \frac{1}{r} = \frac{2\delta s^{1/2}}{1-s}. \tag{34}$$

For p-spin symmetry, we now consider Eq. (27) and will follow similar steps with the spin symmetry case in the deformation of the Dirac theory symmetries. In a similar way as mentioned earlier, Eq. (27) cannot be solved exactly for $k = 0$ and $k = 1$ without tensor interaction, since the two centrifugal terms (proportional to $k\left(k-1\right)r^{-2}$ and $k\left(k-1\right)r^{-4}$) vanish. Applying the approximations Eq. (34) to the centrifugal terms of Eqs.(32) and (33), the general form of the additive potentials $\Sigma_{\rm hcy}^{\rm pert}\left(r\right)$ and $\Delta_{\rm hcy}^{\rm pert}\left(r\right)$ will be as follows:

$$\Sigma_{\rm hcy}^{\rm pert}\left(r\right) = (\frac{L_{nk}^{1\rm sp}s^{3/2}}{\left(1-s\right)^{2}} + \frac{L_{nk}^{2\rm sp}s^{5/2}}{\left(1-s\right)^{3}} + \frac{L_{nk}^{3\rm sp}\delta s^{3}}{\left(1-s\right)^{2}} + \frac{L_{nk}^{4\rm sp}s^{7/2}}{\left(1-s\right)^{3}} + \frac{L_{nk}^{5\rm sp}s^{4}}{\left(1-s\right)^{4}} + \frac{L_{nk}^{6\rm sp}s^{2}}{\left(1-s\right)^{4}} + \frac{L_{nk}^{\rm sp}s^{3/2}}{\left(1-s\right)^{3}})\mathbf{L}\Theta + O\left(\Theta^{2}\right) \tag{35}$$

and

$$\Delta_{\rm hcy}^{\rm pert}\left(r\right) = (\frac{L_{nk}^{1\rm sp}s^{3/2}}{\left(1-s\right)^{2}} + \frac{L_{nk}^{2\rm sp}s^{5/2}}{\left(1-s\right)^{3}} + \frac{L_{nk}^{3\rm sp}\delta s^{3}}{\left(1-s\right)^{2}} + \frac{L_{nk}^{4\rm sp}s^{7/2}}{\left(1-s\right)^{3}} + \frac{L_{nk}^{5\rm sp}s^{4}}{\left(1-s\right)^{4}} + \frac{L_{nk}^{6\rm sp}s^{2}}{\left(1-s\right)^{4}} + \frac{L_{nk}^{\rm sp}s^{3/2}}{\left(1-s\right)^{3}})\widetilde{\mathbf{L}}\Theta + O\left(\Theta^{2}\right) \tag{36}$$

with

$$\begin{cases} L_{nk}^{1\rm sp} = -4Ze^{2}\delta^{3}\left(M + E_{nk}^{\rm sp} - C_{\rm sp}\right), \\ L_{nk}^{2\rm sp} = -4\delta^{3}\left(Ze^{2} + A\right)\left(M + E_{nk}^{\rm sp} - C_{\rm sp}\right), \\ L_{nk}^{3\rm sp} = 8A\delta^{3}\left(M + E_{nk}^{\rm sp} - C_{\rm sp}\right), \\ L_{nk}^{4\rm sp} = -8B\delta^{4}\left(M + E_{nk}^{\rm sp} - C_{\rm sp}\right), \\ L_{nk}^{5\rm sp} = -16B\delta^{4}\left(M + E_{nk}^{\rm sp} - C_{\rm sp}\right), \\ L_{nk}^{6\rm sp} = 16k\left(k+1\right)\delta^{4}, \\ L_{nk}^{\rm sp} = L_{nk}^{7\rm ps} = -4\delta^{3}H. \end{cases} \tag{37}$$

while

$$\begin{cases} L_{nk}^{1\rm ps} = -4Ze^{2}\delta^{3}\left(M - E_{nk}^{\rm ps} + C_{\rm ps}\right), \\ L_{nk}^{2\rm ps} = -4\delta^{3}\left(Ze^{2} + A\right)\left(M - E_{nk}^{\rm ps} + C_{\rm ps}\right), \\ L_{nk}^{3\rm ps} = 8A\delta^{3}\left(M - E_{nk}^{\rm ps} + C_{\rm ps}\right), \\ L_{nk}^{4\rm ps} = -8B\delta^{4}\left(M - E_{nk}^{\rm ps} + C_{\rm ps}\right), \\ L_{nk}^{5\rm ps} = -16B\delta^{4}\left(M - E_{nk}^{\rm ps} + C_{\rm ps}\right), \\ L_{nk}^{6\rm ps} = 16k\left(k-1\right)\delta^{4}. \end{cases} \tag{38}$$

It is important to mention here that the above approximations are valid in short when $\delta r \ll 1$ is satisfied. We have replaced the term $k(k+1)r^{-4}$ and $k(k-1)r^{-4}$ with the approximation in Eq. (34). The Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction is extended by including new terms proportional to the radial terms $\frac{s^{3/2}}{(1-s)^2}$, $\frac{s^{5/2}}{(1-s)^3}$, $\frac{s^3}{(1-s)^2}$, $\frac{s^{7/2}}{(1-s)^3}$, $\frac{s^4}{(1-s)^4}$, $\frac{s^2}{(1-s)^4}$ and $\frac{s^{3/2}}{(1-s)^3}$ to become the improved Hulthén plus a class of Yukawa potential including an improved Coulomb-like tensor interaction in extended RQM symmetries. The generated new effective potentials $\Sigma_{\text{hcy}}^{\text{pert}}(r)$ and $\Delta_{\text{hcy}}^{\text{pert}}(r)$ are also proportional to the infinitesimal vector $\mathbf{\Theta}$. This allows us to consider the new additive parts of the effective potential $\Sigma_{\text{hcy}}^{\text{pert}}(r)$ and $\Delta_{\text{hcy}}^{\text{pert}}(r)$ as a perturbation potential compared with the main potentials $\Sigma_{\text{hcy}}(r)$ and $\Delta_{\text{hcy}}(r)$ (the parent potential operator in the symmetries of extended RQM, that is, the inequality has become achieved $\Sigma_{\text{hcy}}^{\text{pert}}(r) \ll \Sigma_{\text{hcy}}(r)$ and $\Delta_{\text{hcy}}^{\text{pert}}(r) \ll \Delta_{\text{hcy}}(r)$). That is all physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized n^{th} excited states.

B. The expectation values under IHCYPCTI in the DDT for spin symmetry

In this subsection, we want to apply the perturbative theory. In the case of extended RQM symmetries, we find the expectation values $M_{1(nlm)}^{\text{sp-hcy}} \equiv \left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(nlm)}^{\text{sp-hcy}}$, $M_{2(nlm)}^{\text{sp-hcy}} \equiv \left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(nlm)}^{\text{sp-hcy}}$, $M_{3(nlm)}^{\text{sp-hcy}} \equiv \left\langle \frac{s^{3}}{(1-s)^2} \right\rangle_{(nlm)}^{\text{sp-hcy}}$, $M_{4(nlm)}^{\text{sp-hcy}} \equiv \left\langle \frac{s^{7/2}}{(1-s)^3} \right\rangle_{(nlm)}^{\text{sp-hcy}}$, $M_{5(nlm)}^{\text{sp-hcy}} \equiv \left\langle \frac{s^{4}}{(1-s)^4} \right\rangle_{(nlm)}^{\text{sp-hcy}}$, $M_{6(nlm)}^{\text{sp-hcy}} \equiv \left\langle \frac{s^{2}}{(1-s)^4} \right\rangle_{(nlm)}^{\text{sp-hcy}}$ and $M_{7(nlm)}^{\text{sp-hcy}} \equiv \left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(nlm)}^{\text{sp-hcy}}$ for the spin symmetry taking into account the wave function which we have seen previously in Eq. (12). Thus, after straightforward calculations, we obtain the following results:

$$M_{1(nlm)}^{\text{sp-hcy}} = D_{nk}^{\text{sp2}} \int_0^{+\infty} s^{2\beta^{nk}+3/2} (1-s)^{2\zeta^{nk}-2} \left[{}_2F_1\left(-n, 2\beta^{\text{nk}-} + 2\zeta^{nk} + n; 1 + 2\beta^{\text{nk}-}; s\right)\right]^2 dr, \tag{39.1}$$

$$M_{2(nlm)}^{\text{sp-hcy}} = D_{nk}^{\text{sp2}} \int_0^{+\infty} s^{2\beta^{nk}+5/2} (1-s)^{2\zeta^{nk}-3} \left[{}_2F_1\left(-n, 2\beta^{\text{nk}-} + 2\zeta^{nk} + n; 1 + 2\beta^{\text{nk}-}; s\right)\right]^2 dr, \tag{39.2}$$

$$M_{3(nlm)}^{\text{sp-hcy}} = D_{nk}^{\text{sp2}} \int_0^{+\infty} s^{2\beta^{nk}+3} (1-s)^{2\zeta^{nk}-2} \left[{}_2F_1\left(-n, 2\beta^{nk} + 2\zeta^{nk} + n; 1 + 2\beta^{\text{nk}-}; s\right)\right]^2 dr, \tag{39.3}$$

$$M_{4(nlm)}^{\text{sp-hcy}} = D_{nk}^{\text{sp2}} \int_0^{+\infty} s^{2\beta^{nk}+7/2} (1-s)^{2\zeta^{nk}-3} \left[{}_2F_1\left(-n, 2\beta^{nk} + 2\zeta^{nk} + n; 1 + 2\beta^{\text{nk}-}; s\right)\right]^2 dr, \tag{39.4}$$

$$M_{5(nlm)}^{\text{sp-hcy}} = D_{nk}^{\text{sp2}} \int_0^{+\infty} s^{2\beta^{nk}+4} (1-s)^{2\zeta^{nk}-4} \left[{}_2F_1\left(-n, 2\beta^{nk} + 2\zeta^{nk} + n; 1 + 2\beta^{\text{nk}-}; s\right)\right]^2 dr, \tag{39.5}$$

$$M_{6(nlm)}^{\text{sp-hcy}} = D_{nk}^{\text{sp2}} \int_0^{+\infty} s^{2\beta^{nk}+2} (1-s)^{2\zeta^{nk}-4} \left[{}_2F_1\left(-n, 2\beta^{nk} + 2\zeta^{nk} + n; 1 + 2\beta^{\text{nk}-}; s\right)\right]^2 dr, \tag{39.6}$$

and

$$M_{7(nlm)}^{\text{sp-hcy}} = D_{nk}^{\text{sp2}} \int_0^{+\infty} s^{2\beta^{nk}+3/2} (1-s)^{2\zeta^{nk}-3} \left[{}_2F_1\left(-n, 2\beta^{nk} + 2\zeta^{nk} + n; 1 + 2\beta^{\text{nk}-}; s\right)\right]^2 dr. \tag{39.7}$$

We have used useful abbreviations $\langle R \rangle_{(nlm)}^{\text{sp-hcy}} = \langle n, l, m\ R\ n, l, m \rangle$ to avoid the extra burden of writing equations. Furthermore, we have applied the property of the spherical harmonics, which has the form $\int Y_l^m(\Omega') Y_{l'}^{m'}(\Omega)\, d^2\Omega =$

$\delta_{ll'}\delta_{mm'}$. We have $s=\exp(-2\delta r)$, this allows us to obtain $dr=-1/2\frac{ds}{\delta s}$. From the asymptotic behavior of s when $(r\to 0)$ $(y\to +1)$and when $(r\to +\infty)$ $(y\to 0)$, this allows us to reformulate Eqs. $(39, i=\overline{1,7})$ as follows:

$$M_{1(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\int_0^{+1} s^{2\beta^{nk}+3/2-1}(1-s)^{2\zeta^{nk}-1-1}\left[{}_2F_1\left(-n,2\beta^{\text{nk}-}+2\zeta^{nk}+n;1+2\beta^{\text{nk}-};s\right)\right]^2 ds, \tag{40.1}$$

$$M_{2(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\int_0^{+1} s^{2\beta^{nk}+5/2-1}(1-s)^{2\zeta^{nk}-2-1}\left[{}_2F_1\left(-n,2\beta^{\text{nk}-}+2\zeta^{nk}+n;1+2\beta^{\text{nk}-};s\right)\right]^2 ds, \tag{40.2}$$

$$M_{3(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\int_0^{+1} s^{2\beta^{nk}+3-1}(1-s)^{2\zeta^{nk}-1-1}\left[{}_2F_1\left(-n,2\beta^{\text{nk}-}+2\zeta^{nk}+n;1+2\beta^{\text{nk}-};s\right)\right]^2 ds, \tag{40.3}$$

$$M_{4(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\int_0^{+1} s^{2\beta^{nk}+7/2-1}(1-s)^{2\zeta^{nk}-2-1}\left[{}_2F_1\left(-n,2\beta^{\text{nk}-}+2\zeta^{nk}+n;1+2\beta^{\text{nk}-};s\right)\right]^2 ds, \tag{40.4}$$

$$M_{5(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\int_0^{+1} s^{2\beta^{nk}+4-1}(1-s)^{2\zeta^{nk}-3-1}\left[{}_2F_1\left(-n,2\beta^{\text{nk}-}+2\zeta^{nk}+n;1+2\beta^{\text{nk}-};s\right)\right]^2 ds, \tag{40.5}$$

$$M_{6(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\int_0^{+1} s^{2\beta^{nk}+2-1}(1-s)^{2\zeta^{nk}-4}\left[{}_2F_1\left(-n,2\beta^{\text{nk}-}+2\zeta^{nk}+n;1+2\beta^{\text{nk}-};s\right)\right]^2 ds, \tag{40.6}$$

and

$$M_{7(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\int_0^{+1} s^{2\beta^{nk}+3/2-1}(1-s)^{2\zeta^{nk}-2-1}\left[{}_2F_1\left(-n,2\beta^{\text{nk}-}+2\zeta^{nk}+n;1+2\beta^{\text{nk}-};s\right)\right]^2 ds. \tag{40.7}$$

We can evaluate the above integrals either in a recurrence way through the physical values of the principal quantum number $(n=0,1,\ldots)$ and then generalize the result to the general n^{th} excited state, or we use the method proposed by Dong *et al.* [78] and applied by Zhang [79], to obtain the general excited state directly. We calculate the integrals in Eqs. $(40, i=\overline{1,7})$ with the help of the special integral formula:

$$\int_0^{+1} z^{\alpha-1}(1-y)^{\beta-1}\left[{}_2F_1(c_1,c_2;c_3;z)\right]^2 dz=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,{}_3F_2(c_1,c_2,\beta;c_3,\beta+\alpha;1), \tag{41}$$

here ${}_2F_1(c_1,c_2;c_3;y)$ is the generalized hypergeometric function and ${}_3F_2(c_1,c_2,\beta;c_3,\beta+\alpha;1)$ is determined from ${}_3F_2(c_1,c_2,\sigma;c_3,\sigma+\xi;1)=\sum_{n=0}^{+\infty}\frac{(c_1)_n(c_2)_n(\sigma)_n}{(c_3)_n(\sigma+\xi)n!}$ while $\Gamma(\alpha)$ denotes the usual Gamma function. By identified Eq. (41) with the integrals, we obtain the following results:

$$M_{1(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\frac{\Gamma\left(2\beta^{nk}+3/2\right)\Gamma\left(2\zeta^{nk}-1\right)}{\Gamma\left(D_{nk}^{\text{sp}}+1/2\right)}\,{}_3F_2\left(-n,Q_{nk}^{\text{sp}},2\zeta^{nk}-1;1+2\beta^{nk},D_{nk}^{\text{sp}}+1/2;1\right), \tag{42.1}$$

$$M_{2(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\frac{\Gamma\left(2\beta^{nk}+5/2\right)\Gamma\left(2\zeta^{nk}-2\right)}{\Gamma\left(D_{nk}^{\text{sp}}+1/2\right)}\,{}_3F_2\left(-n,Q_{nk}^{\text{sp}},2\zeta^{nk}-2;1+2\beta^{nk},D_{nk}^{\text{sp}}+1/2;1\right), \tag{42.2}$$

$$M_{3(nlm)}^{\text{sp-hcy}}=\frac{D_{nk}^{\text{sp2}}}{2\delta}\frac{\Gamma\left(2\beta^{nk}+3\right)\Gamma\left(2\zeta^{nk}-1\right)}{\Gamma\left(D_{nk}^{\text{sp}}+2\right)}\,{}_3F_2\left(-n,Q_{nk}^{\text{sp}},2\zeta^{nk}-1;1+2\beta^{nk},D_{nk}^{\text{sp}}+2;1\right), \tag{42.3}$$

$$M_{4(nlm)}^{\text{sp-hcy}} = \frac{D_{nk}^{\text{sp}2}}{2\delta} \frac{\Gamma\left(2\beta^{nk}+7/2\right)\Gamma\left(2\zeta^{nk}-2\right)}{\Gamma\left(D_{nk}^{\text{sp}}+3/2\right)}\, {}_3F_2\left(-n, Q_{nk}^{\text{sp}}, 2\zeta^{nk}-2; 1+2\beta^{nk}, D_{nk}^{\text{sp}}+3/2; 1\right), \tag{42.4}$$

$$M_{5(nlm)}^{\text{sp-hcy}} = \frac{D_{nk}^{\text{sp}2}}{2\delta} \frac{\Gamma\left(2\beta^{nk}+4\right)\Gamma\left(2\zeta^{nk}-3\right)}{\Gamma\left(D_{nk}^{\text{sp}}+1\right)}\, {}_3F_2\left(-n, Q_{nk}^{\text{sp}}, 2\zeta^{nk}-3; 1+2\beta^{nk}, D_{nk}^{\text{sp}}+1; 1\right), \tag{42.5}$$

$$M_{6(nlm)}^{\text{sp-hcy}} = \frac{D_{nk}^{\text{sp}2}}{2\delta} \frac{\Gamma\left(2\beta^{nk}+2\right)\Gamma\left(2\zeta^{nk}-4\right)}{\Gamma\left(D_{nk}^{\text{sp}}-2\right)}\, {}_3F_2\left(-n, Q_{nk}^{\text{sp}}, 2\zeta^{nk}-4; 1+2\beta^{nk}, D_{nk}^{\text{sp}}; 1\right), \tag{42.6}$$

and

$$M_{7(nlm)}^{\text{sp-hcy}} = \frac{D_{nk}^{\text{sp}2}}{2\delta} \frac{\Gamma\left(2\beta^{nk}+3/2\right)\Gamma\left(2\zeta^{nk}-2\right)}{\Gamma\left(D_{nk}^{\text{sp}}+1/2\right)}\, {}_3F_2\left(-n, Q_{nk}^{\text{sp}}, 2\zeta^{nk}-2; 1+2\beta^{nk}, D_{nk}^{\text{sp}}+1/2; 1\right) \tag{42.7}$$

with $Q_{nk}^{\text{sp}} = 2\beta^{nk}+2\zeta^{nk}+n$ and $D_{nk}^{\text{sp}} = 2\beta^{nk}+2\zeta^{nk}$.

C. The expectation values under IHCYPCTI in the DDT for p-spin symmetry

In this subsection, we want to apply the perturbative theory. In the case of extended RQM symmetries, we find the expectation values $M_{1(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} \equiv \left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(n\tilde{l}\tilde{m})}^{\text{ps-hcy}}$, $M_{2(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} \equiv \left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(n\tilde{l}\tilde{m})}^{\text{ps-hcy}}$, $M_{3(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} \equiv \left\langle \frac{s^{3}}{(1-s)^2} \right\rangle_{(n\tilde{l}\tilde{m})}^{\text{ps-hcy}}$, $M_{4(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} \equiv \left\langle \frac{s^{7/2}}{(1-s)^3} \right\rangle_{(n\tilde{l}\tilde{m})}^{\text{ps-hcy}}$, $M_{5(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} \equiv \left\langle \frac{s^{4}}{(1-s)^4} \right\rangle_{(n\tilde{l}\tilde{m})}^{\text{ps-hcy}}$, $M_{6(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} \equiv \left\langle \frac{s^{2}}{(1-s)^4} \right\rangle_{(nlm)}^{\text{sp-hcy}}$ and $M_{7(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} \equiv \left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(n\tilde{l}\tilde{m})}^{\text{ps-hcy}}$ for p-spin symmetry with tensor interaction taking into account the wave function which we have seen previously in Eq. (13). By examining the two expressions of the two wave functions shown in Eqs. (12) and (13), we note that there is a possibility of moving from the upper wave function $F_{nk}(r)$ to the other lower wave function $G_{nk}(r)$ by making the following substitutions:

$$D_{nk}^{\text{sp}} \Longleftrightarrow \widetilde{D}_{nk}^{\text{ps}},\ \beta^{nk} \Leftrightarrow \widetilde{\beta}^{nk} \text{and } \zeta^{nk} \Leftrightarrow \widetilde{\zeta}^{nk} \tag{43}$$

which allows us to obtain the expectation values for p-spin symmetry from Eqs. $(42, i=\overline{1,7})$ without re-calculation, as follows

$$M_{1(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} = \frac{\widetilde{D}_{nk}^{\text{sp}2}}{2\delta} \frac{\Gamma\left(2\widetilde{\beta}^{nk}+3/2\right)\Gamma\left(2\zeta^{nk}-1\right)}{\Gamma\left(\widetilde{D}_{nk}^{\text{sp}}+1/2\right)}\, {}_3F_2\left(-n, \widetilde{Q}_{nk}^{\text{sp}}, 2\widetilde{\zeta}^{nk}-1; 1+2\widetilde{\beta}^{nk}, \widetilde{D}_{nk}^{\text{sp}}+1/2; 1\right), \tag{44.1}$$

$$M_{2(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} = \frac{\widetilde{D}_{nk}^{\text{sp}2}}{2\delta} \frac{\Gamma\left(2\widetilde{\beta}^{nk}+5/2\right)\Gamma\left(2\widetilde{\zeta}^{nk}-2\right)}{\Gamma\left(\widetilde{D}_{nk}^{\text{sp}}+1/2\right)}\, {}_3F_2\left(-n, \widetilde{Q}_{nk}^{\text{sp}}, 2\widetilde{\zeta}^{nk}-2; 1+2\widetilde{\beta}^{nk}, \widetilde{D}_{nk}^{\text{sp}}+1/2; 1\right), \tag{44.2}$$

$$M_{3(n\tilde{l}\tilde{m})}^{\text{ps-hcy}} = \frac{\widetilde{D}_{nk}^{\text{sp}2}}{2\delta} \frac{\Gamma\left(2\widetilde{\beta}^{nk}+3\right)\Gamma\left(2\widetilde{\zeta}^{nk}-1\right)}{\Gamma\left(\widetilde{D}_{nk}^{\text{sp}}+2\right)}\, {}_3F_2\left(-n, \widetilde{Q}_{nk}^{\text{sp}}, 2\widetilde{\zeta}^{nk}-1; 1+2\widetilde{\beta}^{nk}, \widetilde{D}_{nk}^{\text{sp}}+2; 1\right), \tag{44.3}$$

$$M_{4(n\widetilde{l}\widetilde{m})}^{\text{ps-hcy}}=\frac{\widetilde{D}_{nk}^{\text{sp}2}}{2\delta}\frac{\Gamma\left(2\widetilde{\beta}^{nk}+7/2\right)\Gamma\left(2\widetilde{\zeta}^{nk}-2\right)}{\Gamma\left(\widetilde{D}_{nk}^{\text{sp}}+3/2\right)}\,{}_{3}F_{2}\left(-n,\widetilde{Q}_{nk}^{\text{sp}},2\widetilde{\zeta}^{nk}-2;1+2\widetilde{\beta}^{nk},\widetilde{D}_{nk}^{\text{sp}}+3/2;1\right), \tag{44.4}$$

$$M_{5(n\widetilde{l}\widetilde{m})}^{\text{ps-hcy}}=\frac{\widetilde{D}_{nk}^{\text{sp}2}}{2\delta}\frac{\Gamma\left(2\widetilde{\beta}^{nk}+4\right)\Gamma\left(2\widetilde{\zeta}^{nk}-3\right)}{\Gamma\left(\widetilde{D}_{nk}^{\text{sp}}+1\right)}\,{}_{3}F_{2}\left(-n,\widetilde{Q}_{nk}^{\text{sp}},2\widetilde{\zeta}^{nk}-3;1+2\widetilde{\beta}^{nk},\widetilde{D}_{nk}^{\text{sp}}+1;1\right), \tag{44.5}$$

$$M_{6(n\widetilde{l}\widetilde{m})}^{\text{ps-hcy}}=\frac{\widetilde{D}_{nk}^{\text{sp}2}}{2\delta}\frac{\Gamma\left(2\widetilde{\beta}^{nk}+2\right)\Gamma\left(2\widetilde{\zeta}^{nk}-4\right)}{\Gamma\left(\widetilde{D}_{nk}^{\text{sp}}-2\right)}\,{}_{3}F_{2}\left(-n,\widetilde{Q}_{nk}^{\text{sp}},2\widetilde{\zeta}^{nk}-4;1+2\widetilde{\beta}^{nk},\widetilde{D}_{nk}^{\text{sp}};1\right), \tag{44.6}$$

and

$$M_{7(n\widetilde{l}\widetilde{m})}^{\text{ps-hcy}}=\frac{\widetilde{D}_{nk}^{\text{sp}2}}{2\delta}\frac{\Gamma\left(2\widetilde{\beta}^{nk}+3/2\right)\Gamma\left(2\widetilde{\zeta}^{nk}-2\right)}{\Gamma\left(\widetilde{D}_{nk}^{\text{sp}}+1/2\right)}\,{}_{3}F_{2}\left(-n,\widetilde{Q}_{nk}^{\text{sp}},2\widetilde{\zeta}^{nk}-2;1+2\widetilde{\beta}^{nk},D_{nk}^{\text{sp}}+1/2;1\right) \tag{44.7}$$

with $\widetilde{Q}_{nk}^{\text{sp}}$ and $\widetilde{D}_{nk}^{\text{sp}}$ are taken the values $2\widetilde{\beta}^{nk}+2\widetilde{\zeta}^{nk}+n$ and $2\widetilde{\beta}^{nk}+2\widetilde{\zeta}^{nk}$, respectively.

D. The corrected energy for the IHCYPCTI in extended RQM symmetries

The global corrected relativistic energy for the IHCYPCTI model in extended RQM symmetries is composed of three principal parts. The first one is generated from the effect of the perturbed spin-orbit effective potentials $\Sigma_{\text{hcy}}^{\text{pert}}(r)$ and $\Delta_{\text{hcy}}^{\text{pert}}(r)$ corresponds to spin symmetry and pseudospin symmetry. These perturbed effective potentials are obtained by replacing the coupling of the angular momentums ($\mathbf{L}$ and $\widetilde{\mathbf{L}}$) operators and the NC vector Θ with the new equivalent couplings $\Theta\mathbf{L}S$ and $\Theta\widetilde{\mathbf{L}}\widetilde{\mathbf{S}}$ for spin-symmetry and p-spin-symmetry, respectively(with $\Theta^2=\Theta_{12}^2+\Theta_{23}^2+\Theta_{13}^2$). This degree of freedom is possible considering that the infinitesimal NC vector Θ is arbitrary. We have oriented the spins-$\left(\mathbf{S},\widetilde{\mathbf{S}}\right)$ of the fermionic particles to become parallel to the vector Θ which interacted with the improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction. Moreover, we replace the new spin-orbit couplings $\Theta\mathbf{L}S$ and $\Theta\widetilde{\mathbf{L}}\widetilde{\mathbf{S}}$ with the corresponding physical form $(\Theta/2)\,\mathbf{G}^2$ and $(\Theta/2)\,\widetilde{\mathbf{G}}^2$, with $\mathbf{G}^2=\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2$ and $\widetilde{\mathbf{G}}^2=\mathbf{J}^2-\widetilde{\mathbf{L}}^2-\widetilde{\mathbf{S}}^2$ for spin-symmetry and p-spin-symmetry, respectively. Furthermore, in RQM, the operators ($\widehat{H}_{rnc}^{\text{hcy}}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physics quantities, the eigenvalues of the operators $\mathbf{G}^2$ and $\widetilde{\mathbf{G}}^2$ are equal to the values $F(j,l,y)=[j(j+1)-l(l+1)-3/4)]/2$ and $F\left(j,\widetilde{l},y\right)=\left[j(j+1)-\widetilde{l}(\widetilde{l}-1)-3/4)\right]/2$, with $|l-1/2|\leq j\leq|l+1/2|$ and $\left|\widetilde{l}-1/2\right|\leq j\leq\left|\widetilde{l}+1/2\right|$ for spin-symmetry and p-spin-symmetry, respectively. As a direct consequence, the partially corrected energies $\Delta E_{\text{hcy}}^{\text{so-sp}}(n,\delta,A,B,H,\Theta,j,l,s)\equiv\Delta E_{\text{hcy}}^{\text{so-sp}}$ and $\Delta E_{\text{hcy}}^{\text{so-ps}}\left(n,\delta,A,B,H,\Theta,j,\widetilde{l},\widetilde{s}\right)\equiv\Delta E_{\text{hcy}}^{\text{so-ps}}$ due to the perturbed effective potentials $\Sigma_{\text{hcy}}^{\text{pert}}(r)$ and $\Delta_{\text{hcy}}^{\text{pert}}(r)$ produced for the n^{th} excited state, in extended RQM symmetries are as follows:

$$\begin{cases}\Delta E_{\text{hcy}}^{\text{so-sp}}=\Theta\left(j(j+1)-k(k+1)-\frac{3}{4}\right)\\ \langle Z\rangle_{(nlm)}^{\text{rhcy}}(n,\delta,A,B,H)\,,\\ \Delta E_{\text{hcy}}^{\text{so-ps}}=\Theta\left(j(j+1)-k(k-1)-\frac{3}{4}\right)\\ \left\langle\widetilde{Z}\right\rangle_{(nlm)}^{\text{rhcy}}(n,\delta,A,B,H)\,.\end{cases} \tag{45}$$

The global two expectation values $\langle Z\rangle_{(nlm)}^{\text{hcy}}(n,\delta,A,B,H)$ and $\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\text{hcy}}(n,\delta,A,B,H)$ for spin-symmetry and p-spin-symmetry, respectively are determined from the following expressions:

$$\begin{cases}\langle Z\rangle_{(nlm)}^{\text{hcy}}(n,\delta,A,B,H)=\sum\limits_{\mu=1}^{7}L_{nk}^{\mu\text{sp}}M_{\mu(nlm)}^{\text{sp-hcy}},\\ \left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\text{hcy}}(n,\delta,A,B,H)=\sum\limits_{\mu=1}^{7}L_{nk}^{\mu\text{ps}}M_{\mu(n\widetilde{l}\widetilde{m})}^{\text{ps-hcy}},\end{cases} \tag{46}$$

where $L_{nk}^{\mu\text{sp}}$ and $L_{nk}^{\mu\text{ps}}$are determined from Eqs. (37) and (38), while $M_{\mu(nlm)}^{\text{sp-hcy}}$ and $M_{\mu(n\widetilde{l}\widetilde{m})}^{\text{ps-hcy}}$ $\left(\mu=\overline{1,7}\right)$ are determined from Eqs. $(42,i=\overline{1,7})$ and Eqs. $(44,i=\overline{1,7}$), respectively. The second main part is obtained from the magnetic effect of the perturbative effective potentials $\Sigma_{\text{hcy}}^{\text{pert}}(r)$ and $\Delta_{\text{hcy}}^{\text{pert}}(r)$ under the IHCYPCTI model in the deformation of the Dirac theory symmetries. These effective potentials are achieved when we replace both ($\mathbf{L}\Theta$ and Θ_{12}) with ($\sigma\aleph L_z$ and $\sigma\aleph$), respectively; here ($\aleph$ and σ) symbolize the intensity of the magnetic field induced by the effect of the deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter Θ_{12} (length)^2 is the same unit of $\sigma\aleph$. We also need to apply $\langle n', l', m' \; L_z \; n, l, m\rangle = m\delta_{m'm}\delta_{l'l}\delta_{n'n}$ and $\left\langle n', \widetilde{l'}, \widetilde{m'} \; \widetilde{L}_z \; n, \widetilde{l}, \widetilde{m}\right\rangle = \widetilde{m}\delta_{\widetilde{m'}\widetilde{m}}\delta_{\widetilde{l'}\widetilde{l}}\delta_{n'n}$ ($-\widetilde{l} \le \widetilde{m} \le \widetilde{l}$ and $-l \le m \le l$) for spin-symmetry and p-spin-symmetry, respectively. All of these data allow for the discovery of the new energy shift $\Delta E_{\text{hcy}}^{\text{mg-sp}}(n, \delta, A, B, H, \sigma, m)$ and $\Delta E_{\text{hcy}}^{\text{mg-ps}}(n, \delta, A, B, H, \sigma, \widetilde{m})$ due to the perturbed Zeeman effect created by the influence of the improved Hulthén plus a class of Yukawa potential model for the n^{th} excited state in extended RQM symmetries as follows:

$$\begin{cases} \Delta E_{\text{hcy}}^{\text{mg-sp}}(n, \delta, A, B, H, \sigma, m) \\ = \sigma\aleph \langle Z\rangle_{(nlm)}^{\text{hcy}}(n, \delta, A, B, H)\, m, \\ \Delta E_{\text{hcy}}^{\text{mg-ps}}(n, \delta, A, B, H, \sigma, \widetilde{m}) \\ = \sigma\aleph \left\langle \widetilde{Z}\right\rangle_{(nl\widetilde{m})}^{\text{hcy}}(n, \delta, A, B, H)\, \widetilde{m}. \end{cases} \tag{47}$$

Now, for our purposes, we are interested in finding a new third automatically important symmetry under the IHCYPCTI model in DDT symmetries. This physical phenomenon is induced automatically from the influence of perturbed effective potentials $\Sigma_{\text{hcy}}^{\text{pert}}(r)$ and $\Delta_{\text{hcy}}^{\text{pert}}(r)$, which we have seen in Eqs. (36) and (37). We consider that the fermionic particles undergo rotation with angular velocity $\mathbf{\Omega}$ if we make the following two simultaneous transformations to ensure that the previous calculations are not repeated:

$$\begin{pmatrix} \mathbf{L}\Theta \\ \widetilde{\mathbf{L}}\Theta \end{pmatrix}$$

will be replace by:

$$\chi \begin{pmatrix} \mathbf{L\Omega}\text{: for spin-sy} \\ \widetilde{\mathbf{L}}\mathbf{\Omega}\text{: for p-spin-sy} \end{pmatrix}. \tag{48}$$

Here χ is just an infinitesimal real proportional constant. We can express the effective potential $\Sigma_{\text{pert}}^{\text{sp-rot}}(s)$ and $\Delta_{\text{pert}}^{\text{ps-rot}}(s)$ which induced the rotational movements of the fermionic particles as follows:

$$\begin{pmatrix} \Sigma_{\text{pert}}^{\text{sp-rot}}(s) \\ \Delta_{\text{pert}}^{\text{ps-rot}}(s) \end{pmatrix} = \begin{pmatrix} \chi \left(\sum_{\mu=1}^{7} L_{nk}^{\mu\text{sp}} M_{\mu(nlm)}^{\text{sp-hcy}}\right) \mathbf{L\Omega} \\ \chi \left(\sum_{\mu=1}^{7} L_{nk}^{\mu\text{sp}} M_{\mu(nlm)}^{\text{ps-hcy}}\right) \widetilde{\mathbf{L}}\mathbf{\Omega} \end{pmatrix}. \tag{49}$$

To simplify the calculations without compromising physical content, we choose the rotational velocity $\mathbf{\Omega}$ parallel to the (Oz) axis. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\begin{pmatrix} \Sigma_{\text{pert}}^{\text{sp-rot}}(s)\, \mathbf{L\Omega} \\ \Delta_{\text{pert}}^{\text{ps-rot}}(s)\, \widetilde{\mathbf{L}}\mathbf{\Omega} \end{pmatrix} = \chi\Omega \begin{pmatrix} \Sigma_{\text{pert}}^{\text{sp-rot}}(s)\, L_z \\ \Delta_{\text{pert}}^{\text{ps-rot}}(s)\, \widetilde{L}_z \end{pmatrix}. \tag{50}$$

All of these data allow for the discovery of the new corrected energy $\Delta E_{\text{hcy}}^{\text{rot-sp}}(n, \delta, A, B, H, \chi, m)$ and $\Delta E_{\text{hcy}}^{\text{rot-sp}}(n, \delta, A, B, H, \chi, \widetilde{m})$ due to the perturbed effective potentials $\Sigma_{\text{pert}}^{\text{sp-rot}}(s)$ and $\Delta_{\text{pert}}^{\text{ps-rot}}(s)$, which is generated automatically by the influence of the improved Hulthén plus a class of Yukawa potential for the n^{th} excited state in DDT symmetries as follows:

$$\begin{pmatrix} \Delta E_{\text{hcy}}^{\text{rot-sp}}(n, \delta, A, B, H, \chi, m) \\ \Delta E_{\text{hcy}}^{\text{rot-sp}}(n, \delta, A, B, H, \chi, \widetilde{m}) \end{pmatrix} = \chi\Omega \begin{pmatrix} \langle Z\rangle_{(nlm)}^{\text{hcy}}(nn, \delta, A, B, H)\, m \\ \left\langle \widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\text{hcy}}(n, \delta, A, B, H)\, \widetilde{m} \end{pmatrix}. \tag{51}$$

It is worth mentioning that the authors in Ref. [80] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two and three-dimensional space at zero temperature, but in this study, the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotation term $\Sigma_{\text{pert}}^{\text{sp-rot}}(s)\,\mathbf{L\Omega}$ and $\Delta_{\text{pert}}^{\text{ps-rot}}(s)\,\widetilde{\mathbf{L}}\mathbf{\Omega}$ automatically appears due to the large symmetries resulting from the deformation of space-space.

We have seen that the eigenvalues of the operators $\mathbf{G}^2 = \mathbf{J}^2 - \mathbf{L}^2 - S^2$ and $\widetilde{\mathbf{G}}^2 = \mathbf{J}^2 - \widetilde{\mathbf{L}}^2 - \widetilde{\mathbf{S}}^2$ are equal to the values $\digamma(j, l, s) = [j(j+1) - l(l+1) - 3/4]/2$ and $\digamma\left(j, \widetilde{l}, \widetilde{s}\right) = \left[j(j+1) - \widetilde{l}(\widetilde{l}-1) - 3/4\right]/2$; thus, for the case of spin-1/2, the possible values of j are $l \pm 1/2$ and $\widetilde{l} \pm 1/2$ for spin symmetry $\digamma(j, l, s)$ and pseudospin symmetry $\digamma\left(j, \widetilde{l}, \widetilde{s}\right)$, which allows us to get their values as follows:

$$\digamma(j = l \pm 1/2, s = 1/2) = \frac{1}{2}\begin{cases} l & \text{Up polarity: } j = l + 1/2 \\ -(l+1) & \text{Down polarity: } j = l - 1/2 \end{cases} \tag{52}$$

and

$$\digamma\left(j = \widetilde{l} \pm 1/2, \widetilde{s} = 1/2\right) = \frac{1}{2}\begin{cases} \widetilde{l} & \text{Up polarity: } j = \widetilde{l} + 1/2 \\ -\left(\widetilde{l}+1\right) & \text{Down polarity: } j = \widetilde{l} - 1/2. \end{cases} \tag{53}$$

The new relativistic energy $E_{\mathrm{nc}}^{\mathrm{sp}}(n,\delta,A,B,\Theta,\sigma,\chi,j,l,s,m)$ and $E_{\mathrm{nc}}^{\mathrm{ps}}(n,\delta,A,B,\Theta,\sigma,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m})$ for the case of spin-1/2 with the improved Hulthén plus a class of Yukawa potential, in the DDT symmetries, corresponding to the generalized n^{th} excited states is:

$$E_{\mathrm{nc}}^{\mathrm{sp}}\left(n,\delta,A,B,H,\Theta,\sigma,\chi,j,l,s,m\right)=\langle Z\rangle_{(nlm)}^{\mathrm{hcy}}\left(n,\delta,A,B,H\right)\left(\sigma\aleph+\chi\Omega\right)m + \langle Z\rangle_{(nlm)}^{\mathrm{sp}}\left(n,\delta,A,B,H\right)\frac{\Theta}{2}\begin{cases} l \quad \text{Up polarity: } j=l+1/2 \\ -\left(l+1\right) \text{ Down polarity: } j=\ l-1/2\end{cases}+E_{nk}^{\mathrm{sp}} \tag{54}$$

and

$$E_{\mathrm{nc}}^{\mathrm{ps}}\left(n,\delta,A,B,H,\Theta,\sigma,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m}\right)=\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\mathrm{ps}}\left(n,\delta,A,B,H\right)\left(\sigma\aleph+\chi\Omega\right)\widetilde{m} + \left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\mathrm{ps}}\left(n,\delta,A,B,H\right)\frac{\Theta}{2}\begin{cases} \widetilde{l} \quad \text{Up polarity: } j=\widetilde{l}+1/2 \\ -\left(\widetilde{l}+1\right) \text{ Down polarity: } j=\ \widetilde{l}-1/2\end{cases}+E_{nk}^{\mathrm{ps}}, \tag{55}$$

where E_{nk}^{sp} and E_{nk}^{ps} are usual relativistic energies under the Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction obtained from the equations of energy in Eqs. (16) and (17), while k and $\widetilde{k}$ are determined from the following relations:

$$k=\begin{cases} k_1=-\left(l+1\right)=l+1/2 \\ \text{For } s_{1/2},\ p_{3/2}\ldots, \text{ etc.} \\ j=l+1/2 \text{ Aligned spin } k<0, \\ k_1=-\left(l+1\right)=l+1/2 \\ \text{For } s_{1/2},\ p_{3/2}\ldots, \text{ etc.} \\ j=l-1/2 \ \text{ Aligned spin } \ k>0,\end{cases} \tag{56.1}$$

and

$$\widetilde{k}=\begin{cases} k_1=-\widetilde{l}=-j-1/2 \\ \text{For } s_{1/2},\ p_{3/2}\ldots, \text{ etc.} \\ j=\widetilde{l}-1/2 \text{ Aligned spin } \ \widetilde{k}<0, \\ k_1=-l-1=l+1/2 \\ \text{For } s_{1/2},\ p_{3/2}\ldots, \text{ etc.} \\ j=\widetilde{l}-1/2 \ \text{ Un aligned spin } \ \widetilde{k}>0.\end{cases} \tag{56.2}$$

E. Study of relativistic particular cases:

In this section, we are about to examine some particular cases regarding the new bound state energy eigenvalues in Eqs. (54) and (55). We could derive some particular potentials, useful for other physical systems, by adjusting relevant parameters of the IHCYPCTI model in the deformation of the Dirac theory symmetries in both cases, such as the improved Dirac–Hulthén problem model, the improved Hulthén potential, the improved Dirac–Yukawa problem, the improved Dirac–Coulomb-like problem, the improved Dirac–inversely quadratic Yukawa problem, and the improved Dirac–Kratzer–Fues problem.

F. Deformed Dirac–improved Hulthén potential

For $A=B=0$, the IHCYPCTI model turns into the improved Hulthén potential $V_{\mathrm{ihp}}\left(r\right)$ in the deformation of theory symmetries:

$$V_{\mathrm{ihp}}\left(r\right)=-\frac{Ze^2\delta\exp\left(-\delta r\right)}{1-\exp\left(-\delta r\right)} - \left[\begin{array}{c}\frac{4Ze^2\delta^2\exp(-2\delta r)}{(1-\exp(-2\delta r))} \\ +\frac{Ze^2\delta^2\exp(-4\delta r)}{(1-\exp(-2\delta r))^2}\end{array}\right]\frac{\mathbf{L}\Theta}{2r}+O\left(\Theta^2\right). \tag{57}$$

It should be noted that the first term denotes the Hulthén potential in usual relativistic QM symmetries which are used in atomic physics, chemical physics, and solid-state physics [81]. Using Eqs. (54) and (55), we make the corresponding parameter replacements and obtain the energy equation for the improved Hulthén potential including the improved Yukawa tensor interaction in the spin and p-spin symmetries of the deformed Dirac theory as:

$$E_{\mathrm{nc}}^{\text{sp-ihp}}\left(n,\delta,H,\Theta,\sigma,\chi,j,l,s,m\right)=\langle Z\rangle_{(nlm)}^{\mathrm{sp}}\left(n,\delta,H\right)\left(\sigma\aleph+\chi\Omega\right)m + \langle Z\rangle_{(nlm)}^{\text{sp-hp}}\left(n,\delta,H\right)\frac{\Theta}{2}\begin{cases} l \quad \text{Up polarity: } j=l+1/2 \\ -\left(l+1\right) \text{ Down polarity: } j=\ l-1/2\end{cases}+E_{nk}^{\text{sp-hp}} \tag{58}$$

and

$$E_{\rm nc}^{\text{ps-ihp}}\left(n,\delta,H,\Theta,\sigma,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m}\right)=\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\rm ps}(n,\delta,H)\left(\sigma\aleph+\chi\Omega\right)\widetilde{m}$$
$$+\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\text{ps-ihp}}(n,\delta,H)\,\frac{\Theta}{2}\begin{cases}\widetilde{l} & \text{Up polarity: } j=\widetilde{l}+1/2\\ -\left(\widetilde{l}+1\right) & \text{Down polarity: } j=\widetilde{l}-1/2\end{cases}+E_{nk}^{\text{ps-hp}}, \tag{59}$$

where $E_{nk}^{\text{sp-hp}}$ and $E_{nk}^{\text{ps-hp}}$ are the energy equations for the Hulthén potential including the Yukawa tensor interaction in the spin and p-spin symmetries of the Dirac theory as obtained from Eqs. (16) and (17) by replacing each $A=B=0$ from[9] and directly from [1]. The new expectation values $\langle Z\rangle_{(nlm)}^{\rm sp}(n,\delta,H)$ and $\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\rm ps}(n,\delta,H)$ are determined from Eqs. (49) and (50) by applying the compensation referred to above at the beginning of the current subsection as follows:

$$\begin{cases}\langle Z\rangle_{(nlm)}^{\text{sp-hp}}(n,\delta,H)=\sum\limits_{\mu=1}^{7}B_{nk}^{\mu\rm sp}M_{\mu(nlm)}^{\text{sp-hp}}\,,\\ \left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\text{ps-hp}}(n,\delta,H)=\sum\limits_{\mu=1}^{7}A_{nk}^{\mu\rm ps}M_{\mu(n\widetilde{l}\widetilde{m})}^{\text{ps-hp}}\end{cases}$$

with $B_{nk}^{1\rm sp}=-4Ze^2\delta^3\left(M+E_{nk}^{\rm sp}-C_{\rm sp}\right)$, $B_{nk}^{2\rm sp}=-4\delta^3Ze^2\left(M+E_{nk}^{\rm sp}-C_{\rm sp}\right)$, $B_{nk}^{3\rm sp}=8A\delta^3\left(M+E_{nk}^{\rm sp}-C_{\rm sp}\right)$,$B_{nk}^{4\rm sp}=B_{nk}^{5\rm sp}=B_{nk}^{4\rm ps}=B_{nk}^{5\rm ps}=0$, $B_{nk}^{6\rm sp}=16k\left(k+1\right)\delta^4$, $B_{nk}^{\rm sp}=B_{nk}^{7\rm ps}=-4\delta^3H$ while ${}_{nk}^{1\rm ps}=-4Ze^2\delta^3\left(M-E_{nk}^{\rm ps}+C_{\rm ps}\right)$, $B_{nk}^{2\rm ps}=-4\delta^3Ze^2\left(M-E_{nk}^{\rm ps}+C_{\rm ps}\right)$ and $B_{nk}^{6\rm ps}=16k\left(k-1\right)\delta^4$.

G. Deformed Dirac-improved Yukawa problem

If we replace $B=0$, the IHCYPCTI model turns into the improved Yukawa potential $V_{\rm ihp}(r)$ in the deformation of the Dirac theory symmetries:

$$V_{\rm iyp}(r)=-\frac{Ae^{-\delta r}}{r}-A\left[\frac{e^{-2\delta r}}{r^2}-\frac{2\delta e^{-2\delta r}}{r}\right]\frac{\mathbf{L}\Theta}{2r}+O\left(\Theta^2\right). \tag{61}$$

It should be noted that the first term denotes the Yukawa potential in usual QM symmetries used in nuclear physics, atomic physics, solid-state physics, and astrophysics[82]. Using Eqs. (54) and (55), we make the corresponding parameter replacements and obtain the energy equation for the improved Yukawa potential including the improved Yukawa tensor interaction in the spin and p-spin symmetries of the deformed Dirac theory as:

$$E_{\rm nc}^{\text{sp-iyp}}(n,\delta,A,H,\Theta,\sigma,\chi,j,l,s,m)=\langle Z\rangle_{(nlm)}^{\rm sp}(n,\delta,A,H)\left(\sigma\aleph+\chi\Omega\right)m$$
$$+\langle Z\rangle_{(nlm)}^{\rm sp}(n,\delta,A,H)\frac{\Theta}{2}\begin{cases}l & \text{Up polarity: } j=l+1/2\\ -(l+1) & \text{Down polarity: } j=l-1/2\end{cases}+E_{nk}^{\text{sp-yp}} \tag{62}$$

and

$$E_{\rm nc}^{\text{ps-iyp}}\left(n,\delta,A,H,\Theta,\sigma,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m}\right)=\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\rm ps}(n,\delta,A,H)\left(\sigma\aleph+\chi\Omega\right)\widetilde{m}$$
$$+\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\rm ps}(n,\delta,A,H)\,\frac{\Theta}{2}\begin{cases}\widetilde{l} & \text{Up polarity: } j=\widetilde{l}+1/2\\ -\left(\widetilde{l}+1\right) & \text{Down polarity: } j=\widetilde{l}-1/2\end{cases}+E_{nk}^{\text{ps-hp}}, \tag{63}$$

where $E_{nk}^{\text{sp-yp}}$and $E_{nk}^{\text{ps-yp}}$ are the energy equation for the Yukawa potential including the Yukawa tensor interaction in the spin and p-spin symmetries of the Dirac theory as obtained from Eqs. (16) and (17) by replacing each $B=0$; it is detailed in the two Eqs. (4.9) and (4.10) in Refs. [81, 82], while the new expectation values $\langle Z\rangle_{(nlm)}^{\text{sp-yp}}(n,\delta,A,H)$ and $\left\langle\widetilde{Z}\right\rangle_{(n\widetilde{l}\widetilde{m})}^{\text{ps-yp}}(n,\delta,A,H)$ are determined from Eqs. (49) and (50) by applying the compensation referred to above at the beginning of the current subsection as follows:

$$\begin{cases} \langle Z\rangle^{\text{hcy}}_{(nlm)}(n,\delta,A,H)=\sum_{\mu=1}^{7}B^{\mu\text{sp}}_{nk}M^{\text{sp-yp}}_{\mu(nlm)}, \\ \left\langle \widetilde{Z}\right\rangle^{hp}_{(n\widetilde{l}\widetilde{m})}(n,\delta,A,H)=\sum_{\mu=1}^{7}B^{\mu\text{ps}}_{nk}M^{\text{ps-hp}}_{\mu(n\widetilde{l}\widetilde{m})} \end{cases} \tag{64}$$

with $A^{1\text{sp}}_{nk}=-4Ze^2\delta^3(M+E^{\text{sp}}_{nk}-C_{\text{sp}})$, $A^{2\text{sp}}_{nk}=-4\delta^3\left(Ze^2+A\right)(M+E^{\text{sp}}_{nk}-C_{\text{sp}})$, $A^{3\text{sp}}_{nk}=L^{4\text{sp}}_{nk}=L^{5\text{sp}}_{nk}=A^{3\text{ps}}_{nk}=A^{4\text{ps}}_{nk}=A^{5\text{ps}}_{nk}=0$, $A^{6\text{sp}}_{nk}=16k(k+1)\delta^4$ and $A^{\text{sp}}_{nk}=L^{7\text{ps}}_{nk}=-4\delta^3H$ while $A^{1\text{ps}}_{nk}=-4Ze^2\delta^3(M-E^{\text{ps}}_{nk}+C_{\text{ps}})$, $A^{2\text{ps}}_{nk}=-4\delta^3\left(Ze^2+A\right)(M-E^{\text{ps}}_{nk}+C_{\text{ps}})$ and $L^{6\text{ps}}_{nk}=16k(k-1)\delta^4$.

H. Deformed Dirac-improved Coulomb-like problem

If we replace $B=0$, we ignore the Hulthén potential, and taking the limit $\delta\to 0$, the IHCYPCTI model turns into the improved Coulomb-like potential $V_{\text{icp}}(r)$ in the deformation of the Dirac theory symmetries:

$$V_{\text{iyp}}(r)=-\frac{A}{r}-\frac{A}{2r^3}\mathbf{L}\Theta+O\left(\Theta^2\right). \tag{65}$$

It should be noted that the first term denotes the Coulomb-like potential in usual QM symmetries. Using Eqs. (54) and (55), we make the corresponding parameter replacements and obtain the energy equation for the improved Coulomb-like potential including the improved Yukawa tensor interaction in the spin and p-spin symmetries of the deformed Dirac theory as:

$$E^{\text{sp-icp}}_{\text{nc}}=\langle Z\rangle^{\text{sp-cp}}_{(nlm)}(n,\delta,A,H)\frac{\Theta}{2}\begin{cases} l \quad \text{Up polarity: } j=l+1/2 \\ -(l+1) \ \text{Down polarity: } j=\ l-1/2\end{cases}$$
$$+\langle Z\rangle^{\text{sp-cp}}_{(nlm)}(n,\delta,A,H)(\sigma\aleph+\chi\Omega)m+\frac{A^2(C_{\text{sp}}-m_e)+4m_e(n+k+H+1)^2}{A^2+4(n+k+H+1)^2} \tag{66}$$

and

$$E^{\text{ps-icp}}_{\text{nc}}=\left\langle \widetilde{Z}\right\rangle^{\text{ps-cp}}_{(n\widetilde{l}\widetilde{m})}(n,\delta,A,H)\frac{\Theta}{2}\begin{cases} \widetilde{l} \quad \text{Up polarity: } j=\widetilde{l}+1/2 \\ -\left(\widetilde{l}+1\right) \ \text{Down polarity: } j=\ \widetilde{l}-1/2\end{cases}$$
$$+\left\langle \widetilde{Z}\right\rangle^{\text{ps-cp}}_{(n\widetilde{l}\widetilde{m})}(n,\delta,A,H)(\sigma\aleph+\chi\Omega)\widetilde{m}+\frac{A^2(C_{\text{ps}}+m_e)-4m_e(n+k+H+1)^2}{A^2-4(n+k+H+1)^2}. \tag{67}$$

The last two terms in Eqs. (66) and (67) are the energy equation for the Coulomb-like potential including the Yukawa tensor interaction in the spin and p-spin symmetries of the Dirac theory obtained from Eqs. (16) and (17) by replacing each $B=0$. It is detailed in the two Eqs. (4.9) and (4.10) in Ref. [85] while the new expectation values $\langle Z\rangle^{\text{sp-yp}}_{(nlm)}(n,\delta,A,H)$and $\left\langle \widetilde{Z}\right\rangle^{\text{ps-yp}}_{(n\widetilde{l}\widetilde{m})}(n,\delta,A,H)$ are determined from Eqs. (49) and (50) by applying the compensation referred to above at the beginning of the current subsection as follows:

$$\begin{cases} \langle Z\rangle^{\text{hcy}}_{(nlm)}(n,\delta,A,H)=\sum_{\mu=1}^{7}B^{\mu\text{sp}}_{nk}M^{\text{sp-yp}}_{\mu(nlm)}, \\ \left\langle \widetilde{Z}\right\rangle^{\text{hp}}_{(n\widetilde{l}\widetilde{m})}(n,\delta,A,H)=\sum_{\mu=1}^{7}B^{\mu\text{ps}}_{nk}M^{\text{ps-hp}}_{\mu(n\widetilde{l}\widetilde{m})} \end{cases} \tag{68}$$

with $B^{1\text{sp}}_{nk}=-4Ze^2\delta^3(M+E^{\text{sp}}_{nk}-C_{\text{sp}})$, $B^{2\text{sp}}_{nk}=-4\delta^3\left(Ze^2+A\right)(M+E^{\text{sp}}_{nk}-C_{\text{sp}})$, $B^{3\text{sp}}_{nk}=B^{4\text{sp}}_{nk}=B^{5\text{sp}}_{nk}=B^{3\text{ps}}_{nk}=B^{4\text{ps}}_{nk}=B^{5\text{ps}}_{nk}=0$, $B^{6\text{sp}}_{nk}=16k(k+1)\delta^4$ and $B^{\text{sp}}_{nk}=B^{7\text{ps}}_{nk}=-4\delta^3H$ while $B^{1\text{ps}}_{nk}=-4Ze^2\delta^3(M-E^{\text{ps}}_{nk}+C_{\text{ps}})$, $B^{2\text{ps}}_{nk}=-4\delta^3\left(Ze^2+A\right)(M-E^{\text{ps}}_{nk}+C_{\text{ps}})$ and $B^{6\text{ps}}_{nk}=16k(k-1)\delta^4$.

I. Deformed Schrödinger equation for improved Hulthén plus a class of Yukawa potential problems in NREQM symmetries

To realize a study of the nonrelativistic limit, in extended nonrelativistic quantum mechanics of the improved Hulthén plus a class of Yukawa potential, two steps must be made. The first step corresponds to the nonrelativistic limit, in usual nonrelativistic quantum energy which is indicated in Ref. [9]. This is done by taking the following steps: we replace H, C_{sp}, $E^{\text{ps}}_{nk}+\mu$, $E^{\text{ps}}_{nk}-\mu$, $2A$, $2B$, $k(k+1)$, $F_{nk}(r)$ by0, 0, 2μ, E^{nr}_{nl}, A, B, $l(l+1)$, $R_{nk}(r)$, respectively, which allows us to obtain the nonrelativistic energy levels as [9]:

$$E_{nl}^{\text{hcy}} = -\frac{1}{2\mu}\left[\frac{-\mu\frac{A+Ze^2}{\delta} + (2n+1)\lambda_{nl} + \gamma_{nl}}{n+1/2+\lambda_{nl}}\right]^2 \quad (69)$$

with $\lambda_{nl}(B) = \sqrt{l^2+l+1/4-2\mu B}$ and $\gamma_{nl} = l(l+1) + n(n+1) + 1/2$. Now, the second step corresponds to the coefficients $L_{nk}^{\text{isp}}\left(i=\overline{1,7}\right)$ in relativistic spin symmetry, those that convert to the new formula R_{nk}^{inr} by applying the data we referred to as:

$$\begin{cases} R_{nk}^{1\text{nr}} = -8Ze^2\delta^3\mu, \\ R_{nk}^{2\text{nr}} = -8\delta^3\left(Ze^2+A\right)\mu, \\ R_{nk}^{3\text{nr}} = 16A\delta^3\mu, \\ R_{nk}^{4\text{nr}} = -16B\delta\mu, \\ R_{nk}^{5\text{nr}} = -32B\delta^4\mu, \\ R_{nk}^{6\text{nr}} = 16l(l+1)\delta^4 \text{ and } R_{nk}^{\text{sp}} = 0. \end{cases} \quad (70)$$

We can reexport the relativistic expectation values $\langle Z\rangle_{(nlm)}^{\text{hcy}}(n,\delta,A,B,H)$ of spin symmetry in Eq. (46) from the corresponding nonrelativistic expectation values $\langle Z\rangle_{(nlm)}^{\text{hcy-nr}}(n,\delta,A,B,H)$ as:

$$\langle Z\rangle_{(nlm)}^{\text{hcy-nr}}(n,\delta,A,B,H) = \sum_{\mu=1}^{7} R_{nk}^{\mu\text{sp}} M_{\mu(nlm)}^{\text{sp-hcy}}. \quad (71)$$

This permuted expressing the nonrelativistic correction energy $\Delta E_{\text{nc-nr}}^{\text{hcy}}(n,\delta,A,B,H,\Theta,\sigma,\chi,j,l,s,m)$ produced by the improved Hulthén plus a class of Yukawa potential problems is:

$$\Delta E_{\text{nc-nr}}^{\text{hcy}}(n,\delta,A,B,H,\Theta,\sigma,\chi,j,l,s,m) = \langle Z\rangle_{(nlm)}^{\text{hcy-nr}}(n,\delta,A,B,H)(\sigma\aleph+\chi\Omega)m + \langle Z\rangle_{(nlm)}^{\text{hcy-nr}}(n,\delta,A,B,H)\frac{\Theta}{2} \begin{cases} l & \text{Up polarity: } j=l+1/2, \\ -(l+1) & \text{Down polarity: } j = l-1/2. \end{cases} \quad (72)$$

The global nonrelativistic energy $E_{\text{nc-nr}}^{\text{hcy}}(n,\delta,A,B,H,\Theta,\sigma,\chi,j,l,s,m)$ produced with the improved Hulthén plus a class of Yukawa potential in ENRQM symmetries as a result of the topological properties of deformation space-space is the sum of usual energy E_{nl}^{hcy} in Eq. (69) under the Hulthén plus a class of Yukawa potential in NRQM symmetries and the obtained correction $\Delta E_{\text{nc-nr}}^{\text{hcy}}(n,\delta,A,B,H,\Theta,\sigma,\chi,j,l,s,m)$ in Eq. (72) as follows:

$$E_{\text{nc-nr}}^{\text{hcy}}(n,\delta,A,B,H,\Theta,\sigma,\chi,j,l,s,m) = -\frac{1}{2\mu}\left(\frac{-\mu\frac{A+Ze^2}{\delta} + (2n+1)\lambda_{nl} + \gamma_{nl}}{n+1/2+\lambda_{nl}}\right)^2 + \langle Z\rangle_{(nlm)}^{\text{hcy-nr}}\left[(\sigma\aleph+\chi\Omega)m + \frac{\Theta}{2}\begin{cases} l & \text{Up polarity: } j=l+1/2 \\ -(l+1) & \text{Down polarity: } j = l-1/2 \end{cases}\right]. \quad (73)$$

It should be noted that the corrected energy $\Delta E_{\text{nc-nr}}^{\text{hcy}}$ expressed in Eq. (72) is due to the effect of the perturbed potential $V_{\text{nr-pert}}^{\text{hcy}}(r)$:

$$V_{\text{nr-pert}}^{\text{hcy}}(r) = \left(l(l+1)r^{-4} - \frac{1}{2r}\frac{\partial V_{\text{hcy}}(r)}{\partial r}\right) \times \mathbf{L}\Theta + O\left(\Theta^2\right). \quad (74)$$

The first term in Eq. (74) is due to the centrifuge term $l(l+1)\widehat{r}^{-2}$ in ENRQM symmetries which equals the usual centrifuge term $l(l+1)r^{-2}$ plus the perturbative centrifuge term $l(l+1)r^{-4}\mathbf{L}\Theta$, while the second term is produced with the effect of the improved Hulthén plus a class of Yukawa potential. This is one of the most important new results of this research. It is worthwhile to mention that for the three simultaneous limits $(\Theta,\sigma,\chi)\to(0,0,0)$, we recover the results in Ref. [9].

IV. SUMMARY AND CONCLUSIONS

In this work, we have solved the deformed Dirac equation of a spin-1/2 particle in the field of the improved Hulthén plus a class of Yukawa potential including a Coulomb-like tensor interaction within the framework of the parametric of Bopp's shift method and standard perturbation theory with an approximation to the centrifugal term. By using a suitable approximation scheme, we have presented in detail the corrected energy eigenvalues for both the spin symmetry and pseudospin symmetry. The corrected eigenvalue appeared sensitive to the quantum numbers $(j, k, l, \widetilde{l}, s, \widetilde{s}, m, \widetilde{m})$, the potential depths (A, B) of the studied potential, the range of the potential δ, and noncommutativity parameters(Θ, σ, χ) under the condition of spin and pseudospin symmetry. Special cases of the potential are also reported. Our results could find usefulness in both atomic and molecular physics. Finally, the new relativistic spin symmetry in the absence of tensor interaction is reduced to the deformed Schrödinger solutions for the improved Hulthén plus a class of Yukawa potential. The present results are in excellent agreement with the previous result. It is worth mentioning that, for all cases, to make the three simultaneous limits $(\Theta, \sigma, \chi) \rightarrow (0, 0, 0)$, the ordinary physical quantities are recovered in refs. [9]. Finally, given the effectiveness of the methods that we used in achieving our goal in this research, we advise researchers to apply the same methods in other studies, whether in the relativ[-6pt]istic or nonrelativistic regimes for other potentials.

[1] M. Hamzavi, A. A. Rajabi, Chin. Phys. B **22**, 080302 (2013); https://doi.org/10.1088/1674-1056/22/8/080302.

[2] O. Aydoğdu, E. Maghsoodi, H. Hassanabadi, Chin. Phys. B **22**, 010302 (2013); https://doi.org/10.1088/1674-1056/22/1/010302.

[3] S. M. Ikhdair, R. Sever, Appl. Math. Comp. **216**, 911 (2010); https://doi.org/10.1016/j.amc.2010.01.104.

[4] S.M. Ikhdair, B.J. Falaye, Phys. Scr. **87**, 035002 (2013); https://doi.org/10.1088/0031-8949/87/03/035002.

[5] S.M. Ikhdair, M. Hamzavi, Few-Body Syst. **53**, 487 (2012); https://doi.org/10.1007/s00601-012-0475-2.

[6] A. N. Ikot, H. Hassanabadi, E. Maghsoodi, S. Zarrinkamer, Ukr. J. Phys. **58**, 915 (2013).

[7] T. O. Magu *et al.*, J. Niger. Assoc. Math. Phys. **42**, 1 (2016).

[8] A. N. Ikot *et al.*, Can. J. Phys. **91**, 560 (2013); https://doi.org/10.1139/cjp-2012-0506.

[9] A. I. Ahmadov *et al.*, Eur. Phys. J. Plus **136**, 208 (2021); https://doi.org/10.1140/epjp/s13360-021-01163-y.

[10] P. M. Ho, H. C. Kao, Phys. Rev. Lett. **88** 151602-1 (2002); https://doi.org/10.1103/physrevlett.88.151602.

[11] A. Connes, M. R. Douglas, A. Schwarz, J. High Energy Phys. **02**, 003 (1998); https://doi.org/10.1088/1126-6708/1998/02/003.

[12] N. Seiberg, E. Witten, J. High Energy Phys. **1999**, 032 (1999); https://doi.org/10.1088/1126-6708/1999/09/032.

[13] S. Capozziello, G. Lambiase, G. Scarpetta, Int. J. Theor. Phys. **39**, 15 (2000); https://doi.org/10.1023/A:1003634814685.

[14] S. Doplicher, K. Fredenhagen, J. E. Roberts, Phys. Lett. B **331**, 39 (1994); https://doi.org/10.1016/0370-2693(94)90940-7.

[15] E. Witten, Phys. Today **49**, 24 (1996); https://doi.org/10.1063/1.881493.

[16] A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. D **52**, 1108 (1995); https://doi.org/10.1103/physrevd.52.1108.

[17] R. J. Adler, D. I. Santigo, Mod. Phys. Lett. A **14** (20), 1371 (1999); https://doi.org/10.1142/s0217732399001462.

[18] T. Kanazawa *et al.*, Eur. Phys. J. C **79**, 95 (2019); https://doi.org/10.1140/epjc/s10052-019-6610-1.

[19] F. Scardigli, Phys. Lett. B **452**, 39 (1999); https://doi.org/10.1016/s0370-2693(99)00167-7.

[20] H. S. Snyder, Phys. Rev. **71**, 38 (1947); https://doi.org/10.1103/PhysRev.71.38; **72**, 68 (1947); https://doi.org/10.1103/PhysRev.72.68.

[21] A. Connes, *Noncommutative Geometry* (Academic Press, 1994).

[22] A. Connes, J. Lott, Nucl. Phys. Proc. Suppl. **18B**, 29 (1991).

[23] N. Seiberg, E. Witten, J. High Energy Phys. **1999**, 032 (1999); https://doi.org/10.1088/1126-6708/1999/09/032.

[24] C. Quesne, V. M. Tkachuk, J. Phys. A: Math. Gen. **38**, 1747 (2005); https://doi.org/10.1088/0305-4470/38/8/011.

[25] C. Quesne, V. M. Tkachuk, J. Phys. A: Math. Gen. **39**, 10909 (2006); https://doi.org/10.1088/0305-4470/39/34/021.

[26] A. Maireche, Int. Front. Sci. Lett. **11**, 29 (2017); https://doi.org/10.18052/www.scipress.com/IFSL.11.29.

[27] A. Maireche, Int. J. Geom. Meth. Mod. Phys. **17**, 2050067 (2020); https://doi.org/10.1142/S021988782050067X.

[28] A. Maireche, Few-Body Syst. **62**, 12 (2021); https://doi.org/10.1007/s00601-021-01596-2.

[29] A. Maireche, Few-Body Syst. **61**, 30 (2020); https://doi.org/10.1007/s00601-020-01559-z.

[30] A. Maireche, Int. J. Geom. Meth. Mod. Phys. **18**, 2150214 (2021); https://doi.org/10.1142/S0219887821502145.

[31] A. Maireche, Afr. Rev. Phys. **15**, 19 (2020).

[32] A. Maireche, Afr. Rev. Phys. **15**, 1 (2020).

[33] S. I. Vacaru, J. Math. Phys. **46**, 042503 (2005); https://doi.org/10.1063/1.1869538.

[34] A. Maireche, Ukr. J. Phys. **65**(11) (2020) 987; https://doi.org/10.15407/ujpe65.11.987.

[35] O. Bertolami *et al.*, Mod. Phys. Lett. A **21**, 795 (2006); https://doi.org/10.1142/s0217732306019840.

[36] A. Maireche, Rev. Mex. de Fis. **67**, 050702 (2021); https://doi.org/10.31349/RevMexFis.67.050702.

[37] E. E. N'Dolo *et al.*, Int. J. Geom. Meth. Mod. Phys. **17** 2050078 (2020); https://doi.org/10.1142/s0219887820500784.

[38] Kh. P. Gnatenko, V. M. Tkachuk, Int. J. Mod. Phys. A **33**(07), 1850037 (2018); https://doi.org/10.1142/s0217751x18500379.

[39] A. Maireche, J. Phys. Stud. **25**, 1002 (2021); https://doi.org/10.30970/jps.25.1002.

[40] Kh. P. Gnatenko, Phys. Lett. A **377**, 3061 (2013); https://doi.org/10.1016/j.physleta.2013.09.036.

[41] E. F. Djemaï, H. Smail, Commun. Theor. Phys. **41**, 837 (2004); https://doi.org/10.1088/0253-6102/41/6/837.

[42] O. G. Valencia, H. L. A. Arias, Int. J. Quant. Chem. **118**, e25589 (2018); https://doi.org/10.1002/qua.25589.

[43] O. Bertolami *et al.*, Phys. Rev. D **72**, 025010 (2005); https://doi.org/10.1103/physrevd.72.025010.

[44] J. Zhang, Phys. Lett. B **584**, 204 (2004); https://doi.org/10.1016/j.physletb.2004.01.049.

[45] M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001); https://doi.org/10.1103/PhysRevLett.86.2716.

[46] E. M. C. Abreu, C. Neves, W. Oliveira, Int. J. Mod. Phys. A **21**, 5359 (2006); https://doi.org/10.1142/s0217751x06034094.

[47] E. M. C. Abreu *et al.*, Int. J. Mod. Phys. A **27**, 1250053 (2012); https://doi.org/10.1142/s0217751x12500534.

[48] J. Wang, K. Li, J. Phys. A Math. Theor. **40**, 2197 (2007); https://doi.org/10.1088/1751-8113/40/9/021.

[49] S. Aghababaei, G. Rezaei, Commun. Theor. Phys. **72**, 125101 (2020); https://doi.org/10.1088/1572-9494/abb7cc.

[50] A. Maireche, Afr. Rev Phys. **12**, 130 (2017).

[51] A. Maireche, J. Nano- Electron. Phys. **8**, 01020 (2016); https://doi.org/10.21272/jnep.8(1).01020.

[52] A. Maireche, J. Nano- Electron. Phys. **8**, 04027 (2016); https://doi.org/10.21272/jnep.8(4(1)).04027.

[53] Y. Yi, K. Kang, W. Jian-Hua, C. Chi-Yi, Chin. Phys. C **34**, 543 (2010); https://doi.org/10.1088/1674-1137/34/5/005.

[54] N. Seiberg, E. Witten, J. High Energy Phys. **1999**, 032 (1999);https://doi.org/10.1088/1126-6708/1999/09/032.

[55] L. Mezincescu, preprint arXiv:hep-th/0007046 (2000); https://arxiv.org/abs/hep-th/0007046.

[56] L. Gouba, Int. J. Mod. Phys. A **31**, 1630025 (2016); https://doi.org/10.1142/s0217751x16300258.

[57] F. Bopp, Ann. Inst. Henri Poincaré **15**, 81 (1956); http://www.numdam.org/item/AIHP_1956__15_2_81_0.

[58] J. Gamboa, M. Loewe, J. C. Rojas, Phys. Rev. D **64**, 067901 (2001); https://doi.org/10.1103/PhysRevD.64.067901.

[59] A. Maireche, Sri Lankan J. Phys. **21**, 11 (2020); https://doi.org/10.4038/sljp.v21i1.8069.

[60] A. Maireche, J. Nano- Electron. Phys. **10**, 06015 (2018); https://doi.org/10.21272/jnep.10(6).06015.

[61] A. Maireche, Yanbu J. Eng. Sci. **17**, 51 (2019).

[62] A. Maireche, Int. Lett. Chem. Phys. Astron. **76**, 1 (2017); https://doi.org/10.18052/www.scipress.com/ILCPA.76.1.

[63] A. Maireche, Open Acc. J. Math. Theor. Phys. **1**, 232 (2018); https://doi.org/10.15406/oajmtp.2018.01.00040.

[64] A. Maireche, J. Nano- Electron. Phys. **9**, 03021 (2017); https://doi.org/10.21272/jnep.9(3).03021.

[65] A. Maireche, Jordan J. Phys. **14**, 59 (2021); https://doi.org/10.47011/14.1.6.

[66] A. Maireche, Sri Lankan J. Phys. **22**, 1 (2021); https://doi.org/10.4038/sljp.v22i1.8079.

[67] A. Maireche, Lat. Am. J. Phys. Educ. **14**, 3310 (2020).

[68] A. Maireche, Mod. Phys. Lett. A.**35**, 052050015 (2020); https://doi.org/10.1142/s0217732320500157.

[69] H. Motavalli, A. R. Akbarieh, Mod. Phys. Lett. A **25**, 2523 (2010); https://doi.org/10.1142/s0217732310033529.

[70] M. Darroodi, H. Mehraban, H. Hassanabadi, Mod. Phys. Lett. A **33**, 1850203 (2018); https://doi.org/10.1142/s0217732318502036.

[71] A. Maireche, Mod. Phys. Lett. A **36**, 2150232 (2021); https://doi.org/10.1142/S0217732321502321.

[72] A. Maireche, SPIN J. **11**, 2150029 (2021); https://doi.org/10.1142/S2010324721500296.

[73] A. Maireche, J. Phys. Stud. **25**, 4301 (2021); https://doi.org/10.30970/jps.25.4301.

[74] A. Maireche, Yanbu J. Eng. Sci. **18**, 10 (2021); https://doi.org/10.53370/001c.28090.

[75] A. Saidi, M.B. Sedra, Mod. Phys. Lett. A **35**, 2050014 (2020); https://doi.org/10.1142/s0217732320500145.

[76] A. Houcine, B. Abdelmalek, Phys. Part. Nucl. Lett. **16**, 195 (2019).

[77] R. L. Greene, C. Aldrich, Phys. Rev. A **14**, 2363 (1976); https://doi.org/10.1103/physreva.14.2363.

[78] S. H. Dong, W. C. Qiang, G. H. Sun, V. B. Bezerra, J. Phys. A **40**, 10535 (2007); https://doi.org/10.1088/1751-8113/40/34/010.

[79] Y. Zhang, Phys. Scr. **78**, 015006 (2008); https://doi.org/10.1088/0031-8949/78/01/015006.

[80] K. Bencheikh, S. Medjedel, G. Vignale, Phys. Lett. A **89**, 063620 (2014); https://doi.org/10.1103/physreva.89.063620.

[81] L. Hulthén, Mat. Astron. Fys. A **28**, 5 (1942).

[82] H. Yukawa, Proc. Phys. Math. Soc. Jpn. **17**, 48 (1935).

[83] O. Aydoğdu, R. Sever, Phys. Scr. **84**, 025005 (2011); https://doi.org/10.1088/0031-8949/84/02/025005.

[84] M. R. Setare, S. Haidari, Phys. Scr. **81**, 065201 (2010); https://doi.org/10.1088/0031-8949/81/06/065201.

[85] O. Aydoğdu, R. Sever, Ann. Phys. **325**, 373 (2010); https://doi.org/10.1016/j.aop.2009.10.009.

РЕЛЯТИВІСТСЬКІ СИМЕТРІЇ ДЕФОРМОВАНОГО РІВНЯННЯ ДІРАКА ЧЕРЕЗ ПОКРАЩЕНИЙ ПОТЕНЦІАЛ ГЮЛЬТЕНА ПЛЮС КЛАС ПОТЕНЦІАЛІВ ЮКАВИ З КУЛОНОПОДІБНОЮ ТЕНЗОРНОЮ ВЗАЄМОДІЄЮ В ДЕФОРМОВАНІЙ КВАНТОВІЙ МЕХАНІЦІ

Абдельмаджід Майреш
Лабораторія фізики та хімії матеріалів, фізичне відділення,
Факультет природничих наук, Університет Мсіли, BP 239 Шебілія-Мсіла, Алжир,
abdelmadjid.maireche@univ-msila.dz

Ми наближено розв'язуємо деформоване рівняння Дірака для нового запропонованого покращеного потенціалу Гюльтена плюс клас потенціалів Юкави з кулоноподібною тензорною взаємодією (з довільним спін-орбітальним квантовим числом k) у контексті симетрій розширеної релятивістської квантової механіки. У межах спінової та псевдоспінової симетрії ми отримуємо ґлобальне нове власне значення енерґії, яке дорівнює власному значенню енерґії у звичайній релятивістській квантовій механіці плюс поправка, індукована трьома нескінченно малими адитивними частинами гамільтоніана, які відповідають спін-орбітальній взаємодії, новому модифікованому зеєманівському доданкові й обертальному доданкові Фермі. Для отримання цієї поправки використано методу зсуву Боппа й стандартну теорію збурень із наближенням для відцентрового члена. Отримані нові значення виявились чутливими до квантових чисел $(j, k, l, \widetilde{l}, s, \widetilde{s}, m, \widetilde{m})$, змішані глибини потенціалів (A, B), діапазон потенціалу δ та параметри некоммутативності (Θ, σ, χ). Отримано змішаний потенціал, який у деяких конкретних випадках дає розв'язки для різних потенціалів, покращеного потенціалу Гюльтена, покращеного потенціалу Юкави та покращеної кулоноподібної задачі разом з енерґіями відповідних зв'язаних станів.

Ключові слова: рівняння Дірака, потенціал Гюльтена плюс клас потенціалів Юкави, некомутативна геометрія, метод зсуву Боппа та зіркові добутки.