SOLVABLE TWO-PARTICLE SYSTEMS WITH TIME-ASYMMETRIC INTERACTIONS IN DE SITTER SPACE

A. Duviryak
Institute for Condensed Matter Physics of NAS of Ukraine,
1, Svientsitskii St., Lviv, UA–79011, Ukraine
e-mail:duviryak@icmp.lviv.ua

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The two-particle models in de Sitter space-time with time-asymmetric retarded-advanced interactions are constructed. Particular cases of the field-type electromagnetic and scalar interactions are considered. The manifestly covariant descriptions of the models within the Lagrangian and Hamiltonian formalisms with constraints are proposed. It is shown that the models are de Sitter-invariant and integrable. An explicit solution of equations of motion is derived in quadratures by means of the projection operator technique.

Key words: de Sitter space, time-asymmetric models, integrable systems.

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I. INTRODUCTION

It is known that one has to deal with complex difference-differential equations when considering a relativistic classical dynamics of a system of interacting charges [1, 2]. This is even more the case for scalar [2], gravitational [3] or non-Abelian [4] interactions where the dynamics is governed by integro-differential equations. Such a hereditary dynamics is neither solvable nor appropriate for the Hamiltonian description. In order to avoid these difficulties, Staruszkiewicz [5], Rudd and Hill [6] invented the model describing the following time-asymmetric interaction of two pointlike charged particles: the advanced field of the first particle acts on the second particle, the retarded field of the second particle acts on the first particle, and a radiation reaction is neglected. This model is built of the action-at-a-distance Tetrode–Fokker variational functional [7, 8] via replacing its integrand, the symmetric Green function of d'Alembert equation, with the retarded (or advanced) one. In this way the model was reformulated to the Lagrangian form and then to the Hamiltonian form [9] which was shown integrable [10] due to exact Poincaré-invariance. The Staruszkiewicz–Rudd–Hill model was generalized for a variety of non-electromagnetic time-asymmetric interactions (scalar, gravitational, confining etc.) [11–13], and corresponding quantum versions [14, 15] revealed their physical adequacy, despite an artificially broken causality of interactions.

The purpose of this work is a generalization of the Staruszkiewicz–Rudd–Hill model, formulated primarily in a flat space-time, to the case of de Sitter space-time. The construction of this model presumes that the interaction between particles spreads at the speed of light, i.e., along the light-cone surface. This is not the case in the curved space-time where an additional slow "tail" component of the interaction arises [16]. The present paper shows that in the special case of de Sitter space-time the "tail" contribution from the electromagnetic Green function can be reduced to an equivalent on-light-cone contribution. This fact suggests a relevant two-particle model with the time-asymmetric electromagnetic interaction [17]. Actually, here a family of two-particle models in de Sitter space-time is presented. It includes also the system with scalar interaction and models with various phenomenological interactions.

An even more important issue raised in this paper is the integrability of the presented models. This point is relevant since, to the author's knowledge, solvable examples of dynamics of interacting particles in a curved space-time are unknown. The time-asymmetric models are, by construction, invariant with respect to the de Sitter group O(1,4), and formulated by means of a variation principle which is reduced to the Lagrangian form and then to the Hamiltonian form. Based on the Noether theorem and the structure of the Lie algebra of O(1,4), there exist a sufficient number of integrals of motion to ensure the integrability of a two-particle system in quadratures. In practice, however, the problem appears too cumbersome to be solved by means of commonly used methods, such as the Hamilton–Jacobi one. Instead, the representation of de Sitter space-time as a hyperboloid in the 5-dimensional Minkowski space $\mathbb{M}_5$ is used to apply Dirac's canonical formalism with constraints. Besides, the technique of projection operators built in terms of conserved canonical generators of O(1,4) is elaborated. These tools are used to solve the Hamiltonian equations of motion in quadratures.

The paper is organized as follows. In Section II, a single particle dynamics is used to introduce elements of a 5-dimension representation. In Section III, the Tetrode–Fokker variational principle for 2-particle systems with electromagnetic, scalar, and other interactions in de Sitter space is formulated. It is then appropriately modified in Section IV to generate a family of time-asymmetric models and to put their description into the Lagrangian formalism and then (in Section V) into the canonical formalism with constraints. This transform is detailed in Appendix A. The system of two free particles as a time-asymmetric model is not manifestly separable and thus it is particularly considered in Appendix B. In Section VI, the canonical equations of motion are derived and solved by means of the projection operator techniques developed in Appendix C. Main results and prospects of the work are presented in Section VII.

II. MANIFESTLY COVARIANT TEST PARTICLE MECHANICS IN DE SITTER SPACE

Let us start with the action integral determining the dynamics of a test particle of the mass m in a curved space-time:

$$I = -m \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \, \sqrt{g_{\mu\nu}(x(\tau))\dot{x}^{\mu}(\tau)\dot{x}^{\nu}(\tau)} \; ; \qquad (2.1)$$

here τ parameterizes points $x(\tau)$ of a particle world line, i.e., the geodesic, $x^{\mu}(\tau)$ ($\mu = 0, \dots, 3$) are particle coordinates, and $g_{\mu\nu}(x)$ is a metric tensor in a chosen chart of the space-time considered. The action (2.1) is invariant with respect to an arbitrary change of the evolution parameter: $\tau \to \tau' = f(\tau)$, since the parametrization of geodesics has no physical meaning. For de Sitter space-time [18], geodesics were studied from different viewpoints [18–21] in many coordinate charts introduced for this space-time [21–23].

It is convenient to consider de Sitter space-time as a 4-dimensional hyperboloid $\mathbb{H}$:

$$\eta_{MN}y^M y^N := (y^0)^2 - (y^1)^2 - \cdots - (y^4)^2 = -R^2 \quad (2.2)$$

in the 5-dimensional Minkowski space $\mathbb{M}_5$ with coordinates y^M ($M = 0, 1, \dots, 4$) and the metrics $||\eta_{MN}|| = \mathrm{diag}(+, -, \dots, -)$; [21, 24]. The constant R determines the scalar curvature $\mathcal{R}$ of the de Sitter space, and it is related to the cosmological Λ-constant: $\mathcal{R} = 12/R^2 = 4\Lambda$; the speed of light is put $c = 1$.

The hyperboloid $\mathbb{H}$ is invariant with respect to de Sitter group O(1,4) represented in $\mathbb{M}_5$ by standard linear pseudoorthogonal transformations. Thus, we will use standard notations for O(1,4)-invariants $y \cdot z := \eta_{MN}y^M z^N$ and $y^2 := y \cdot y$ built of arbitrary 5-vectors y, $z \in \mathbb{M}_5$.

The embedding $\mathbb{H} \hookrightarrow \mathbb{M}_5$ implies, in terms of local coordinates x^{μ} in de Sitter space, a set of appropriate functions $y^M(x)$ turning the equation (2.2) into identity [21–23]. Then the pseudo-Euclidian O(1,4)-invariant metrics is pulled back naturally from $\mathbb{M}_5$ onto $\mathbb{H}$:

$$\rho(x, x') := (y - y')^2\big|_{\mathbb{H}} . \qquad (2.3)$$

This endows de Sitter space with a causal structure of the ambient Minkowski space:

- the interval between points $x, x' \in \mathbb{H}$ is timelike if $\rho(x, x') > 0$, i.e., if $y' \in \mathbb{H} \subset \mathbb{M}_5$ lies inside the light cone with a vertex $y \in \mathbb{H} \subset \mathbb{M}_5$ (or the same with y and y' permuted);
- the interval is spacelike if $\rho(x, x') < 0$, i.e., if y' lies outside the light cone;
- the interval is isotropic if $\rho(x, x') = 0$, i.e., if y' lies on the light cone hypersurface.

For infinitely closed 5-vectors y and $y' = y + \mathrm{d}y$, the function (2.3) yields the pseudo-Riemannian metrics involved in the action integral (2.1) for the case of de Sitter space:

$$\mathrm{d}s^2 := \eta_{MN}\mathrm{d}y^M \mathrm{d}y^N\big|_{\mathbb{H}} = g_{\mu\nu}(x)\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}.$$

Thus, the test particle dynamics in de Sitter space can be reformulated to some variational principle with a constraint, defined in the configuration space $\mathbb{M}_5$ [21, 25, 26]. The simplest version is [21]:

$$I = -\int \mathrm{d}\tau \, \left\{ m\sqrt{\dot{y}^2(\tau)} - \lambda(\tau)(y^2(\tau) + R^2) \right\}, \quad (2.4)$$

where the condition (2.2) is taken into account as a holonomic constraint by means of the Lagrange multiplier $\lambda(\tau)$. The Euler–Lagrange equation for 5-vector $y(\tau)$ representing the particle position $x(\tau) \in \mathbb{H}$ can be written down in the following manifestly covariant form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\dot{y}}{\sqrt{\dot{y}^2}} - \sqrt{\dot{y}^2}\frac{y}{R^2} = 0 \qquad (2.5)$$

which is invariant with respect to both the O(1,4) group and an arbitrary change of the evolution parameter τ. The solution of the geodesic equation (2.5) is

$$y(\tau) = y(0)\cosh\frac{s(\tau)}{R} + R\frac{\dot{y}(0)}{\sqrt{\dot{y}^2(0)}}\sinh\frac{s(\tau)}{R}, \qquad (2.6)$$

where the constant 5-vectors $y(0)$ and $\dot{y}(0)$ are subjected to constraint (2.2) and its differential consequence $y \cdot \dot{y} = 0$, and $s(\tau)$ is the proper time elapsed from $y(0)$ to $y(\tau)$:

$$s(\tau) := \int_0^{\tau} \mathrm{d}\tau \, \sqrt{\dot{y}^2(\tau)} \, . \qquad (2.7)$$

The proper time as a function of τ cannot be determined from equation (2.5), due to reparametrization invariance, but it can be chosen manually for convenience. For example, with the proper time parametrization $s(\tau) := \tau$ we have $\dot{y}^2 = 1$, and the equation (2.6) reproduces the de Sitter geodesic found in Ref. [21].

Due to de Sitter symmetry, there exist 10 integrals of motion collected in the skew-symmetric angular 5-momentum tensor:

$$J_{MN} = y_M\pi_N - y_N\pi_M = -J_{NM}, \qquad (2.8)$$

where

$$\pi_M = m\dot{y}_M/\sqrt{\dot{y}^2} \qquad (2.9)$$

are components of 5-momentum.

At this point one can develop the covariant Hamiltonian description on the phase space $\mathrm{T}^*\mathbb{M}_5$ with variables y^M, π_N ($M, N = 0, \dots, 4$) and standard Poisson brackets: $\{y^M, y^N\} = 0$, $\{\pi_M, \pi_N\} = 0$, $\{y^M, \pi_N\} = \delta^M_N$. The integrals of motion J_{MN} become canonical generators of O(1,4) group while the Legendre transform (2.9) is degenerated due to the reparametrization invariance of action (2.4). Thus, the canonical Hamiltonian vanishes while the *mass-shell* constraint arises, $\pi^2 - m^2 = 0$, apart from the holonomic constraint (2.2). Both constrains are primary ones according to Dirac's terminology of canonical formalism with constraints [27]. They form Dirac's primary Hamiltonian: $H'_{\rm D} = \lambda(\pi^2 - m^2) + \lambda_1(y^2 + R^2)$, where λ and λ_1 are Lagrange multipliers. The compatibility conditions

$$\{y^2 + R^2, H'_{\rm D}\} = 4\lambda y \cdot \pi \approx 0,$$

$$\{\pi^2 - m^2, H'_{\rm D}\} = -4\lambda_1 y \cdot \pi \approx 0,$$

give rise to the secondary constraint $y \cdot \pi = 0$, so that Dirac's Hamiltonian at this stage takes the form: $H''_{\rm D} = H'_{\rm D} + \lambda_2\, y \cdot \pi$. Reexamining compatibility conditions gives no new constraints but fixes partially Lagrange multipliers: $\lambda_1 = 0$. Putting then $\lambda_2 = -y \cdot \pi/y^2$ yields Dirac's final Hamiltonian $H_{\rm D} = \lambda(\tau)\phi(y, \pi)$ with the unspecified Lagrange multiplier $\lambda(\tau)$ (due to the reparametrization invariance) and the function $\phi(y, \pi)$ which determines the modified mass-shell constraint

$$\phi := \pi_\perp^2 - m^2 \equiv \tfrac{1}{2} J^2/y^2 - m^2 = 0; \qquad (2.10)$$

here $\pi_{\perp M} := \pi_M - \dfrac{y \cdot \pi}{y^2} y_M \approx \dfrac{J_M^{\ N} y_N}{R^2}$ (so that $y \cdot \pi_\perp \equiv 0$) and $J^2 := J_{MN}J^{MN}$. Symbol " $\approx$ " denotes a "weak equality", i. e. by virtue of the holonomic constraint (2.2); [27].

Let us note that the set of constraints (2.2) and (2.10) are the 1st class [27], i.e., they satisfy the identity: $\{y^2 + R^2, \phi\} \equiv 0$. Together with Dirac's Hamiltonian $H_{\rm D} = \lambda\phi$, these constraints endow effectively the system with three degrees of freedom (as it should). Henceforth the quantity $y \cdot \pi$ is not involved in the dynamics, and the secondary constraint $y \cdot \pi = 0$ can be abandoned.

The Hamiltonian equation for the particle position 5-vector y reads:

$$\dot{y} = \lambda\{y, \phi\} = 2\lambda\pi_\perp \approx \frac{2\lambda}{R^2} \mathsf{J}\, y. \qquad (2.11)$$

Note that the matrix $\mathsf{J} := \|J^M_{\ \ N}\| := \|\eta^{ML}J_{LN}\|$ is conserved, thus, equation (2.11) is linear. Its formal solution follows immediately: $y(\tau) = {\rm e}^{\frac{s(\tau)}{mR^2}\mathsf{J}} y(0)$, where the unspecified function $s(\tau) = 2m\int_0^\tau {\rm d}\tau\, \lambda(\tau)$ is the Hamiltonian image for the proper time function (2.7). The Cauchy problem becomes solved after matrix J is expressed in terms of initial values $y(0)$ and $\dot{y}(0)$ by the equalities (2.8), (2.9) and their consequences $\mathsf{J}y \approx mR^2 v$, $\mathsf{J}v = my$, where $v = \dot{y}/\sqrt{\dot{y}^2}$. Then expanding the exponent in power series reproduces solution (2.6).

It may seem unreasonable to use of a 5-dimensional reparametrization invariant description together with Dirac's formalism with constraints in order to derive geodesics in de Sitter space. These tools, however, appear effective when considering two-body problems in the following sections.

III. ACTION-AT-A-DISTANCE DYNAMICS OF TWO PARTICLES IN DE SITTER SPACE

Within framework of the Wheeler–Feynman electrodynamics [1, 2, 28, 29], a system of charged point-like particles is described by the Tetrode–Fokker action-at-a-distance variational principle [7, 8]. This formalism was generalized for a curved space-time by Hoyle and Narlikar [28] and others [29, 30].

For a system of two charged particles of masses m_a and charges e_a ($a = 1, 2$), the Tetrode–Fokker action integral has the form:

$$I = I_{\rm free} + I_{\rm int}, \quad \text{where} \quad I_{\rm free} = -\sum_{a=1}^{2} m_a \int {\rm d}s_a\,, \quad (3.1)$$

$${\rm d}s_a := \sqrt{g_{\mu\nu}(x_a(\tau_a))\dot{x}_a^\mu(\tau_a)\dot{x}_a^\nu(\tau_a)}\, {\rm d}\tau_a\,, \qquad (3.2)$$

$$I_{\rm int} = -4\pi e_1 e_2 \iint {\rm d}x_1^\mu\, {\rm d}x_2^\nu\, G_{\mu\nu}(x_1, x_2); \qquad (3.3)$$

here $x_a^\mu(\tau_a)$ ($\mu = 0, \ldots, 3$) are space-time coordinates of particle world lines parameterized by evolution parameters τ_a ($a = 1, 2$). Free-motion terms $I_{\rm free}$ of action (3.1) have form (2.1) for each particle. An integrand of the interaction term (3.3) is the symmetric Green function $G_{\mu\nu'}(x, x')$ of the covariant wave equation $\Box A_\mu + \mathcal{R}_\mu^{\ \nu} A_\nu = 0$ for the electromagnetic potential A_μ [16, 31]; here $\Box$ is the d'Alembertian in a curved space-time considered, and $\mathcal{R}_\mu^{\ \nu}$ is the Ricci tensor. For a curved space-time, $G_{\mu\nu'}(x, x')$ is a bi-vector function, whose construction in general is a complicated problem [16].

For de Sitter space-time, the symmetric Green function is known from Ref. [32][1]. It is presented here in geometric terms, which are indifferent to the choice of a coordinate chart:

$$G_{\mu\nu'}(x, x') = G^\delta_{\mu\nu'}(x, x') + G^\Theta_{\mu\nu'}(x, x'); \qquad (3.4)$$

here

$$G^\delta_{\mu\nu'}(x, x') := \frac{1}{4\pi}\bar{g}_{\mu\nu'}(x, x')\, \delta(\rho(x, x')), \qquad (3.5)$$

$$G^\Theta_{\mu\nu'}(x, x') := -\frac{1}{24\pi R^2}\Big\{ \Big(\frac{1}{Z} + \frac{1}{2Z^2}\Big) \bar{g}_{\mu\nu'} + \frac{R^2}{Z^3}(\partial_\mu Z)(\partial_{\nu'} Z)\Big\}\Theta(\rho(x, x')); \qquad (3.6)$$

$$\bar{g}_{\mu\nu'}(x, x') := -2R^2\Big\{\partial_\mu\partial_{\nu'} Z - \frac{1}{Z}(\partial_\mu Z)(\partial_{\nu'} Z)\Big\}, \qquad (3.7)$$

$$Z(x, x') := 1 + \tfrac{1}{4}\rho(x, x')/R^2, \qquad (3.8)$$

where $\bar{g}_{\mu\nu'}(x, x')$ is the parallel propagator [16, 31], and the metric function $\rho(x, x')$ is defined by (2.3). We note

[1] An earlier proposal [31] is unappropriate as it does not meet demands of de Sitter-covariance.

that the Green function (3.4) consists of two parts. The local part (3.5) is proportional to the Dirac δ-function and, thus, supported by the light cone surface $\rho(x,x')=0$. The non-local part (3.6) is proportional to the Heaviside Θ-function and, thus, supported by the light cone interior $\rho(x,x')>0$. This is a common feature of curved space-times [16], contrary to the Minkowski space-time, where Green functions of massless fields have a local part only. But in the present case of de Sitter space-time, the non-local contribution (3.6) of the Green function (3.4) in integral (3.3) can be effectively reduced to a local one [17].

In order to show this, let us first introduce the relative position 5-vector $r\equiv y_1-y_2$, the particle unit 5-velocities $v_a\equiv\dot y_a/\sqrt{\dot y_a^2}$, and the dimensionless scalars of these 5-vectors $v_1\cdot v_2$ and $r\cdot v_a/R$ $(a=1,2)$, which are homogeneous functions of degree zero of derivatives $\dot y_1$ and $\dot y_2$. It is convenient for a subsequent interim calculation to present these scalars as follows:

$$\omega := v_1\cdot v_2|_{\mathbb{H}} = -\frac{1}{2}\frac{\mathrm{d}^2\rho(x_1,x_2)}{\mathrm{d}s_1\,\mathrm{d}s_2},\qquad \nu_a := \frac{r\cdot v_a}{R}\Big|_{\mathbb{H}} = -\frac{(-)^a}{2R}\frac{\mathrm{d}\rho(x_1,x_2)}{\mathrm{d}s_a},\quad a=1,2, \tag{3.9}$$

where the function $\rho(x_1,x_2)$ and the interval elements $\mathrm{d}s_a$ are defined by eqs. (2.3) and (3.2), respectively. Note that the differentiation over $\mathrm{d}s_1$ (or $\mathrm{d}s_2$) acts on $x_1(\tau_1)$ (or $x_2(\tau_2)$).

In these terms, the integrand of the interaction term (3.3) of action (3.1) reads:

$$\mathrm{d}x_1^\mu\,\mathrm{d}x_2^\nu\,G_{\mu\nu}(x_1,x_2) = \frac{\mathrm{d}s_1\,\mathrm{d}s_2}{4\pi}\left\{\left(\omega-\frac{\nu_1\nu_2}{2Z}\right)\delta(\rho)-\left(\frac{2Z+1}{Z^2}\omega-\frac{Z+1}{Z^3}\nu_1\nu_2\right)\frac{\Theta(\rho)}{12R^2}\right\}.$$

Then, applying the integration-by-part formula:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}s_1\,\mathrm{d}s_2\,\omega F(\rho) = -\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}s_1\,\mathrm{d}s_2\,\frac{\mathrm{d}^2\rho}{\mathrm{d}s_1\,\mathrm{d}s_2}F(\rho)$$
$$= -2R^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}s_1\,\mathrm{d}s_2\,\nu_1\nu_2\,\frac{\mathrm{d}F(\rho)}{\mathrm{d}\rho}-\frac{1}{2}\left[\int\mathrm{d}\rho\,F(\rho)\right]_{\substack{s_1=-\infty\\ s_2=-\infty}}^{\substack{s_1=+\infty\\ s_2=+\infty}},$$

which holds for any function $F(\rho)$, to the Tetrode–Fokker integral (3.3), one obtains:

$$I_{\text{int}} = -4\pi e_1e_2\iint\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\dot x_1^\mu\dot x_1^\nu\,G_{\mu\nu}(x_1,x_2) \simeq -e_1e_2\iint\mathrm{d}s_1\,\mathrm{d}s_2\,\omega\delta(\rho), \tag{3.10}$$

where symbol “ $\simeq$ ” denotes an equality up to boundary terms which do not contribute in variational problem.

It is remarkable that the only local (i.e., light cone surface) contribution of the Green function remains in the Tetrode–Fokker integral (3.10); this structure is a necessary starting point for a construction of the model of Staruszhkiewicz–Rudd–Hill type in the next section.

Similarly, one can consider a particle system with the scalar interaction. The interaction term of the Fokker-type action (3.1) in this case has the form [28]:

$$I_{\text{int}} = -4\pi g_1g_2\iint\mathrm{d}s_1\,\mathrm{d}s_2\,G(x_1,x_2), \tag{3.11}$$

where g_a $(a=1,2)$ are scalar “charges” of particles, and the bi-scalar function $G(x,x')$ is the symmetric Green function of the wave equation $\Box\varphi=0$ for a scalar field φ mediating the interaction and *minimally* coupled to gravitation [16]. For de Sitter space-time, the Green function $G(x,x')$ was found by Narlikar [31]:

$$G(x,x') = G^\delta(x,x')+G^\Theta(x,x') := \frac{1}{4\pi}\left\{\delta(\rho)+\frac{1}{2R^2}\Theta(\rho)\right\}. \tag{3.12}$$

In contrast to the case of electromagnetic interaction, the nonlocal contribution $G^\Theta(x,x')$ of the Green function (3.12) is essential: it cannot be removed from action (3.11) by means of the integration by parts or another equivalent transformation.

The Penrose–Chernikov–Tagirov equation $(\Box+\mathcal{R}/6)\varphi=0$ corresponds to a *conformal* coupling of the scalar field to gravitation [33, 34]. In the case of de Sitter space-time, the scalar curvature $\mathcal{R}=12/R^2$ is constant, and the Green function can be found easily using of distributional methods [16]. It appears purely local:

$$G(x,x') = G^{\delta}(x,x') := \frac{1}{4\pi}\delta(\rho). \qquad (3.13)$$

The electromagnetic (3.10) and scalar (3.11), (3.13) interaction terms of the Fokker-type action admit the obvious de-Sitter-invariant generalization:

$$I_{\text{int}} = -\iint \mathrm{d}s_1\,\mathrm{d}s_2\; f(\nu_1,\nu_2,\omega)\delta(\rho), \qquad (3.14)$$

where $\mathrm{d}s_a$ are defined in (3.2), and $f(\nu_1,\nu_2,\omega)$ may be an arbitrary function of its three scalar arguments (3.9), so it is a homogeneous function of degree zero of $\dot{y}_1$ and $\dot{y}_2$. Thus, expression (3.14) possesses both the de Sitter invariance and the double reparametrization invariance. It comprises a variety of interactions which may have a field-theoretical nature or can be introduced phenomenologically.

IV. TIME-ASYMMETRIC MODELS IN DE SITTER SPACE-TIME

Staruszkiewicz [5, 9], Rudd and Hill [6] replaced in the Tetrode–Fokker action integral the symmetric Green function G of d'Alembert equation with the retarded $G^{(+)}$ or advanced $G^{(-)}$ Green function: $G^{(\pm)}(x_1,x_2) = 2\Theta[\pm(x_1^0 - x_2^0)]G(x_1,x_2)$. This led them to a two-particle model with the time-asymmetric retarded-advanced interaction. Following this idea, one should insert the factor $2\Theta[\eta(x_1^0 - x_2^0)] = 2\Theta[\eta(y_1^0 - y_2^0)]$, where $\eta = +1$ or -1 in the general interaction term (3.14) of the Fokker-type action (3.1). Then, similarly to the singe-particle case considered in Section II, it is convenient to present this Fokker-type action via global variables in the ambient Minkowski space $\mathbb{M}_5$. One, thus, obtains:

$$I_{\text{int}} = -\iint \mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ \times\; \sqrt{\dot{y}_1^2}\sqrt{\dot{y}_2^2}\, f(\nu_1,\nu_2,\omega)\, 2\Theta(\eta r^0)\,\delta(r^2)|_{\mathbb{H}^2}, \quad (4.1)$$

where the integrand in r.-h.s. of (4.1) is constrained on $\mathbb{H}^2 = \mathbb{H}\times\mathbb{H}$, i.e., the particle position 5-vectors $y_a(\tau_a)$ $(a = 1, 2)$ are subjected to the *hyperboloid* conditions for each particle:

$$y_a^2 + R^2 = 0, \qquad a = 1, 2. \qquad (4.2)$$

An integrand of the double integral I_{int} in (4.1) is non-zero provided

$$r^2 := (y_1 - y_2)^2 = 0, \qquad \eta r^0 := \eta(y_1^0 - y_2^0) > 0. \quad (4.3)$$

This condition can be treated as the equation of the past or future *light cone*, depending on the value $\eta = \pm 1$ and on which point, y_1 or y_2, is the vertex of the cone. If the time-symmetric action (3.14) is invariant under a particle permutation, the invariance of the corresponding time-asymmetric action (4.1) is provided by the additional change $\eta \to -\eta$.

From a physics viewpoint, the choice of the sign factor $\eta = \pm 1$ is unimportant. Both cases correspond to the electromagnetic interaction with a "spoiled" causality. They lead to distinguished two-body problems which differ from one another and from those of the Wheeler–Feynman or retarded electrodynamics. It is worth noting that in the case of the flat-space Staruszkiewucz–Rudd–Hill model, the particle world lines corresponding to different $\eta = \pm 1$ are distinguishable only in a highly relativistic domain [5, 13].

The Fokker-type action integral (3.1), (4.1) is invariant with respect to an arbitrary change of each parameter τ_1 and τ_2. Thus, two of the ten variables $y_1^M(\tau_1)$, $y_2^M(\tau_2)$ $(M = 0, \ldots, 4)$ to be found remain undetermined within the variational problem. It is profitable to fix partially this functional arbitrariness manually as follows. Let us choose one of the variables, say $y_2^0(\tau_2)$, in such a way that condition (4.3) turns into an identity at $\tau_1 = \tau_2$. This implies that both particle world lines are parameterized by a common evolution parameter, say τ_1, and the simultaneous events $y_1(\tau_1)$ and $y_2(\tau_1)$ lie on the isotropic light cone surface (4.3). Using the equality (see [9])

$$2\,\Theta\Big[\eta\Big(y_1^0(\tau_1) - y_2^0(\tau_2)\Big)\Big]\,\delta\Big[\Big(y_1(\tau_1) - y_2(\tau_2)\Big)^2\Big] \\ = \frac{\delta(\tau_1 - \tau_2)}{\Big|\dot{y}_2(\tau_2)\cdot\Big(y_1(\tau_1) - y_2(\tau_2)\Big)\Big|}$$

in the interaction term (4.1) and integrating over τ_2 reduces the Fokker-type action (3.1) to the single-time form

$$I = \int \mathrm{d}\tau\; \tilde{L} \qquad (4.4)$$

with the Lagrangian $\tilde{L} := L|_{\mathrm{T}\mathbb{K}}$, where

$$L \;=\; -\sum_{a=1}^{2} m_a\sqrt{\dot{y}_a^2} \;-\; \sqrt{\dot{y}_1^2}\sqrt{\dot{y}_2^2}\,\frac{f(\nu_1,\nu_2,\omega)}{|\dot{y}_2\cdot r|}. \quad (4.5)$$

The Lagrangian $\tilde{L}$ is defined on the tangle bundle $\mathrm{T}\mathbb{K}$ over the 7-dimensional configuration manifold $\mathbb{K} \subset \mathbb{H}^2 \subset \mathbb{M}_5^2 \equiv \mathbb{M}_5 \times \mathbb{M}_5$ described by conditions (4.2), (4.3). The corresponding variational problem gives rise to the second order differential equations of motion and, thus, the transition to the usual Hamiltonian description is straightforward.

The Lagrangian $\tilde{L}$ (as well as L) is the first degree homogeneous function of particle velocities. Thus, action (4.4) has a residual invariance with respect to an arbitrary change in the common evolution parameter: τ. This symmetry allows one to fix the remaining timelike variable manually and, together with conditions (4.2), (4.3), enables to arrive at the ordinary Lagrangian description in the 6-dimensional configuration space $\mathbb{Q}$. In practice, however, the explicit elimination of redundant variables, say y_1^0, y_1^4, y_2^0, y_2^4, breaks a manifest 5-dimensional Lorentz-covariance, and makes the subsequent treatment cumbersome. As usual, a success in solving equations of motion is predetermined by an appropriate parametrization of the configuration space, which is not evident in the case of $\mathbb{Q}$.

An alternative way is the use of a manifestly covariant Lagrangian description in the 10-dimensional configuration space $\mathbb{M}_5^2$. In this case, an unconditional extremum problem of action (4.4) is modified in favor of an equivalent conditional extremum problem of

$$I = \int d\tau \left\{ L + \lambda_0 r^2 + \sum_{a=1}^{2} \lambda_a (y_a^2 + R^2) \right\} \qquad (4.6)$$

with the Lagrangian function (4.5) defined on $\mathrm{T}\mathbb{M}_5^2$. The Lagrangian multipliers $\lambda_0(\tau)$, $\lambda_a(\tau)$ take conditions (4.3), (4.2) into account as holonomic constraints; the unilateral constraint $\eta r^0 > 0$ is implied as well.

De Sitter invariance of Lagrangian (4.5) and constraints (4.2), (4.3) provides the existence of ten Noether integrals of motion, collected in the angular 5-momentum tensor:

$$J_{MN} = \sum_{a=1}^{2} (y_{aM}\pi_{aN} - y_{aN}\pi_{aM}), \qquad (4.7)$$

where

$$\pi_{aM} = \partial L / \partial \dot{y}_a^M, \qquad a = 1, 2. \qquad (4.8)$$

Besides, Lagrangian (4.5) satisfies the identity:

$$\sum_{a=1}^{2} \dot{y}_a \cdot \pi_a - L = 0, \qquad (4.9)$$

due to the reparametrization invariance of action (4.6).

V. CANONICAL FORMALISM WITH CONSTRAINTS

The Lagrangian description in the configuration space $\mathbb{M}_5^2$ enables a natural transition to the manifestly covariant Hamiltonian description with constraints [27]. The corresponding 20-dimensional phase space $\mathrm{T}^*\mathbb{M}_5^2$ with the particle canonical variables y_a^M, π_{bN} $(a, b = 1, 2;$ $M, N = 0, \ldots, 4)$ is endowed with the standard Poisson brackets: $\{y_a^M, y_b^N\} = 0$, $\{\pi_{aM}, \pi_{bN}\} = 0$, $\{y_a^M, \pi_{bN}\} = \delta_{ab}\delta_N^M$.

Components of the conserved angular 5-momentum tensor (4.7) become, within the Hamiltonian description, the generators J_{MN} of the canonical realization of the de Sitter group, i.e., they satisfy the canonical relations of the Lie algebra of O(1,4):

$$\{J_{MN}, J_{LK}\} = \eta_{ML}J_{NK} + \eta_{NK}J_{ML} - \eta_{MK}J_{NL} - \eta_{NL}J_{MK}. \qquad (5.1)$$

Due to identity (4.9), the Legendre transformation (4.8) is degenerated, the canonical Hamiltonian vanishes, while the additional constraint arises [27], similarly to the mass-shell constraint in the single particle case. The function determining this constraint constitutes (together with the holonomic constraints (4.2), (4.3)) Dirac's primary Hamiltonian.

The subsequent procedure is similar to that of the single particle case in Section II. The compatibility conditions of the dynamics with primary constraints give rise to secondary constraints which then are combined with the primary constraints in the secondary Dirac' Hamiltonian etc. In the final compatible form, the dynamics is generated by Dirac's Hamiltonian $H_\mathrm{D} = \lambda(\tau)\Phi(y_a, \pi_b)$ where $\lambda(\tau)$ is an unspecified Lagrange multiplier (due to the reparametrization invariance), and constraint $\Phi(y_a, \pi_b) = 0$ is the first class with respect to the holonomic constraints (4.2), (4.3), i.e., the function $\Phi(y_a, \pi_b)$ satisfies the equalities:

$$\{\Phi, r^2\} = 0, \qquad \{\Phi, y_a^2 + R^2\} = 0, \quad a = 1, 2. \quad (5.2)$$

Besides, this constraint must be de Sitter invariant since the angular momentum tensor (4.7) must be conserved.

We will refer to $\Phi(y_a, \pi_b) = 0$ as the *dynamical* constraint for two reasons. Firstly, the function $\Phi(y_a, \pi_b)$ generates an evolution via Dirac's Hamiltonian. Secondly, a specific form of $\Phi(y_a, \pi_b)$ is determined by Lagrangian (4.5), in particular, by the form of the interaction function $f(\nu_1, \nu_2, \omega)$ chosen. However, equations (5.2) and de Sitter invariance requirements are sufficient to outline a general structure of the dynamical constraint and the corresponding Hamiltonian mechanics.

Let functions of canonical variables $\varphi(y_a, \pi_b)$ which satisfy conditions (5.2) be referred to as *observables* in Dirac's meaning [27]. We will use sometimes the collective canonical variables:

$$Y^M = \tfrac{1}{2}(y_1^M + y_2^M), \quad r^M = y_1^M - y_2^M,$$

$$\Pi_M = \pi_{1M} + \pi_{2M}, \qquad \pi_M = \tfrac{1}{2}(\pi_{1M} - \pi_{2M}). \quad (5.3)$$

The components of position 5-vectors Y, r are the observables. Solving equations (5.2) yields other observables, the momentum-type 5-vectors $\Pi_\perp$, $\pi_\perp$ with the components:

$$\Pi_{\perp M} := \frac{Y^L J_{LM}}{Y^2} \approx \Pi_M + \frac{(Y\cdot\Pi)Y_M + (Y\cdot\pi)r_M}{R^2}, \quad (5.4)$$

$$\pi_{\perp M} := \left(\delta_M^N - \frac{r_M \Pi_\perp^N}{\Pi_\perp \cdot r}\right)\left(\delta_N^L - \frac{Y_N Y^L}{Y^2}\right)\pi_L \qquad (5.5)$$

which are not all independent due to the identities: $\Pi_\perp \cdot Y \equiv 0$, $\pi_\perp \cdot Y \equiv 0$, $\Pi_\perp \cdot \pi_\perp \equiv 0$.

A set of functions $\varphi(Y, r, \Pi_\perp, \pi_\perp)$ constitutes a complete algebra of observables, which is closed with respect to Poisson brackets. Indeed, if φ_1 and φ_2 are observables then $\{\varphi_1, \varphi_2\}$ is observable due to the Jacobi identity. The particle positions y_a and the dynamical constraint $\Phi(Y, r, \Pi_\perp, \pi_\perp)$ are observable, thus, the algebra of observables is sufficient to formulate equations of motion generated by Dirac's Hamiltonian $H_\mathrm{D} = \lambda\Phi$. Aforementioned secondary constraints are not observable and can be abandoned, similarly to the single particle case.

The requirement of the dynamical constraint to be de Sitter invariant yields the following general structure:

$$\Phi(\Pi_\perp^2,\ \pi_\perp^2,\ \Pi{\cdot}r,\ \pi{\cdot}r) = 0, \tag{5.6}$$

where Φ may be an arbitrary function of its scalar arguments: $\Pi_\perp^2$, $\pi_\perp^2$ and $\Pi\cdot r \approx \Pi_\perp\cdot r$, $\pi\cdot r \approx \pi_\perp\cdot r$; here the use of weak equality "$\approx$" by virtue of the holonomic constraints (3.1), (3.4) simplifies the dynamical constraint but does not affect the dynamics of observables.

The dynamical constraint (5.6) determines implicitly one of the argument of Φ as a function of three other arguments. Since this function can be regarded arbitrary within a general consideration, the Hamiltonian formalism with constraints (4.2), (4.3), (5.6) embraces a variety of two-particle systems as wide as the original Fokker-type or Lagrangian formalism with the arbitrary function $f(\nu_1,\nu_2,\omega)$ does, except, perhaps, for some special cases.

The procedure of how to obtain the dynamical constraint, given the interaction function $f(\nu_1,\nu_2,\omega)$ in Lagrangian (4.5), is described in detail in Appendix A. In practice, however, elementary algebraic operations implied there can be rarely finished in a closed form.

Fortunately, two physically motivated examples, considered in Section III, are the cases. For the electromagnetic time-asymmetric interaction, one puts in (4.5) $f = \alpha_\mathrm{e}\,\omega$, where $\alpha_\mathrm{e} := e_1e_2$, and arrives at the following dynamical constraint:

$$\begin{aligned}\Phi_\mathrm{e} := \pi_\perp^2 + \tfrac14\Pi_\perp^2 + \frac{1}{4R^2}\left(\frac{(\pi_1{\cdot}r)(\pi_2{\cdot}r)}{\eta\Pi{\cdot}r} - \alpha_\mathrm{e}\right)(\eta\Pi{\cdot}r - 4\alpha_\mathrm{e}) - \frac{m_1^2\,\pi_2{\cdot}r + m_2^2\,\pi_1{\cdot}r}{\Pi{\cdot}r}\\
- \alpha_\mathrm{e}\frac{\Pi_\perp^2 - m_1^2 - m_2^2}{\eta\Pi{\cdot}r} + \alpha_\mathrm{e}^2\left(\frac{m_1^2}{\eta\pi_1{\cdot}r - \alpha_\mathrm{e}} + \frac{m_2^2}{\eta\pi_2{\cdot}r - \alpha_\mathrm{e}}\right) = 0.\end{aligned} \tag{5.7}$$

For the scalar interaction $f = \alpha_\mathrm{s} := g_1g_2$, and the dynamical constraint has the form:

$$\Phi_\mathrm{s} := \pi_\perp^2 + \tfrac14\Pi_\perp^2 + \frac{(\pi_1{\cdot}r)(\pi_2{\cdot}r)}{4R^2} - \frac{m_1^2\,\pi_2{\cdot}r + m_2^2\,\pi_1{\cdot}r + 2\eta\alpha_\mathrm{s}m_1m_2}{\Pi{\cdot}r\left(1 - \dfrac{\alpha_\mathrm{s}^2}{(\pi_1{\cdot}r)(\pi_2{\cdot}r)}\right)} = 0. \tag{5.8}$$

Both constraints (5.7) and (5.8) reduce in the free-particle limit $\alpha\to0$ to the constraint:

$$\Phi_\mathrm{free} := \pi_\perp^2 + \tfrac14\Pi_\perp^2 + \frac{(\pi_1{\cdot}r)(\pi_2{\cdot}r)}{4R^2} - \frac{m_1^2\,\pi_2{\cdot}r + m_2^2\,\pi_1{\cdot}r}{\Pi{\cdot}r}. \tag{5.9}$$

This case of the time-asymmetric system with no interaction (i.e., $f = 0$) deserves a particular consideration. The free-particle dynamical constraint (5.9) is not an additive function in variables of different particles. The reason is that this constraint is concerted with the light cone constraint (4.3), which, in turn, binds in an isotropic interval the positions of even free particles. In Appendix B, the dynamics of two free particles is manifested from this tangled description.

The system determined by the set of 1st-class constraints (4.2), (4.3), and (5.6) has 6 degrees of freedom. Besides, as it follows from the structure of the Lie algebra (5.1) of de Sitter group [35], of ten components of the conserved angular momentum tensor (4.7) one can construct six integrals of motion, which are in an involution in terms of Poisson brackets. This is sufficient for the system to be integrable in the Liouville sense. The next natural step would be a transition to the description on a reduced 12-dimensional phase space, and separating degrees of freedom by choosing appropriate canonical variables. It turned out more constructive to analyze the system within the manifestly covariant description on the 20-dimensional phase space $\mathrm{T}^*\mathbb{M}_5^2$ where de Sitter symmetry is realized in a transparent way.

VI. EQUATIONS OF MOTION AND THEIR INTEGRATION

Useful integrals of motion arise from two Casimir functions of the de Sitter algebra (5.1):

$$J^2 := -\mathrm{tr}(\mathsf{J}^2) = J_{MN}J^{MN}, \qquad V^2 := V_MV^M, \tag{6.1}$$

where the following 5-pseudo-vector

$$V_M := \frac18\epsilon_{MABCD}J^{AB}J^{CD} \tag{6.2}$$

is introduced by means of the Levi-Chivita symbol ϵ_{MABCD}. Then using the equalities

$$\begin{aligned}\Pi_\perp^2 &\approx -\frac{1}{R^2}\left(\tfrac12J^2 + (\pi{\cdot}r)^2\right),\\
\pi_\perp^2 &\approx -\frac{1}{(\Pi{\cdot}r)^2R^2}\left(V^2 - \tfrac12(\pi{\cdot}r)^2J^2 - (\pi{\cdot}r)^4\right)\end{aligned} \tag{6.3}$$

recasts arguments of the dynamical constraint (5.6) into an equivalent set,

$$\Phi(\Pi{\cdot}r,\ \pi{\cdot}r;\ J^2, V^2) = 0, \tag{6.4}$$

which is more convenient for a dynamical analysis.

Let us consider the equation of motion for the relative position 5-vector r:

$$\dot{r} = \lambda\left\{r, \Phi(\Pi{\cdot}r,\ \pi{\cdot}r;\ J^2, V^2)\right\}$$
$$= \lambda\left(\frac{\partial\Phi}{\partial\,\pi{\cdot}r} - 4\frac{\partial\Phi}{\partial J^2}\mathsf{J} - \frac{\partial\Phi}{\partial V^2}\mathsf{K}\right) r; \tag{6.5}$$

here $\mathsf{J} := ||J^M{}_N||$ and

$$\mathsf{K} := ||K^M{}_N|| := ||\epsilon^M{}_{NABC}V^A J^{BC}|| \tag{6.6}$$

are conserved matrices while the Lagrangian multiplier $\lambda(\tau)$ as a function of τ is unspecified and can be chosen for convenience.

If the variable $\Pi\cdot r = \Psi(\tau)$ was known as a function of τ, then $\pi\cdot r = \psi(\tau)$ could be found from the dynamical constraint (6.4) as a solution of the algebraic equation: $\Phi(\Psi(\tau), \psi(\tau); J^2, V^2) = 0 \quad \Longrightarrow \quad \psi(\tau) := \psi(\Psi(\tau);\ J^2, V^2)$ (since J^2, V^2 are conserved).

In turn, the Hamiltonian equation for $\Psi(\tau)$,

$$\dot{\Psi} = \lambda\left\{\Psi, \Phi\right\}$$
$$= \lambda\,\frac{\partial\Phi(\Psi,\ \psi(\Psi; J^2, V^2);\ J^2, V^2)}{\partial\psi}\,\Psi, \tag{6.7}$$

is self-sufficient, separable in τ and Ψ, and reduces obviously to quadratures. Note that the resulting function $\Psi(\tau)$ depends on the choice of the Lagrange multiplier $\lambda(\tau)$. Alternatively, one can choose $\Psi(\tau)$ and then find $\lambda(\tau)$ from eq. (6.7) without integration. The choice of the function $\Psi(\tau)$ implies a fixing of the evolution parameter τ.

At this point, equation (6.5) becomes a closed linear equation with respect to 5-vector y, with known τ-dependent matrix coefficients. The substitution

$$r(\tau) = \frac{\Psi(\tau)}{\Psi(0)}\,q(\tau) \tag{6.8}$$

simplifies this equation to the form:

$$\dot{q} = -\lambda\left(4\frac{\partial\Phi}{\partial J^2}\mathsf{J} + \frac{\partial\Phi}{\partial V^2}\mathsf{K}\right) q. \tag{6.9}$$

A subsequent integration procedure is based on the projection operator techniques described in Appendix C. The structure and the action of projection operators depend on eigenvalues of matrix J, which, in turn, depend on the values of the Casimir functions (6.1). Here we suppose $J^2 < 0$ and $V^2 < 0$ so that J possesses the following eigenvalues: $\pm\Sigma := \pm\sqrt{\Sigma_+^2}$, $\pm\mathrm{i}\,S := \pm\sqrt{\Sigma_-^2}$ and 0, where $\Sigma_\pm^2$ are defined in (C.3). Other cases can be treated similarly; they are omitted here.

Let us decompose the 5-vector q (and then other position 5-vectors) by means of projection operators (C.12)–(C.13) defined in Appendix C:

$$q = (\mathcal{O}^{(\Sigma)} + \mathcal{O}^{(S)} + \mathcal{P}^{(0)})q := q^{(\Sigma)} + q^{(S)} + q^{(0)}. \tag{6.10}$$

Projectors (C.12)–(C.13) commute with matrix J. Using this fact and the properties (C.16) of matrix (6.6) permits one to split equation (6.9) into the set:

$$\dot{q}^{(i)}(\tau) = f^{(i)}(\tau)\mathsf{J}\,q^{(i)}(\tau), \qquad i = \Sigma, S, 0, \tag{6.11}$$

where $f^{(0)}(\tau) \equiv 0$,

$$f^{(\Sigma)}(\tau) := -\lambda\left(4\frac{\partial\Phi}{\partial J^2} + 2S^2\frac{\partial\Phi}{\partial V^2}\right), \quad f^{(S)}(\tau) := -\lambda\left(4\frac{\partial\Phi}{\partial J^2} - 2\Sigma^2\frac{\partial\Phi}{\partial V^2}\right). \tag{6.12}$$

Then formal solutions of equations (6.11)–(6.12) are:

$$q^{(i)}(\tau) = \exp\{F^{(i)}(\tau)\mathsf{J}\}r^{(i)}(0), \quad \text{where } F^{(i)}(\tau) := \int_0^\tau \mathrm{d}\tau\, f^{(i)}(\tau), \quad i = \Sigma, S, 0. \tag{6.13}$$

Matrix exponents in these solutions can be unraveled by means of eqs. (C.14):

$$q^{(\Sigma)}(\tau) = \left(\cosh\left(\Sigma F^{(\Sigma)}(\tau)\right) + \frac{\mathsf{J}}{\Sigma}\sinh\left(\Sigma F^{(\Sigma)}(\tau)\right)\right) r^{(\Sigma)}(0), \tag{6.14}$$

$$q^{(S)}(\tau) = \left(\cos\left(SF^{(S)}(\tau)\right) + \frac{\mathsf{J}}{S}\sin\left(SF^{(S)}(\tau)\right)\right) r^{(S)}(0), \tag{6.15}$$

$$q^{(0)}(\tau) = r^{(0)}(0). \tag{6.16}$$

A convolution of 5-vector r with the angular momentum tensor (4.7) expressed in terms of collective variables (5.3) yields the equality for the 5-vector Y:

$$Y \approx \frac{\mathsf{J} - \psi}{\Psi} r. \tag{6.17}$$

Then eqs. (6.8), (6.10), (6.14)–(6.17) lead to the expressions for particle positions

$$y_a^{(i)}(\tau) = \frac{1}{\Psi(0)} \{\mathsf{J} - \psi(\tau) - \tfrac{1}{2}(-)^a \Psi(\tau)\}\, q^{(i)}(\tau), \tag{6.18}$$
$$a = 1, 2, \quad i = \Sigma, S, 0,$$

where all the quantities in r.-h.s. are known functions of τ at this point.

In order to have a complete solution of the Cauchy problem, it is sufficient to express the angular momentum matrix J and its invariants Σ, S in terms of initial values $y_a(0)$, $\dot{y}_a(0)$ by eqs. (4.7), (4.8) and (6.1), (6.2), (C.3). If the initial point belongs to $\mathrm{T}\,\mathbb{K}$, i.e., the initial values $y_a(0)$, $\dot{y}_a(0)$ are subjected to the conserved holonomic constraints (4.2), (4.3) and their differential consequences (see also (A.7), (A.8)), then the particle world lines (6.18) lie in $\mathbb{K}$ by construction. The momentum-type variables (5.4), (5.5) are subsidiary and not important within the classical consideration.

VII. CONCLUSION

Green functions of massless fields in the Minkowski space-time are located on the light cone surface. This field-theoretical outcome was basic for a construction of the original Staruszkiewicz–Rudd–Hill model and its non-electromagnetic generalizations.

In a curved space-time, the Green function of electromagnetic and other massless fields has a nonlocal tail spread in the light cone interior [16]. It is shown here that in particular case of de Sitter space-time the nonlocal contribution of the electromagnetic Green function in the Tetrode–Fokker action integral can be converted to a dynamically equivalent local contribution. The nonlocal contribution of the scalar Green function is unavoidable, if the theory of minimal coupling is implied. Instead, the Green function of the scalar field conformally coupled to de Sitter metrics is shown to be purely local. These two examples of field-theoretical nature are included in a wide class of time-asymmetric models built from general requirements of de Sitter symmetry and self-consistency of the Hamiltonian dynamics.

Every time-asymmetric model has 6 degrees of freedom and 6 integrals of motion in involution, which are independent functions of canonical generators J_{MN} of O(1,4) group [35]. Thus, these dynamical systems are integrable in the Liouville sense. In practice, the integrability presupposes a choice of appropriate canonical variables in terms of which degrees of freedom separate. In the case of curved de Sitter space-time, this task encounters technical difficulties when constructing the description in a 12-dimensional phase space.

Thus, in the present paper the time-asymmetric models are treated as constrained systems in 20-dimensional phase space $\mathrm{T}^*\mathbb{M}_5^2$. de Sitter invariance of all the constraints admits a formulation of equations of motion in a manifestly covariant 5-dimensional form. Moreover, there exists some analogy between the dynamics of a relativistic particle in a constant electromagnetic field [36, 37] and the present problem. As the Maxwell tensor in the first case, the conserved angular 5-momentum tensor in the second case is skew-symmetric, treated as constant and covariantly "mounted" into equations of motion. Thus the projection operator technique, used in the first case [36, 37], is adapted here to the present 5-dimensional case. In such a way, the equations of motion are split and solved in quadratures.

It was noted in the Introduction that the Staruszkiewicz–Rudd–Hill model in a flat space-time endows corresponding two-particle systems with physically meaningful features. What distinguishes the model from the retarded or Wheeler–Feynman electrodynamics is the time-asymmetric retarded-advanced causal structure of interaction, a price for the solvability of the model. Even so, the classical model represents properly relativistic effects in a system of two charged particles within the moderately relativistic domain where the radiation reaction is minor. The quantum versions of this model and some other time-asymmetric models yield relativistic spectra, which accord well with the results of the quantum field theory [14] and actual meson spectroscopy [15].

A study of de-Sitter-relativistic effects in systems of single gravitating bodies and test particles [21, 25, 26] deepen our understanding of the expanding Universe. The next step in this direction would be a prospective elaboration of de Sitter invariant two-particle models with electromagnetic and other interactions. Quantization of time-asymmetric models in de Sitter space can be focus of future works.

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APPENDIX

A. The relation between the Lagrangian function and the dynamical constraint of time-asymmetric models

Having chosen the sign $\eta = 1$ or $\eta = -1$ in the model (see Section IV), let us present the Lagrangian (4.5) in the equivalent form:

$$L = \vartheta F(\nu_1, \nu_2, \omega), \tag{A.1}$$

$$\vartheta := \eta \dot{y}_1 \cdot r = \eta \dot{y}_2 \cdot r = \eta(\dot{y}_1 + \dot{y}_2) \cdot r/2 > 0, \tag{A.2}$$

$$F := -\sum_{a=1}^{2} \frac{\eta m_a}{R\nu_a} - \frac{f(\nu_1, \nu_2, \omega)}{R^2 \nu_1 \nu_2}, \tag{A.3}$$

where F is a function of the scalar arguments (3.9) and, thus, is a homogeneous function of degree zero of particle velocities $\dot{y}_a$. The scalar factor ϑ is homogeneous of degree one and positive on timelike world lines. It is presented in (A.2) diversely by accounting a differential consequence $\dot{y}_1 \cdot r = \dot{y}_2 \cdot r$ of the light cone constraint (4.3). In this regards, an apparent particle asymmetry of the interaction term of the Lagrangian (4.5) is seen.

In terms of functions (A.2), (A.3) and the collective variables (5.3), the Legendre transform (4.8) acquires the manifestly covariant 5-vector form:

$$\Pi = \frac{\partial L}{\partial \dot{Y}} = A(\nu_1, \nu_2, \omega)\frac{\dot{Y}}{\vartheta} + B(\nu_1, \nu_2, \omega)\frac{\dot{r}}{\vartheta} + D(\nu_1, \nu_2, \omega)\eta r, \tag{A.4}$$

$$\pi = \frac{\partial L}{\partial \dot{y}} = B(\nu_1, \nu_2, \omega)\frac{\dot{Y}}{\vartheta} + C(\nu_1, \nu_2, \omega)\frac{\dot{r}}{\vartheta}, \tag{A.5}$$

where

$$A := A_1 + A_2, \quad B := \frac{A_1 - A_2}{2}, \quad C := \frac{A}{4} + \frac{\partial f}{\partial \omega},$$

$$D := \frac{1}{R^2}\left[\frac{f}{\nu_1 \nu_2} - \frac{1}{\nu_2}\frac{\partial f}{\partial \nu_1} - \frac{1}{\nu_1}\frac{\partial f}{\partial \nu_2}\right],$$

$$A_a := -\eta R m_a \nu_a - \frac{\nu_a}{\nu_{\bar{a}}} f + \frac{\nu_a^2}{\nu_{\bar{a}}}\frac{\partial f}{\partial \nu_a} + \left[\frac{\nu_a}{\nu_{\bar{a}}}\omega - 1\right]\frac{\partial f}{\partial \omega}; \qquad a = 1, 2, \quad \bar{a} = 3 - a. \tag{A.6}$$

The right-hand side of eqs. (A.4), (A.5) is evidently zero-degree homogeneous in $\dot{Y}$, $\dot{y}$; thus, the Legendre transform is degenerated.

We are interested in relations between scalars on $\mathrm{T}^*\mathbb{M}_5^2$ and $\mathrm{T}\mathbb{K}$, where $\mathrm{T}\mathbb{K} \subset \mathrm{T}\mathbb{M}_5^2$ is described by the holonomic constraints (4.2), (4.3) and their differential consequences expressed for convenience in terms of the collective variables (5.3):

$$Y^2 = -R^2; \qquad Y \cdot r = 0; \qquad r^2 = 0, \ \eta r^0 > 0; \tag{A.7}$$

$$\dot{Y} \cdot Y = 0; \qquad \dot{Y} \cdot r = -\dot{r} \cdot Y; \qquad \dot{r} \cdot r = 0. \tag{A.8}$$

Multiplying eqs. (A.4), (A.5) by r and Y and accounting (A.7), (A.8) yields the relations:

$$\Pi \cdot r = \eta A(\nu_1, \nu_2, \omega); \qquad \pi \cdot r = \eta B(\nu_1, \nu_2, \omega); \tag{A.9}$$

$$\Pi \cdot Y = -\eta B(\nu_1, \nu_2, \omega); \qquad \pi \cdot Y = -\eta C(\nu_1, \nu_2, \omega). \tag{A.10}$$

Among these scalars on $\mathrm{T}^*\mathbb{M}_5^2$, two of them, $\Pi \cdot r$ and $\pi \cdot r$, are observables, and they are arguments of dynamical constraint (5.6). Squaring eq. (A.4), one can express scalar Π^2 in terms of ν_1, ν_2, ω. Scalars $\Pi \cdot Y$, $\pi \cdot Y$ and Π^2 are not observables, but they are related to the third argument $\Pi_\perp^2$ of (5.6) via the following equality derived by squaring eq. (5.4): $\Pi_\perp^2 = \Pi^2 + [(\Pi \cdot Y)^2 + 2(\Pi \cdot r)(\pi \cdot Y)]/R^2$. Using this and previous equations yields:

$$\Pi_\perp^2 = \frac{1}{R^2}\left[\frac{A_1^2}{\nu_1^2} + \frac{A_2^2}{\nu_2^2} + 2\omega\frac{A_1 A_2}{\nu_1 \nu_2} + B^2 - 2AC\right] + 2AD. \tag{A.11}$$

In general, three equations (A.9) and (A.11) can be inverted yielding ν_1, ν_2 and ω as functions of $\Pi\cdot r$, $\pi\cdot r$ and $\Pi_\perp^2$. For these functions we will use notations $\bar\nu_1$, $\bar\nu_2$, $\bar\omega$, and $\bar A := A(\bar\nu_1,\bar\nu_2,\bar\omega) = \eta\Pi\cdot r, \ldots, \bar D := D(\bar\nu_1,\bar\nu_2,\bar\omega)$ etc. At this point, the set of equations (A.4), (A.5) can be formally inverted yielding velocities in terms of canonical variables:

$$\frac{\dot Y}{\vartheta} = \frac{\bar C}{\bar\Delta}(\Pi - \bar D\eta r) - \frac{\bar B}{\bar\Delta}\pi, \qquad \frac{\dot r}{\vartheta} = \frac{\bar A}{\bar\Delta}\pi - \frac{\bar B}{\bar\Delta}(\Pi - \bar D\eta r), \tag{A.12}$$

where $\Delta := AC - B^2$. Then the l.h.-s. of eq. (4.9) can be regarded as the Hamiltonian, proportional to the dynamical constraint: $H_{\rm D} \propto \Phi = 0$. Inserting expressions (A.12) for particle velocities $\dot y_a = \dot Y - \frac{1}{2}(-)^a\dot r$ $(a = 1,2)$ into l.h.-s. of eq. (4.9) yields the dynamical constraint of the form (5.6):

$$\pi_\perp^2 + \frac{\bar C^2}{R^2} + \frac{\bar C}{\eta\Pi\cdot r}\left[\Pi_\perp^2 - \frac{(\pi\cdot r)^2}{R^2}\right] - \frac{\bar\Delta(\bar F + \bar D)}{\eta\Pi\cdot r} = 0. \tag{A.13}$$

It determines the scalar observable $\pi_\perp^2$ via three other arguments $\Pi\cdot r$, $\pi\cdot r$ and $\Pi_\perp^2$ of dynamical constraint (5.6). In the free-particle case $f = 0$, we arrive at eq. (5.9).

One can obtain other relations between canonical variables, such as $\Pi\cdot Y + \pi\cdot r = 0$, following from (A.9), (A.10). These relations represent secondary constraints, mentioned in Section V, which involve unobservable quantities and, thus, do not have a physical meaning.

B. The free-particle system

The free-particle dynamical constraint (5.9) can be presented diversely:

$$\Phi_{\rm free} \approx \frac{\pi_2\cdot r}{\Pi\cdot r}\phi_1 + \frac{\pi_1\cdot r}{\Pi\cdot r}\phi_2 = 0, \tag{B.1}$$

where $\phi_a := \pi_{a\perp}^2 - m_a^2$ and $\pi_{a\perp} := \pi_a - \dfrac{y_a\cdot\pi_a}{y_a^2}y_a$ $(a = 1,2)$. This form is more convenient here. It does not imply, however, that both expressions ϕ_1 and ϕ_2 vanish (as one could opine from Section II), so that $\phi_- := \frac{1}{2}(\phi_1 - \phi_2) \neq 0$.

The Hamilton equations for the position 5-vectors read:

$$\dot y_a = \lambda\{y_a, \Phi_{\rm free}\} \approx 2\lambda\frac{\pi_{\bar a}\cdot r}{\Pi\cdot r}\left(\pi_{a\perp} + (-)^a\frac{\phi_-}{\Pi\cdot r}r\right), \qquad \begin{array}{l} a = 1,2, \\ \bar a = 3 - a, \end{array} \tag{B.2}$$

and yield the expressions for the unit 5-velocities of particles:

$$v_a \equiv \frac{\dot y_a}{\sqrt{\dot y_a^2}} \approxeq \frac{1}{m_a}\left(\pi_{a\perp} + (-)^a\frac{\phi_-}{\Pi\cdot r}r\right) \tag{B.3}$$

which are free of the unspecified Lagrangian multiplier λ; here symbol " $\approxeq$ " denotes a weak equality by virtue of all the constraints (4.2), (4.3) and (5.9).

Differentiating equalities (B.3) and using the Hamiltonian equations (B.2) and corresponding equations for π_a yields the expressions for derivatives $\dot v_a$:

$$\dot v_a \approxeq 2\lambda\frac{\pi_{\bar a}\cdot r}{\Pi\cdot r}\frac{\sqrt{\dot y_a^2}}{m_a}R^2 y_a. \tag{B.4}$$

From (B.2) and (B.4) the 2nd-order equations of motion follow:

$$\frac{\rm d}{{\rm d}\tau}\frac{\dot y_a}{\sqrt{\dot y_a^2}} - \sqrt{\dot y_a^2}\frac{y_a}{R^2} = 0, \qquad a = 1,2. \tag{B.5}$$

They are split in variables of different particles, and coincide for each particle with the test body equation (2.5). The solutions $y_a(\tau)$ have forms (2.6), and (2.7) for each particle $a = 1,2$.

C. The angular 5-momentum tensor and projection operators

Components of the angular 5-momentum tensor form the skew-symmetric odd-dimensional matrix $||J_{MN}||$, thus, one of its eigenvalues is zero. The same is true for the matrix $\mathsf{J} := ||J^M{}_N|| := ||\eta^{ML}J_{LN}||$. In order to find other eigenvalues of J, one can use the Hamilton-Cayley theorem and construct the characteristic equation for J. It obviously includes odd degrees of J up to five with de Sitter invariant coefficients. Then by direct calculations one arrives at the desirable identity:

$$\mathsf{J}^5 + \tfrac{1}{2}J^2\mathsf{J}^3 + V^2\mathsf{J} \equiv 0, \tag{C.1}$$

where J^2 and V^2 are two Casimir functions of de Sitter algebra, defined by eqs. (6.1), (6.2). The l.-h.s. of (C.1) can be formally factorized:

$$(\mathsf{J}^2 - \Sigma_+^2)(\mathsf{J}^2 - \Sigma_-^2)\mathsf{J} = 0, \tag{C.2}$$

where

$$\Sigma_\pm^2 := -\tfrac{1}{4}J^2 \pm \sqrt{\mathcal{D}}, \qquad \mathcal{D} := J^4/16 - V^2. \tag{C.3}$$

Thus, matrix J has 5 eigenvalues $\pm\Sigma_+, \pm\Sigma_-, 0$.

Projection operators onto 1-dimensional subspaces corresponding to eigenvalues j of J can be introduced by a standard technique; see for example [37]:

$$\mathcal{P}^{(j)} = \prod_{j'\neq j} \frac{\mathsf{J} - j'}{j - j'}, \qquad j = \pm\Sigma_+, \pm\Sigma_-, 0; \tag{C.4}$$

here j' in the product runs over all eigenvalues except j.

In general, the Casimir functions J^2 and V^2 and, thus, the discriminant $\mathcal{D}$ can acquire arbitrary real (positive or negative) values, so that the eigenvalues j can be real or complex. Here, however, we limit this arbitrariness by natural physical restrictions.

For the single-particle case $J^2 \approx -m^2R^2$, while $V = 0$. In the case of two free particles, one obtains from (4.7), (4.8) and (4.5) (with $f = 0$):

$$\varkappa := -\frac{J^2}{4m_1m_2R^2} = \mu + \omega - \nu_1\nu_2, \tag{C.5}$$

$$\chi := \frac{V^2}{m_1^2m_2^2R^4} = \nu_1^2 + \nu_2^2 - 2\nu_1\nu_2\omega + \nu_1^2\nu_2^2, \tag{C.6}$$

$$\delta := \frac{\mathcal{D}}{m_1^2m_2^2R^4} = \varkappa^2 - \chi = (\mu+\omega)^2 - \nu_1^2 - \nu_2^2 - 2\mu\nu_1\nu_2, \tag{C.7}$$

where ω and ν_a $(a = 1, 2)$ are defined by eq. (3.9), and $\mu := \frac{1}{2}[\frac{m_1}{m_2} + \frac{m_2}{m_1}] \geq 1$.

Let us evaluate J^2 and $\mathcal{D}$ (or $\varkappa$ and δ) on the time-like world lines, for which $v_a^2 = 1$, $v_a^0 \geq 1$ $(a = 1, 2)$. Since the Casimir functions are integrals of motion and O(1,4)-invariants, it is sufficient to evaluate r.h.-s. of (C.5)–(C.7) at the initial moment $\tau = 0$ in an arbitrary reference frame.

We will use the 3-vector notations for 5-vectors: $y = \{y^0, y^1, y^2, y^3, y^4\} := \{y^0, \boldsymbol{y}, y^4\}$.

Let us start with the case $\eta = +1$, i.e., $y_1^0 > y_2^0$.

The action of the group O(1,4) on the hyperboloid $\mathbb{H}$ is transitive [21]. Thus, there exists a reference frame where the starting 5-position y_1 of the 1st particle and its 5-velocity v_1 are as follows:

$$y_1 = \{0, \mathbf{0}, R\}, \qquad v_1 = \{1, \mathbf{0}, 0\}. \tag{C.8}$$

Thus, $\omega = v_2^0 \geq 1$. Besides, it follows from (C.8) and constrains (4.2), (4.3) that: $y_2^4 = R$, $y_2^0 = -|\boldsymbol{y}_2|$ with arbitrarily chosen 3-vector $\boldsymbol{y}_2$, i.e.,

$$y_2 = \{-|\boldsymbol{y}_2|, \boldsymbol{y}_2, R\}. \tag{C.9}$$

Now, using the differential consequence $y_2 \cdot v_2 = 0$ of constrains (4.2) yields $v_2^4 < 0$. Thus, $\nu_1 = -y_2 \cdot v_1 = |\boldsymbol{y}_2|/R > 0$, $\nu_2 = y_1 \cdot v_2 = -v_2^4 > 0$, and $\omega^2 - \nu_2^2 \geq 1$, $\omega - \nu_2 > 0$.

Finally we impose the additional condition $y^0 + y^4 > 0$. It selects a half of the hyperboloid $\mathbb{H}$ which corresponds to the flat exponentially expanding Friedmann universe [21]. It is obviously from (C.8) $y_1^0 + y_1^4 > 0$. If the second particle belongs to the same universe, i.e., $y_2^0 + y_2^4 > 0$, then the restriction $|\boldsymbol{y}_2| < R$ follows from (C.9). Thus $\nu_1 < 1$. Using all these inequalities yields the estimates:

$$\varkappa = \mu + \omega - \nu_1\nu_2 > \mu + \omega - \nu_2 > \mu,$$

$$\delta = (\mu+\omega)^2 - \nu_1^2 - \nu_2^2 - 2\mu\nu_1\nu_2 > (\mu+\omega)^2 - \omega^2 - 2\mu\omega = \mu^2,$$

so that

$$J^2 < -2(m_1^2+m_2^2)R^2 < 0, \quad \mathcal{D} > (m_1^2+m_2^2)^2R^4/4 > 0 \Rightarrow \ \Sigma_+^2 > (m_1^2+m_2^2)R^2 > 0 \tag{C.10}$$

while Σ_-^2 can be negative or positive.

For $\eta = -1$ the same estimates can be obtained by the particle permutation $1 \leftrightarrow 2$.

If an interaction of particles is present but not too strong to close up the gaps $\propto m_1^2+m_2^2$ in (C.10), the inequalities $J^2 < 0$, $\mathcal{D} > 0$ may hold, and we have again $\Sigma_+^2 > 0$ and $\Sigma_-^2 \lesseqgtr 0$.

Here we consider the case $\Sigma_+^2 := \Sigma^2 > 0$, $\Sigma_-^2 := -S^2 < 0$ in detail. The matrix J has 5 eigenvalues: $\pm\Sigma$, $\pm\mathrm{i}\,S$ (where $\Sigma > S > 0$) and 0.

Projection operators (C.4) onto 1-dimensional subspaces corresponding to these eigenvalues have the form:

$$\mathcal{P}^{(\pm\Sigma)} := \frac{(\mathsf{J}\pm\Sigma)(\mathsf{J}^2+S^2)\mathsf{J}}{2\Sigma^2(\Sigma^2+S^2)}, \quad \mathcal{P}^{(\pm\mathrm{i}\,S)} := \frac{(\mathsf{J}\pm\mathrm{i}\,S)(\mathsf{J}^2-\Sigma^2)\mathsf{J}}{2S^2(\Sigma^2+S^2)}, \tag{C.11}$$

$$\mathcal{P}^{(0)} := -\frac{(\mathsf{J}^2+S^2)(\mathsf{J}^2-\Sigma^2)}{\Sigma^2S^2}. \tag{C.12}$$

Instead of projectors (C.11), it is convenient to use analogs of Fradkin operators [36, 37]:

$$\mathcal{O}^{(\Sigma)} := \mathcal{P}^{(+\Sigma)} + \mathcal{P}^{(-\Sigma)} = \frac{(\mathsf{J}^2+S^2)\mathsf{J}^2}{\Sigma^2(\Sigma^2+S^2)},$$

$$\mathcal{O}^{(S)} := \mathcal{P}^{(+\mathrm{i}\,S)} + \mathcal{P}^{(-\mathrm{i}\,S)} = \frac{(\mathsf{J}^2-\Sigma^2)\mathsf{J}^2}{S^2(\Sigma^2+S^2)} \tag{C.13}$$

which project onto the corresponding 2-dimensional subspaces. We note the important properties of these operators:

$$\mathsf{J}^2\mathcal{O}^{(\Sigma)} = \Sigma^2\mathcal{O}^{(\Sigma)}, \qquad \mathsf{J}^2\mathcal{O}^{(S)} = -S^2\mathcal{O}^{(S)}, \qquad \mathsf{J}\,\mathcal{P}^{(0)} = 0. \tag{C.14}$$

In order to derive important properties of matrix K defined by eq. (6.6), it should be simplified. Accounting (6.2) in (6.6) and unraveling the convolution of Levi-Civita symbols $\epsilon_{.....}\epsilon^{.....}$ in terms of products of Kronecker symbols $\delta^{\cdot}_{\cdot}\cdots\delta^{\cdot}_{\cdot}$ yields the formula:

$$\mathsf{K} = 2\mathsf{J}^3 + J^2\mathsf{J}. \tag{C.15}$$

The action of projectors (C.12), (C.13) onto (C.15) results in the relations:

$$\mathcal{O}^{(\Sigma)}\mathsf{K} = 2S^2\mathcal{O}^{(\Sigma)}\mathsf{J}, \qquad \mathcal{O}^{(S)}\mathsf{K} = -2\Sigma^2\mathcal{O}^{(S)}\mathsf{J}, \qquad \mathcal{P}^{(0)}\mathsf{K} = 0. \tag{C.16}$$

Properties (C.14) and (C.16) are used in Section VII for the integration of the system.

The case $\Sigma_+^2 > 0$, $\Sigma_-^2 > 0$ can be considered similarly.

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ІНТЕҐРОВНІ ДВОЧАСТИНКОВІ СИСТЕМИ З ЧАСОАСИМЕТРИЧНИМИ ВЗАЄМОДІЯМИ В ПРОСТОРІ ДЕ СІТТЕРА

А. Дувіряк

Інститут фізики конденсованих систем НАН України,
вул. Свєнціцького 1, Львів, UA–79011, Україна

У межах електродинаміки Вілера–Фейнмана релятивістську систему взаємодійних двох зарядів описано різницево-диференційними рівняннями руху, які випливають із дії Тетроде–Фоккера. З такою варіаційною проблемою важко впоратися, вона погано пристосована до гамільтонізації та квантування.

Щоб уникнути цих труднощів, Старушкевич, а також Рудд і Гілл замінили в дії симетричну функцію Ґріна рівнянь Максвелла на спізнену(або випередну). Це дало змогу переформулювати динаміку в лаґранжеву, а також і гамільтонову форму, яка виявилась інтеґрованою завдяки точній пуанкаре-інваріантності. Фізично ця модель описує таку часоасиметричну взаємодію двох заряджених частинок: випередне поле першої частинки діє на другу частинку, спізнене поле другої частинки діє на першу частинку, а реакцією випромінювання знехтувано. Модель Старушкевича–Рудда–Гілла була узагальнена для інших часоасиметричних взаємодій (скалярної, ґравітаційної, утримної тощо), а відповідні квантові версії виявили фізичну змістовність.

Варіаційний принцип Тетроде–Фоккера можна узагальнити на криві часопростори, якщо відома відповідна функція Ґріна. Для простору де Сіттера електромагнетну функцію Ґріна побудували Гіґучі і Чеонґ, вона складається з двох частин: локальної з носієм на гіперповерхні світлого конуса і нелокальної — у його об'ємі. У цій роботі показано, що внесок нелокальної частини в дію можна звести до еквівалентного локального внеску. Це дало змогу своєю чергою сконструювати десіттерівський аналог моделі Старушкевича–Рудда–Гілла та узагальнити її на широкий клас часосиметричних взаємодій, включно з електромагнетною, скалярною та іншими.

Зображення простору де Сіттера як гіперболоїда в 5-вимірному просторі Мінковського допускає формулювання часоасиметричних моделей у межах лаґранжевого, а також і гамільтонового формалізму з в'язями. Динаміка інваріантна щодо групи де Сіттера O(1,4). Тому є 10 інтеґралів руху, зібраних у 5-вимірній кососиметричній матриці моменту імпульсу J. Розв'язок рівнянь руху загальної часоасиметричної моделі побудовано у квадратурах за допомогою проєкційних операторів, сконструйованих у термінах матриці J.

Ключові слова: простір де Сіттера, часоасиметричні моделі, інтеґровні системи.