MANDELSTAM PROBLEM

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The theory of thermal fluctuations in open hydrodynamic steady states (OHSS) is presented exclusively within the framework of hydrodynamics. The history of studies of fluctuations in a continuous medium with a stationary flux is described. It is shown that the application the fluctuation-dissipation theorem (FDT) to the OHSS with the requirement of fulfilling the Onsager's reciprocal relations (fluctuating hydrodynamics), is erroneous. The reason is that the flux, changing the dynamics and initial values of the fluctuations, violates the detailed balance existing in equilibrium. This is demonstrated by the example of the Mandelstam problem on fluctuations in a medium with a heat flux. For this problem, the structure dynamic factor is calculated for an isotropic solid and a liquid. The loss of time symmetry by the correlation functions of fluctuations and the asymmetry of their spectral representations in this problem is due to the spatial temperature variation, which determines the flux. In order to show the generality of this result for all OHSS with spatial heterogeneity, the Kelvin problem on thermal fluctuations of the interface displacements between two liquids is also considered. The upper moving liquid has velocity potential changes as the temperature in the Mandelstam problem. Reciprocal relations for both the Mandelstam and the Kelvin problems are pointed out.

Key words: Mandelstam, open hydrodynamic steady state (OHSS), fluctuations, flux, reciprocal relations, fluctuating hydrodynamics.

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I. INTRODUCTION

Ninety one years ago, Onsager established reciprocal relations obtained from the symmetry of the correlation functions of thermodynamic (hydrodynamic) fluctuations (detailed balance) [1]. To explain the origin of such symmetry, he put forward the principle of microscopic reversibility, based on the fact that the movement of molecules is carried out by equations that are even with respect to time. Since all substances are composed of molecules, it was believed that Onsager's reciprocal relations will always be executed.

Below, however, a problem is discussed for which this is not the case. We are talking about hydrodynamic fluctuations in steady states with a stationary flux. There are many examples of such states in hydrodynamics. Investigations of these systems, which began about half a century ago, intensively use various kinds of statistical substantiation for the fulfillment of Onsager's reciprocal relations in such systems. In this article, we want to show that there is no need to use them, since all necessary statistical justifications in hydrodynamics have already been made. Thus, hydrodynamics is quite self-sufficient for solving problems of hydrodynamic fluctuations in the OHSS.

II. HISTORICAL OVERVIEW

The author's interest in thermal hydrodynamic fluctuations in the OHSS was sparked by Professor I. Z. Fisher in the early seventies of the last century. Fisher believed that the flux would yield peculiarities in the light scattering spectra, and thus, it would be possible to draw conclusions about the flux from the spectra.

As it turned out later, a similar idea was put forward much earlier by Mandelstam. In [2], he proposed to study the scattering of light by thermal fluctuations in a medium subjected to a stationary heat flux. Mandelstam expected that the scattering intensity from a certain small region would be determined not only by the temperature of the region itself, but also by its distribution. Thus, we can say that Mandelstam is the founder of the study of hydrodynamic fluctuations in the medium with fluxes. Note that a hundred years ago (from 1918 to 1922) L. I. Mandelstam headed the department of physics at our university, which was then called the Odesa Industrial Institute.

For the problem formulated by Mandelstam, Leontovich calculated heat flux fluctuations at some point of a crystal plate, whose temperature varies linearly in the direction perpendicular to the plate [3]. The basis of Leontovich's theory was the simple assumption that fluctuations are determined by the local temperature in the point under consideration. It follows from the results obtained by Leontovich that the fluctuation elastic acoustic waves observed in the scattered light should have different intensities, which was noticed by Vladimirskii [4], see also [5].

The steady state of the medium with the flux is in general non-equilibrium, so it is natural that fluctuations in such states are called non-equilibrium ones. Precisely due to the non-equilibrium, there was no strict base for the theory of such fluctuations from the very beginning of studies. This led to an abundance of various research methods. In particular, before [3], Leontovich considered, in his own words, "a one-dimensional and moreover rather special model of a crystal" [6].

The experiments performed at that time to detect the asymmetry of satellites were unsuccessful [7], obviously due to imperfections in the used equipment (experimental confirmation of the asymmetry occurred much later [8]). For this reason, and also in connection with the death of Mandelstam, the active study of non-equilibrium hydrodynamic fluctuations in the USSR stopped. A detailed history of the study of light scattering spectra until this period is described in [9].

A new impulse for the study of non-equilibrium hydrodynamic fluctuations was generated by Uhlenbeck in his review lectures [10], where he presented his vision of the transition from a microscopic description to a macroscopic one in statistical physics and in particular for the steady state with a flux. In detail the problem is considered in works [11, 12], where Fox and Uhlenbeck derived the Langevin FDT for fluctuating forces. This was done macroscopically by preserving the Onsager's reciprocal relations in the derivation of the FDT [11] and microscopically on the basis of the Boltzmann kinetic equation [12]. For the Navier–Stokes liquid, the main conclusion made here is that the sources of fluctuations in the equations of hydrodynamics will be the same as in an equilibrium liquid, regardless of the specific form of the considered non-equilibrium steady state, i.e. sources of fluctuations are universal for all OHSS. The same conclusion was made a year earlier by Bixon and Zwanzig [13] derived fluctuating forces from the Boltzmann equation using different method than [12].

The same kind of statistical justification has also been used in many works devoted to obtaining the FDT for systems with flux, see for example [14–16]. All of them are based on the fact that the source of fluctuations is randomly moving molecules. The macroscopic fields causing the non-equilibrium have scales significantly exceeding the molecular scales. Therefore, they cannot change significantly the molecular motion, and, consequently, the intensities of the sources of hydrodynamic fluctuations. The influence of such fields is reduced only to the modulation of the intensities of the equilibrium fluctuating sources.

Fluctuating sources for the equilibrium (flux less) liquid were established by Landau and Lifshitz [17], who applied the Callen–Welton FDT to the system of hydrodynamic equations for fluctuation perturbations. Earlier, a similar approach was used by Rytov for electromagnetic fluctuations [18]. Landau and Lifshitz found the correlation functions of fluctuating forces (or random fluxes) added to the equations of the Navier–Stokes hydrodynamics. Subsequently, they received the name of the Landau–Lifshitz fluctuating forces.

The finite intensity of the Landau–Lifshitz fluctuating forces and the statistical justification for their universal application to any OHSS allowed Uhlenbeck to suggest an increase of hydrodynamic fluctuations in non-equilibrium dissipative hydrodynamic steady state systems near stability thresholds. Zaitsev and Schliomis [19] were the first to consider the relatively simple convective Rayleigh–Benard instability with the Landau–Lifshitz fluctuating forces for a certain average temperature of the liquid layer. Their results confirmed the assumption of a fluctuation increase when approaching the stability threshold.

The subsequent modernization of the non-equilibrium Langevin FDT for the OHSS was the use of local values of thermodynamic parameters in the equilibrium formulas for the sources without changing the formulas themselves. It was believed that in this form the FDT takes into account local equilibrium in hydrodynamics. As a result, a whole new area has arisen in statistical physics, called fluctuating hydrodynamics, see [20] and references therein. Adherents of fluctuating hydrodynamics, referring to the non-equilibrium of the problem, reject other possible methods of solving the problem of fluctuations in the OHSS. Thus, the results of calculating correlation functions of fluctuations obtained by solving the Cauchy problem with the averaging of the initial conditions [21] for the same problem as in [19] were not recognized. Fluctuating hydrodynamics continues to develop rapidly [22].

The combination of kinetics to justify the sources of fluctuations and the equations of hydrodynamics to describe the evolution of fluctuations in fluctuating hydrodynamics makes the concept of non-equilibrium terra incognita for stationary hydrodynamic states. The terms "strong non-equilibrium", "weak non-equilibrium", "out of equilibrium", etc. are used in the literature. In fact, there can be no combined ("joint") description. Kinetics and hydrodynamics are clearly separated and have different temporal limits of applicability, which was shown by Bogoliubov back in 1946 [23]. Bogoliubov's work also gives an answer to the question of what constitutes a non-equilibrium in hydrodynamics, namely: hydrodynamics is valid in the approximation of local equilibrium. This conclusion and Onsager's regression hypothesis are the base that has been lacking since the Mandelstam problem arose, and that is the starting point in the whole theory presented below.

III. EVOLUTION TO STEADY STATE

Bogoliubov microscopically examined how a transition from a certain deviated state to an equilibrium state occurs. The set of coupled equations for distribution functions obtained by him from the exact Liouville equation was also studied in the works of Born, Green, Kirkwood, Yvon and was called the BBGKY hierarchy.

The solution of the corresponding system of equations is based on the difference in evolutionary time at different stages. There are three time scales determined by the collision time τ_0, the time between collisions t_0, and the time the molecule travels the inner size T_0. The orders of these quantities for gas are 10^{-12} s, 10^{-9} s, and split of a second, so that inequalities arise:

$$\tau_0 \ll t_0 \ll T_0. \tag{1}$$

Bogoliubov named the stage of the evolution of the system in the time interval $\tau_0 < t < t_0$ kinetic. The hydrodynamic stage sets in for times $t > t_0$. In liquids, where the time between collisions is the same as the collision time, the hydrodynamic stage is set in $10^{-12} - 10^{-9}$ s.

From the BBGKY hierarchy it follows that at the kinetic stage the one-particle distribution function is decisive. All multiparticle distribution functions are expressed in terms of one-particle. At the hydrodynamic stage, when enough collisions have occurred, a local equilibrium is established, and we should talk only about macroscopic hydrodynamic variables depending on the space-time coordinates and their fluctuations.

Thus, the stochasticity that occurs at the kinetic stage for a single molecule is not related to the hydrodynamic stochasticity of a liquid particle, when the state of the system is described by the field quantities, such as density, temperature, pressure. At the hydrodynamic stage, the thermodynamic limit is fulfilled for the number of molecules, so that the substance is a continuous medium for which there is no other "non-equilibrium" other than local equilibrium. Hydrodynamic fluctuations are fluctuations of locally equilibrium field quantities, or else, hydrodynamics is valid in the zero order by the Knudsen number; there are no molecules in it.

From a mathematical point of view, the assertion that hydrodynamic fluctuations are locally equilibrium is a consequence of the central limit theorem. Since any hydrodynamic variable is the result of the summation of such random variables as the number of molecules N, their momenta or energies in the limit $N \to \infty$, the distribution of fluctuations of hydrodynamic quantities should occur according to the normal law.

There is no reason to think that the picture of the evolution of the perturbations in the OHSS given by local hydrodynamic variables will be different. In this case, for large times $t > t_0$, it is generally accepted to use hydrodynamic equations for such perturbations (Onsager's regression hypothesis), while the hydrodynamic description that arises in the thermodynamic limit always implies local equilibrium.

The theory of hydrodynamic fluctuations under the conditions of local equilibrium formally does not differ from the theory of equilibrium fluctuations. In hydrodynamics the fulfillment of the limit $N \to \infty$ for any arbitrarily small liquid particle ensures the fulfillment of the law of large numbers and that fluctuations are small. The dynamics of small fluctuation deviations x_i from certain stationary values is determined by the linearized Navier–Stokes hydrodynamic equations

$$\dot{x}_i = -\lambda_{ij} x_j, \tag{2}$$

here and below we use the notation and definitions adopted in [24]. The statistical properties of fluctuations are specified by the locally equilibrium Gaussian distribution function of the initial values

$$f(\mathbf{x}) \propto \exp\left(-\frac{1}{2}\beta_{ij} x_i x_j\right). \tag{3}$$

The matrices λ and β depend on the macroscopic fields that determine the OHSS, and this is the main difference from the equilibrium version. Inhomogeneous fields lead to spatial dispersion of the corresponding matrices. Equations (2), (3) determine the random Ornstein–Uhlenbeck process, whose stationarity gives the possibility of its spectral, in time, representation. The spectral densities of the correlation functions of fluctuations are equal

$$(x_i x_j)_\omega = \int\limits_{-\infty}^{\infty} \langle x_i(t)\, x_j(0)\rangle e^{i\omega t} dt, \tag{4}$$

where angle brackets mean locally equilibrium averaging. We can calculate (4) using three equivalent methods. The first is direct calculation. The other two methods are more indirect and represent fluctuation-dissipation theorems — the first (Callen–Welton) and the second (Langevin). In this spirit, the work [25] was carried out, which continued the work [21].

IV. FLUCTUATING HYDRODYNAMICS

Here local equilibrium is considered as equilibrium, and states with a stationary flux are classified as non-equilibrium. Hence the use of the term "Non-equilibrium steady state" (NESS) and desire to consider such states from the first principles.

The Langevin fluctuation-dissipation theorem is the main method for studying fluctuations in fluctuating hydrodynamics. As is known, it consists in adding fluctuating forces with correlation functions

$$\langle y_i(t)\, y_j(0)\rangle = Q_{ij}\delta(t) \tag{5}$$

to the right-hand side of (2). To find correlation functions of fluctuations or their spectral densities, we must average the solution of inhomogeneous equations using (5). In fluctuating hydrodynamics, the intensity of fluctuating forces Q is determined so that Onsager's reciprocal relations must be fulfilled. As already said, microscopic and macroscopic substantiations of the choice of such intensities are used.

In the first case the starting point in [12, 13] is the solution of the usual Boltzmann equation in the form of a series over deviations from equilibrium. The zero term in this expansion is the equilibrium Maxwellian distribution function. It is immediately clear that as the zero state, the equilibrium state is used, where there are no fluxes. The dynamics of fluctuations for this expansion will be determined by the matrix λ_{eq} corresponding equilibrium, and the fluctuating forces will be the same as those of Landau and Lifshitz. If instead of the equilibrium Maxwellian distribution function we use the locally equilibrium one, keeping the same expansion, we again obtain the Landau and Lifshitz formulas, only with local temperature.

In the second case, the same result is achieved by referring to the symmetry of the intensity matrix Q [11, 16]. It is believed that only the symmetric part of the matrix λ, i.e., the matrix λ_{eq}, contributes to the intensity. This conclusion is illustrated by considering fluctuations of the oscillator relative to the zero position and hydrodynamic fluctuations in a fluid at a rest. That is, non-flux systems are considered.

Thus, the above works substantiated application of the Langevin FDT with sources determined by the equilibrium matrix λ_{eq} to the steady states with flux. According to fluctuating hydrodynamics, the matrix λ that determines the dynamics of fluctuations depends on stationary fluxes, but the intensity of Langevin sources, determined by the matrix λ_{eq}, does not.

The fallacy of this kind of conclusions is that the FDT, both the first and the second, are mathematical methods for solving systems of linear differential equations with random initial conditions and absolutely do not need any statistical justification. Indeed, formula (4) for the spectral density can be represented as

$$(x_i x_j)_\omega = \int\limits_0^\infty \langle x_i(t)\, x_j(0)\rangle\, e^{i\omega t} dt + \int\limits_0^\infty \langle x_j(t)\, x_i(0)\rangle\, e^{-i\omega t} dt, \qquad (6)$$

and the same spectral density obtained from equations (2) with fluctuating forces can be represented in the form

$$(x_i x_j)_\omega = (-i\omega\delta_{ki} + \lambda_{ki})^{-1} (i\omega\delta_{mj} + \lambda_{mj})^{-1} Q_{km}. \qquad (7)$$

The second FDT answers the question of what should be Q so that (6) and (7) coincide, and, therefore, has exclusively algebraic, and not statistical, origin. From equality (6) and (7) we obtain the only one formula for the second FDT

$$Q_{ij} = \gamma_{ij} + \gamma_{ji}, \qquad (8)$$

where $\gamma_{ij} = \lambda_{ik}\beta^{-1}_{kj}$ are kinetic coefficients, and $\beta^{-1}_{ij} = \langle x_i x_j\rangle$ are simultaneous correlation functions of fluctuations. Similarly, the first FDT follows from the definition of susceptibility and the same kind of algebraic transformations.

It is quite another matter that statistical physics determines the matrices λ and β^{-1} involved in the FDT. If, however, we are talking about hydrodynamics, then all the problems have already been solved: λ is determined by Onsager's regression hypothesis, and β^{-1} by local equilibrium, in accordance with the BBGKY hierarchy. The fact that in hydrodynamics non-equilibrium reduces to local equilibrium motivated the author to introduce the term containing the word "hydrodynamic" — "Open hydrodynamic steady states" (OHSS) [26] instead of the term "Non-equilibrium steady state" (NESS).

We will discuss now the concept of dissipative non-equilibrium phase transitions in fluctuating hydrodynamics. The "statistical justification" of the application of Landau–Lifshitz equilibrium forces to the OHSS means that Q is specified in the second FDT (8). Then (8) serves as an equation for determining the matrix β^{-1}, which naturally, will no longer be locally equilibrium. In particular, near the state of dissipative instability for some unstable mode from the FDT

$$Q = 2\lambda\beta^{-1} \qquad (9)$$

follows unlimited growth β^{-1} when dissipative quantity λ tends to zero.

In fact, the FDT determines not simultaneous correlation functions, but the intensity of random forces. That is why the interpretation of formula (9) is actually quite different. When the dissipative properties of the medium change, the matrix β^{-1} does not change, since it is determined by elastic properties that do not change. Therefore, instead of $\beta^{-1} \to \infty$, one should speak about reducing the source intensity Q to zero when $\lambda \to 0$. Herewith, in the spectrum of the unstable mode, an unlimited narrowing of the line occurs at a finite integral intensity in accordance with the results of the work [21].

And finally, something must be said regarding the use of local thermodynamic parameters in equilibrium formulas of Landau and Lifshitz. It would seem that this idea makes sense when there is sufficient reason to assume that fluxes do not change the dynamic equations for fluctuations. However, this is not the case. According to the FDT, the entire matrix of simultaneous correlation functions of fluctuations must be local equilibrium, but not only thermodynamic parameters, in particular, temperature. Mandelstam problem just illustrates this fact.

V. FLUCTUATIONS IN A SOLID UNDER A TEMPERATURE GRADIENT

Mandelstam problem of fluctuations in the medium under a temperature gradient turns out to be the simplest and, at the same time, the most fundamental. We consider the case of unbounded medium. This corresponds to the study of fluctuations in a certain volume far enough from the boundaries, so that their influence is insignificant due to the damping of fluctuations due to dissipative processes. We assume, as in [3], that the gradient does not change the dynamic equations, so all changes are due to the local temperature.

The main interest from the experimental point of view is the dynamic structure factor — the spatial-temporal Fourier transform of the autocorrelation function of some fluctuation quantity $\varphi(\mathbf{r}, t)$

$$S_{\varphi,\varphi}(\mathbf{k},\omega) = \frac{1}{TV}\int\limits_V d\mathbf{r} \int\limits_V d\mathbf{r}' \int\limits_{-T/2}^{T/2} dt \int\limits_{-T/2}^{T/2} dt' \times \langle \varphi(\mathbf{r}, t)\, \varphi(\mathbf{r}', t')\rangle\, e^{i\omega(t-t') - i\mathbf{k}(\mathbf{r}-\mathbf{r}')}, \qquad (10)$$

where the observation time T is quite long, and the volume V is large. According to our theory, the angle brackets mean locally equilibrium averaging.

First we use the direct method to calculate (10). The stationarity of a random process allows us to represent (10) in the form

$$S_{\varphi,\varphi}(\mathbf{k},\omega)=\frac{1}{V}\int\limits_V d\mathbf{r}\int\limits_V d\mathbf{r}' e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\left(\int\limits_0^\infty dt\left\langle\varphi(\mathbf{r},t)\varphi(\mathbf{r}',0)\right\rangle e^{i\omega t}+\int\limits_0^\infty dt\left\langle\varphi(\mathbf{r}',t)\varphi(\mathbf{r},0)\right\rangle e^{-i\omega t}\right). \tag{11}$$

Let the solution to the Cauchy problem for the evolution of fluctuations obtained from hydrodynamic equations have the form

$$\varphi(\mathbf{r},t)=\int\limits_V d\mathbf{r}' G(\mathbf{r}-\mathbf{r}',t)\,\varphi(\mathbf{r}',0), \tag{12}$$

where $G(\mathbf{r}-\mathbf{r}',t)$ is Green's function, and the simultaneous locally equilibrium fluctuation function determined by the local temperature is

$$\left\langle\varphi(\mathbf{r},0)\varphi(\mathbf{r}',0)\right\rangle=\alpha_\varphi(\mathbf{r})\,T(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}'), \tag{13}$$

where $\alpha_\varphi(\mathbf{r})$ is the static susceptibility, and

$$T(\mathbf{r})=T_0+\mathbf{r}\boldsymbol{\nabla}T. \tag{14}$$

So that

$$\begin{aligned}&\left\langle\varphi(\mathbf{r},0)\varphi(\mathbf{r}',0)\right\rangle\\&=\left\langle\varphi(\mathbf{r},0)\varphi(\mathbf{r}',0)\right\rangle^{\mathrm{eq}}(1+\mathbf{qr}),\end{aligned} \tag{15}$$

where the index eq means the equilibrium average at temperature T_0 and

$$\mathbf{q}=\frac{\boldsymbol{\nabla}T}{T_0}. \tag{16}$$

We assume that the change in temperature due to the gradient is small. The modulus of $\mathbf{r}$ is limited by $1/\alpha$ where α is the spatial attenuation coefficient. As a result, we obtain an inequality $q\ll\alpha$ which implies that the relative gradient should be much smaller than the attenuation coefficient. We also assume that, as usual, the attenuation is small, so that $\alpha\ll k$. Thus, we have the following inequalities $q\ll\alpha\ll k$.

Substituting (12),(15) into (11), we find that the dynamic structure factor is the sum of the equilibrium result with temperature T_0 and the contribution from the temperature gradient

$$S_{\varphi,\varphi}(\mathbf{k},\omega)=S^{\mathrm{eq}}_{\varphi,\varphi}(\mathbf{k},\omega)+\frac{1}{V}\int\limits_V d\mathbf{r}\int\limits_V d\mathbf{r}' e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\times\begin{pmatrix}\displaystyle\int\limits_0^\infty \mathbf{qr}'\left\langle\varphi(\mathbf{r},t)\varphi(\mathbf{r}',0)\right\rangle^{\mathrm{eq}}e^{i\omega t}dt\\ \displaystyle+\int\limits_0^\infty \mathbf{qr}\left\langle\varphi(\mathbf{r}',t)\varphi(\mathbf{r},0)\right\rangle^{\mathrm{eq}}e^{-i\omega t}dt\end{pmatrix}. \tag{17}$$

The last term can be transformed given that equilibrium correlation function $\left\langle\varphi(\mathbf{r},t)\varphi(\mathbf{r}',0)\right\rangle^{\mathrm{eq}}$ depends on the absolute value of the difference $\mathbf{r}-\mathbf{r}'$. Therefore, passing to the variables $(\mathbf{x},\mathbf{R})=\left(\mathbf{r}-\mathbf{r}',\frac{\mathbf{r}+\mathbf{r}'}{2}\right)$, we'll get

$$\Delta S_{\varphi,\varphi}(\mathbf{k},\omega)=-i\frac{\mathbf{q}}{2}\frac{\partial}{\partial\mathbf{k}}\left(\left\langle\varphi_{\mathbf{k},\omega}\varphi_{-\mathbf{k}}\right\rangle^{\mathrm{eq}}-\left\langle\varphi_{-\mathbf{k},-\omega}\varphi_{\mathbf{k}}\right\rangle^{\mathrm{eq}}\right)=\mathbf{q}\frac{\partial}{\partial\mathbf{k}}\mathrm{Im}\left\langle\varphi_{\mathbf{k},\omega}\varphi_{-\mathbf{k}}\right\rangle^{\mathrm{eq}}. \tag{18}$$

The final result is

$$S_{\varphi,\varphi}(\mathbf{k},\omega)=S^{\mathrm{eq}}_{\varphi,\varphi}(\mathbf{k},\omega)+\Delta S_{\varphi,\varphi}(\mathbf{k},\omega)=\left(2\,\mathrm{Re}+\mathbf{q}\frac{\partial}{\partial\mathbf{k}}\mathrm{Im}\right)\left\langle\varphi_{\mathbf{k},\omega}\varphi_{-\mathbf{k}}\right\rangle^{\mathrm{eq}}. \tag{19}$$

which is a generalization of the expression for the dynamic structure factor found in [27] for the case of constant α_φ using another method.

If we represent (18) in the form

$$\Delta S_{\varphi,\varphi}(\mathbf{k},\omega)=\frac{i}{2}\left\{\left\langle\varphi_{-\mathbf{k}-\mathbf{q},-\omega}\varphi_{\mathbf{k}+\mathbf{q}}\right\rangle^{\mathrm{eq}}-\left\langle\varphi_{\mathbf{k}+\mathbf{q},\omega}\varphi_{-\mathbf{k}-\mathbf{q}}\right\rangle^{\mathrm{eq}}+\left\langle\varphi_{\mathbf{k},\omega}\varphi_{-\mathbf{k}}\right\rangle^{\mathrm{eq}}-\left\langle\varphi_{-\mathbf{k},-\omega}\varphi_{\mathbf{k}}\right\rangle^{\mathrm{eq}}\right\}, \tag{20}$$

then the result can be interpreted as follows. In addition to modes with wave vectors $\pm\mathbf{k}$, modes with $\pm\mathbf{k}\pm\mathbf{q}$ are involved in scattering. This can be explained by the fact that the open system under consideration has a flow of thermal phonons with a momentum $-\mathbf{q}$ in the direction of the heat flux. Modes with wave vectors $\pm\mathbf{k}\pm\mathbf{q}$, emitting or absorbing a similar phonon, take part in the scattering.

If we represent the correction in the form

$$\Delta S_{\varphi,\varphi}(\mathbf{k},\omega)=\frac{i}{2}\left\{\left\langle\varphi_{\mathbf{k}-\frac{\mathbf{q}}{2},\omega}\varphi_{-\mathbf{k}+\frac{\mathbf{q}}{2}}\right\rangle^{\mathrm{eq}}-\left\langle\varphi_{-\mathbf{k}+\frac{\mathbf{q}}{2},-\omega}\varphi_{\mathbf{k}-\frac{\mathbf{q}}{2}}\right\rangle^{\mathrm{eq}}+\left\langle\varphi_{-\mathbf{k}-\frac{\mathbf{q}}{2},-\omega}\varphi_{\mathbf{k}+\frac{\mathbf{q}}{2}}\right\rangle^{\mathrm{eq}}-\left\langle\varphi_{\mathbf{k}+\frac{\mathbf{q}}{2},\omega}\varphi_{-\mathbf{k}-\frac{\mathbf{q}}{2}}\right\rangle^{\mathrm{eq}}\right\}. \tag{21}$$

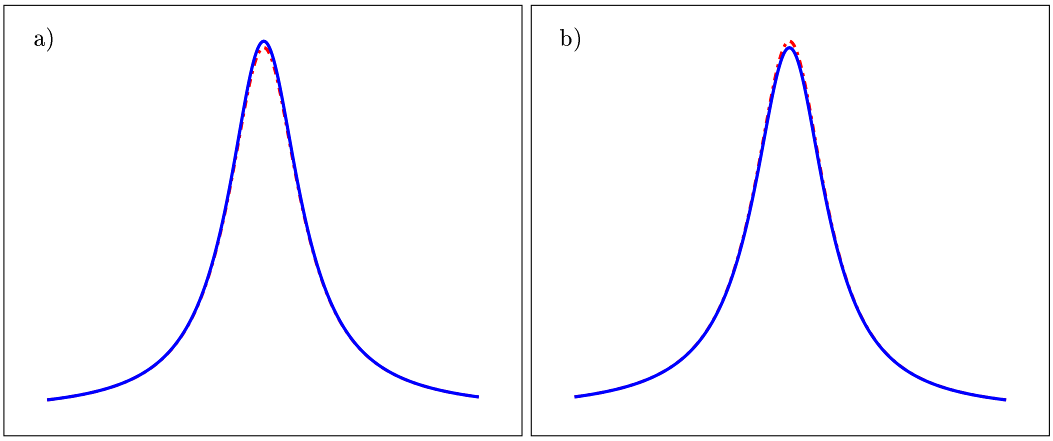

Fig. 1. Features of the Brillouin peaks: (a) peak from a wave along flux (solid line), (b) peak from a wave against flux (solid line). Peaks in the equilibrium are shown by a dash-dotted line. Owing to the sharpness of the peaks, only a narrow frequency box about $\omega = \pm ck$ is plotted

It can be said that, in addition to modes with wave vectors $\pm\mathbf{k}$, modes with $\pm\mathbf{k} \pm \frac{\mathbf{q}}{2}$ in pairs are involved in scattering, again interacting with the flow of thermal phonons with momentum $-\mathbf{q}$.

We now write the Stokes wave equation for Fourier transforms of fluctuation displacements $\xi_{\mathbf{k}}$ and displacement velocities $\dot{\xi}_{\mathbf{k}}$ in an isotropic solid in the absence of a temperature gradient

$$\begin{cases} \frac{\partial \xi_{\mathbf{k}}}{\partial t} = \dot{\xi}_{\mathbf{k}} \\ \frac{\partial \dot{\xi}_{\mathbf{k}}}{\partial t} = -2\delta\dot{\xi}_{\mathbf{k}} - c^2k^2\xi_{\mathbf{k}} \end{cases}, \tag{22}$$

where $\delta = \alpha c$ is the temporal attenuation coefficient of the sound wave, c is the speed of sound. Matrix λ is equal to

$$\lambda = \begin{pmatrix} 0 & -1 \\ c^2k^2 & 2\delta \end{pmatrix}, \tag{23}$$

and matrix β^{-1}

$$\beta^{-1} = \begin{pmatrix} \frac{1}{c^2kk'} & 0 \\ 0 & 1 \end{pmatrix} \frac{T_0}{\rho_0}\delta_{\mathbf{k},-\mathbf{k}'}. \tag{24}$$

As a result, we find

$$\begin{aligned} \langle \xi_{\mathbf{k},\omega}\xi_{-\mathbf{k}} \rangle^{\mathrm{eq}} &= G(\mathbf{k},\omega) \langle \xi_{\mathbf{k}}\xi_{-\mathbf{k}} \rangle^{\mathrm{eq}} \\ &= \frac{-i\omega + 2\delta}{-\omega^2 + c^2k^2 - 2i\omega\delta} \frac{T_0}{\rho_0 c^2 k^2}. \end{aligned} \tag{25}$$

Let's apply formula (19) to (25). Due to the inequality $\alpha \ll k$, the main contribution when differentiating with respect to $\mathbf{k}$ will be from c^2k^2 in Green's function, while differentiating the factor k^{-2} in $\langle \xi_{\mathbf{k},\omega}\xi_{-\mathbf{k}} \rangle^{\mathrm{eq}}$, as well as the terms with δ, if δ depends on k, will give a small result.

Finally, for the dynamic structure factor we'll get

$$\begin{aligned} S_{\xi,\xi}(\mathbf{k},\omega) &= \frac{4T_0}{\rho_0} \\ &\times \left\{ \begin{aligned} &\frac{\delta}{\left(\omega^2 - c^2k^2\right)^2 + 4\omega^2\delta^2} \\ &\times \left(1 - \frac{\mathbf{qk}}{k^2} \frac{2\omega^3\delta}{\left(\omega^2 - c^2k^2\right)^2 + 4\omega^2\delta^2}\right) \end{aligned} \right\}. \end{aligned} \tag{26}$$

In the absence of a temperature gradient $S_{\xi,\xi}(\mathbf{k},\omega)$ has two Brillouin peaks at frequencies $\pm ck$. Since the correction is odd in frequency, heights of peaks and integral intensities will be different. The magnitude of correction has maximum at frequencies $\pm ck$, and it should be small due to $q \ll \alpha$.

Features of the Brillouin peaks in the presence of a temperature gradient are shown in Fig. 1.

Result (26) is identical to formula (45) from the work [27] for the isothermal propagation of sound fluctuations in liquid. It is only necessary to omit in the numerator of the correction in (45) the term $\left(\omega^2 - c^2k^2\right)^2$, which is negligible and does not play any role due to the indicated inequality $\alpha \ll k$, put $Dk^2 = 2\delta$ (D being generalized viscosity), and replace the simultaneous correlation function of density fluctuations to the function of fluctuation displacements.

We now consider the same problem using the Langevin method. The matrix λ (23) remains the same. The matrix β^{-1} must be determined from locally equilibrium estimates of the Fourier transforms of fluctuation displacements $\xi_{\mathbf{k}}$ and displacement velocities $\dot{\xi}_{\mathbf{k}}$ in an isotropic solid under a temperature gradient. It is obtained after the temperature (14) transformation to

$$T = T_0(1 + \mathbf{qr}) \approx T_0(1 + \sin(\mathbf{qr})). \tag{27}$$

As a result,

$$\beta^{-1} = \begin{pmatrix} \frac{1}{c^2kk'} & 0 \\ 0 & 1 \end{pmatrix} \frac{T_0}{\rho_0} \Delta_{\mathbf{k},\mathbf{k}'}, \tag{28}$$

where

$$\Delta_{\mathbf{k},\mathbf{k}'} = \delta_{\mathbf{k},-\mathbf{k}'} + \frac{i}{2}\left(\delta_{\mathbf{k}+\mathbf{q},-\mathbf{k}'} - \delta_{\mathbf{k}-\mathbf{q},-\mathbf{k}'}\right). \tag{29}$$

It is important that, in addition to the equilibrium correlations between the modes with $\mathbf{k}$ and $\mathbf{k}'$, correlations between the modes with $\mathbf{k}$ and $\mathbf{k}' \pm \mathbf{q}$ take place. For kinetic coefficients and force intensities from (23) and (28) we obtain

$$\gamma = \begin{pmatrix} 0 & -1 \\ \frac{k}{k'} & 2\delta \end{pmatrix} \frac{T_0}{\rho_0} \Delta_{\mathbf{k},\mathbf{k}'},$$
$$Q = \begin{pmatrix} 0 & -1 + \frac{k'}{k} \\ -1 + \frac{k}{k'} & 4\delta \end{pmatrix} \frac{T_0}{\rho_0} \Delta_{\mathbf{k},\mathbf{k}'}. \tag{30}$$

Let's add forces (30) to (22). If the solution of inhomogeneous equations (22) for displacements is substituted in (11) we get

$$S_{\xi,\xi}(\mathbf{k},\omega) = \frac{1}{V}\int\limits_V d\mathbf{r} \int\limits_V d\mathbf{r}' e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}$$
$$\times \sum_{\mathbf{p},\mathbf{p}'} \left\langle \frac{y_{1,\mathbf{p},\omega}(-i\omega + 2\delta) + y_{2,\mathbf{p},\omega}}{-\omega^2 + c^2p^2 - 2i\omega\delta} \frac{y_{1,\mathbf{p}',-\omega}(i\omega + 2\delta) + y_{2,\mathbf{p}',-\omega}}{-\omega^2 + c^2p'^2 + 2i\omega\delta} \right\rangle e^{i\mathbf{p}\mathbf{r}} e^{i\mathbf{p}'\mathbf{r}'}. \tag{31}$$

Passing then to the variables $(\mathbf{x}, \mathbf{R})$ and integrating, assuming that $e^{\pm i\mathbf{q}\mathbf{R}} \approx 1$, we have

$$S_{\xi,\xi}(\mathbf{k},\omega) = \sum_{\mathbf{p},\mathbf{p}'} \frac{\left(-1 + \frac{p'}{p}\right)(-i\omega + 2\delta) + \left(-1 + \frac{p}{p'}\right)(i\omega + 2\delta) + 4\delta}{(-\omega^2 + c^2p^2 - 2i\omega\delta)(-\omega^2 + c^2p'^2 + 2i\omega\delta)}$$
$$\times \left(\delta_{\mathbf{p},\mathbf{k}}\delta_{\mathbf{p}',-\mathbf{k}} + \frac{i}{2}\delta_{\mathbf{p},\mathbf{k}-\frac{\mathbf{q}}{2}}\delta_{\mathbf{p}',-\mathbf{k}-\frac{\mathbf{q}}{2}} - \frac{i}{2}\delta_{\mathbf{p},\mathbf{k}+\frac{\mathbf{q}}{2}}\delta_{\mathbf{p}',-\mathbf{k}+\frac{\mathbf{q}}{2}}\right). \tag{32}$$

Performing summation and selecting the terms linear in $\mathbf{q}$, we arrive to (26). The condition $\mathbf{q}\mathbf{R} \ll 1$ used here is equivalent to $q \ll \alpha$.

Thus, Langevin and direct calculations of the dynamic structure factor for the Mandelstam problem are equivalent. It is seen from (30) that the kinetic coefficients providing the same result (26) as the direct method violate Onsager's reciprocal relation. Obviously, the flux breaks the detailed balance of the correlation functions of fluctuations in the OHSS. The violation of Onsager's reciprocal relations in systems with a flux was first established in [25]. There, for the Rayleigh–Benard problem, fluctuating forces were found in accordance with formula (8). Their use has fully confirmed the results of the work [21].

Reciprocal relations for fluctuation quantities in the OHSS were obtained in [26]. In the Mandelstam problem considered here, these reciprocal relations for non-correlated fluctuation displacements and displacement velocities have the form:

$$\frac{\left\langle \xi_{\mathbf{k}}(t)\,\dot{\xi}_{\mathbf{k}'}(0)\right\rangle}{\left\langle \dot{\xi}_{\mathbf{k}}(t)\,\xi_{\mathbf{k}'}(0)\right\rangle} = \frac{\gamma_{12}}{\gamma_{21}}, \tag{33}$$

where γ is determined by (30). It was also pointed out in [26] that from the thermodynamic point of view Onsager's reciprocal relations arise due to the absence of fluxes in the system.

The solution of the Mandelstam problem using the methods of fluctuating hydrodynamics is identical to the solution of the problem of the sound isothermal fluctuations using the same methods [28]. Namely, instead of (30), equilibrium expressions are used for the kinetic coefficients and intensities of fluctuating forces with the replacement of the equilibrium temperature by the local temperature (27)

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 2\delta \end{pmatrix} \frac{T_0}{\rho_0} \Delta_{\mathbf{k},\mathbf{k}'}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 4\delta \end{pmatrix} \frac{T_0}{\rho_0} \Delta_{\mathbf{k},\mathbf{k}'}. \tag{34}$$

With such kinetic coefficients, Onsager's reciprocal relation holds. At the same time, (34) gives a non-equilibrium ensemble, which contradicts the local equilibrium (28). Indeed, with λ (23) and γ (34) we obtain

$$\beta^{-1} = \lambda^{-1}\gamma = \begin{pmatrix} \frac{1}{c^2k^2} & 0 \\ 0 & 1 \end{pmatrix} \frac{T_0}{\rho_0} \Delta_{\mathbf{k},\mathbf{k}'}. \tag{35}$$

As we see, simultaneous correlations take place between the same modes as above, but they no longer depend on kk', but on k^2. If the dynamic structure factor is calculated with such forces as (34), then in the numerator of formula (32) remains only 4δ and there are no terms with factors $\left(-1 + \frac{p'}{p}\right)$ and $\left(-1 + \frac{p}{p'}\right)$. As a result, the correction value is twice as large as in (26).

Thus, even if the stationary flux does not change the equations of the dynamics of fluctuation perturbations, fluctuating hydrodynamics using local temperature rather than local simultaneous correlation functions leads to an error, since it contradicts the FDT and local equilibrium.

VI. ASYMMETRY OF FLUCTUATIONS IN THE OHSS

The dynamic structure factor (10) has obvious symmetry

$$S_{\varphi,\varphi}(\mathbf{k},\omega) = S_{\varphi,\varphi}(-\mathbf{k},-\omega) = S^{*}_{\varphi,\varphi}(\mathbf{k},\omega) \tag{36}$$

due to the symmetry of the autocorrelation function relative to the replacement $\mathbf{r}, t \longleftrightarrow \mathbf{r}', t'$. In the equilibrium, the dynamic structure factor depends on the modulus of the wave vector, since there is no distinguished direction in space; therefore, the first of the written equations ensures the frequency symmetry of the dynamic structure factor

$$S^{\mathrm{eq}}_{\varphi,\varphi}(k,\omega) = S^{\mathrm{eq}}_{\varphi,\varphi}(k,-\omega). \tag{37}$$

The same equality (36) prohibits in the OHSS the fulfillment of the relation

$$S_{\varphi,\varphi}(\mathbf{k},\omega) = S_{\varphi,\varphi}(\mathbf{k},-\omega). \tag{38}$$

We now calculate the asymmetry in the Mandelstam problem. We assume that the direction of the temperature gradient and vertical axis z coincide. Let's denote the intensities of the waves propagating in the crystal at angles $\theta+\pi$ and θ to the vertical axis with I_+ and I_-. The angle θ is acute. The asymmetry is usually determined now by the formula

$$\varepsilon_I = \frac{I_+ - I_-}{I_+ + I_-}. \tag{39}$$

Asymmetry can also be determined using the maximum height of the Brillouin peaks instead of the intensity. We denote this asymmetry with ε_ω.

Integrating (26) over the frequency, we obtain

$$I_\pm = I_0\left(1 \pm \frac{q\cos\theta}{4\alpha}\right), \tag{40}$$

where I_0 is the wave intensity in a medium with constant temperature T_0. Correspondingly

$$\varepsilon_I = \frac{\nabla T\cos\theta}{4\alpha T_0}. \tag{41}$$

Compare the result (41) with Leontovich's theory. It considers the energy flux $K d\Omega d\omega$ in a frequency interval $(\omega, \omega + d\omega)$ carried by elastic waves through a unit area normal to the direction of their propagation, which is determined by a unit vector lying inside the solid angle element $d\Omega$. The original equation of the theory for the same geometry as above is

$$\cos\theta\frac{\partial K}{\partial z} + aK = C\left(T_0 + z\nabla T\right). \tag{42}$$

Here C is a certain constant, a is the energy absorption coefficient, twice the amplitude absorption coefficient α. As seen from (42), the source of fluctuation elastic waves is the local temperature. Propagation occurs at an angle θ to the vertical axis and is accompanied by attenuation.

Leontovich found a solution to equation (42) with boundary conditions on the surfaces of the sample. Assuming that the dimensions of the sample are large and the point where the energy flux is considered is far enough from the boundaries, it follows from it that

$$K(z,\theta) = CT(z) - C\frac{\nabla T\cos\theta}{a}. \tag{43}$$

This solution also follows directly from (42) for an unbounded sample.

In the work [4], Vladimirskii defined the asymmetry as $\left[K(z,\theta+\pi) - K(z,\theta)\right]/K$, where K corresponds to a medium with temperature T_0. Taking into account (43), he found that it is equal $\frac{\nabla T\cos\theta}{\alpha T_0}$. If the asymmetry was determined by the ratio of the difference between the quantities $K(z,\theta+\pi)$ and $K(z,\theta)$ to their sum, then it would be $\frac{\nabla T\cos\theta}{2\alpha T_0}$.

At first glance, it may seem that a contradiction with formula (41) arises. However, it should be taken into account that the quantity K is not an integral characteristic, since by definition it represents the frequency distribution of the energy flux in an angle $d\Omega$, i.e. it is a dynamic structure factor. Therefore, the comparison should not be carried out with ε_I (39), but with ε_ω. Calculating the asymmetry ε_ω from (26), we obtain

$$\varepsilon_\omega = \frac{S_{\xi,\xi}(\mathbf{k},-ck) - S_{\xi,\xi}(\mathbf{k},+ck)}{S_{\xi,\xi}(\mathbf{k},-ck) + S_{\xi,\xi}(\mathbf{k},+ck)} = \frac{\nabla T\cos\theta}{2\alpha T_0}. \tag{44}$$

Thus we get the same result, as from Leontovich's theory.

Fluctuating hydrodynamics with local temperature gives twice as much value ε_I as (41), see the paragraph after (35). Measurements of asymmetry in experiments with small temperature gradients indicate a quantitative discrepancy with predictions of fluctuating hydrodynamics, the asymmetry should be smaller [8, 29, 30].

We show below with a simple example that the amplification of fluctuations in the direction of flux and their weakening in the opposite direction, discovered by Leontovich and Vladimirskii for heat flux, is a common property of the OHSS containing spatial inhomogeneity, causing the flux.

Consider the behavior of gravity-capillary fluctuation waves at the interface between two semi-infinite layers of liquid, the lower of which has a density ρ and the upper one ρ'. Let the upper liquid be ideal, and the lower one have a kinematic viscosity ν, which we will consider small. Let the upper layer move at speed $\mathbf{U}$ relative to the motionless lower one. We choose a coordinate system whose axis Ox coincides with the direction of speed, the axis Oz is directed vertically upward, so that the upper layer occupies half-space $z > 0$, and the lower one $z < 0$, respectively. Thus, the heterogeneity of the system is due to the velocity potential of the upper liquid. This problem for ideal liquids is known as the Kelvin problem [31].

For low-viscosity lower liquid, it is not difficult to obtain an equation for the modes of fluctuation displacements

$$\frac{\rho+\rho'}{k}\left[\ddot{\xi}_{\mathbf{k}}+2\left(i\Omega_*+\delta\right)\dot{\xi}_{\mathbf{k}}+\left(\Omega^2-\Omega_*^2\right)\xi_{\mathbf{k}}\right]=0, \quad (45)$$

where

$$\mathbf{V}=\frac{\rho'}{\rho+\rho'}\mathbf{U},\quad \Omega_*=\mathbf{Vk},\quad \delta=2\frac{\rho}{\rho+\rho'}\nu\, k^2,$$
$$\Omega_0^2=\frac{\alpha k^3+g\left(\rho-\rho'\right)}{\rho+\rho'},\quad \Omega^2=\Omega_0^2-\frac{\rho}{\rho'}\Omega_*^2. \quad (46)$$

Here $\mathbf{V}$ is the velocity of the center of mass, α is the coefficient of the surface tension, g hereinafter referred to as acceleration of gravity. The solution of the dispersion equation corresponding to (45) gives the frequencies of fast and slow gravity-capillary waves propagating at an acute and obtuse angle to the vector $\mathbf{U}$

$$\omega_{1,2}=\Omega_*\pm\Omega_1-i\left(\delta\pm\delta_1\right), \quad (47)$$

where

$$\Omega_1=\sqrt{\left[\Omega^2-\delta^2+\sqrt{\left(\Omega^2-\delta^2\right)^2+4\left(\delta\Omega_*\right)^2}\right]/2},$$
$$\delta_1=\sqrt{\left[-\left(\Omega^2-\delta^2\right)+\sqrt{\left(\Omega^2-\delta^2\right)^2+4\left(\delta\Omega_*\right)^2}\right]/2}. \quad (48)$$

In addition to the difference in frequency, the waves differ in attenuation: in a fast wave, the attenuation coefficient is greater. In the absence of viscosity, instead of (47) we have

$$\omega_{1,2}=\Omega_*\pm\Omega, \quad (49)$$

and we can immediately write expressions for simultaneous correlations of displacements and displacement velocities in a reference frame associated with the center of mass:

$$\left\langle\zeta_{\mathbf{k}}\zeta_{\mathbf{k}'}\right\rangle=\frac{Tk}{\left(\rho+\rho'\right)\Omega^2}\delta_{\mathbf{k},-\mathbf{k}'},\quad \left\langle\zeta_{\mathbf{k}}\dot{\zeta}_{\mathbf{k}'}\right\rangle=0,$$
$$\left\langle\dot{\zeta}_{\mathbf{k}}\dot{\zeta}_{\mathbf{k}'}\right\rangle=\Omega^2\left\langle\zeta_{\mathbf{k}}\zeta_{\mathbf{k}'}\right\rangle. \quad (50)$$

Given that

$$\xi_{\mathbf{k}}=\zeta_{\mathbf{k}}e^{-i\Omega_* t}, \quad (51)$$

we get β^{-1} the matrix that interests us

$$\beta^{-1}=\begin{pmatrix}1 & i\Omega_*\\ -i\Omega_* & \Omega^2+\Omega_*^2\end{pmatrix}\left\langle\xi_{\mathbf{k}}\xi_{\mathbf{k}'}\right\rangle,$$
$$\left\langle\xi_{\mathbf{k}}\xi_{\mathbf{k}'}\right\rangle=\left\langle\zeta_{\mathbf{k}}\zeta_{\mathbf{k}'}\right\rangle. \quad (52)$$

For the variables $\xi_{\mathbf{k}}$ and $\dot{\xi}_{\mathbf{k}}$ we represent (45) in the form (2) with the matrix λ

$$\lambda=\begin{pmatrix}0 & -1\\ \Omega^2-\Omega_*^2 & 2\left(i\Omega_*+\delta\right)\end{pmatrix}. \quad (53)$$

Using expressions (52), (53) and any of calculation methods, we find the dynamic structure factor

$$S_{\xi,\xi}\left(\mathbf{k},\omega\right)=4\delta\left\langle\xi_{\mathbf{k}}\xi_{-\mathbf{k}}\right\rangle \quad (54)$$
$$\times\frac{\Omega^2+\omega_*\Omega_*}{\left[\left(\omega_*-\Omega_1\right)^2+\left(\delta+\delta_1\right)^2\right]\left[\left(\omega_*+\Omega_1\right)^2+\left(\delta-\delta_1\right)^2\right]}.$$

The notation $\omega_*=\omega-\Omega_*$ was introduced in (54).

Note that the linear heterogeneity of the velocity potential, which determines the flow, leads to a correction in the structure dynamic factor proportional to $\omega\Omega_*$ in accordance with formula (36). The same should be noted for the Mandelstam problem where the correction is proportional to $\omega^3\mathbf{qk}$.

For the case $\Omega_0\gg\Omega_*$, (54) can be written as

$$S_{\xi,\xi}\left(\mathbf{k},\omega\right)=\left\langle\xi_{\mathbf{k}}\xi_{-\mathbf{k}}\right\rangle \quad (55)$$
$$\times\left\{\frac{\left(1-\Omega_*/\Omega_0\right)}{\left(\omega_*+\Omega_0\right)^2+\delta^2}+\frac{\left(1+\Omega_*/\Omega_0\right)}{\left(\omega_*-\Omega_0\right)^2+\delta^2}\right\}.$$

In this form, the dynamic structure factor consists of two Lorentzians, the first corresponds to a fluctuation wave propagating against the flow, and the second along the flow. The integrated intensity of the second is greater. The magnitude of the asymmetry ε_I will be

$$\varepsilon_I=\frac{\Omega_*}{\Omega_0}. \quad (56)$$

Thus, just as in the case of the Mandelstam problem, where there is a heat flux, we conclude that fluctuations in the direction of the flow have a greater intensity than in the direction opposite to the flow.

It is easy to obtain the reciprocal relation in the Kelvin problem. Using (51) from (45) we write the equation for non-correlated quantities $\zeta_{\mathbf{k}}$ and $\dot{\zeta}_{\mathbf{k}}$ in the form (2) with the matrix

$$\lambda=\begin{pmatrix}0 & -1\\ \Omega^2-2i\delta\Omega_* & 2\delta\end{pmatrix}. \quad (57)$$

Formulas (50) and (57) give the matrix

$$\gamma=\begin{pmatrix}0 & -\Omega^2\\ \Omega^2-2i\delta\Omega_* & 2\delta\Omega^2\end{pmatrix}\left\langle\zeta_{\mathbf{k}}\zeta_{-\mathbf{k}}\right\rangle, \quad (58)$$

and reciprocal relation (33) for $\zeta_{\mathbf{k}}$ and $\dot{\zeta}_{\mathbf{k}}$ takes the form

$$\frac{\left\langle\zeta_{\mathbf{k}}(t)\dot{\zeta}_{-\mathbf{k}}(0)\right\rangle}{\left\langle\dot{\zeta}_{\mathbf{k}}(t)\zeta_{-\mathbf{k}}(0)\right\rangle}=-\frac{\Omega^2}{\Omega^2-2i\delta\Omega_*}. \quad (59)$$

Returning to the variables $\xi_{\mathbf{k}}$ and $\dot{\xi}_{\mathbf{k}}$ we obtain

$$\frac{\left\langle\xi_{\mathbf{k}}(t)\dot{\xi}_{-\mathbf{k}}(0)\right\rangle-i\Omega_*\left\langle\xi_{\mathbf{k}}(t)\xi_{-\mathbf{k}}(0)\right\rangle}{\left\langle\dot{\xi}_{\mathbf{k}}(t)\xi_{-\mathbf{k}}(0)\right\rangle+i\Omega_*\left\langle\xi_{\mathbf{k}}(t)\xi_{-\mathbf{k}}(0)\right\rangle}=-\frac{\Omega^2}{\Omega^2-2i\delta\Omega_*} \quad (60)$$

As can be seen from the results (29),(30), (33) and (60), the reciprocal relations in the OHSS essentially depend on the flux, and also that Onsager's reciprocal relations hold only in the absence of a flux (in the equilibrium).

VII. FLUCTUATIONS IN A LIQUID UNDER A TEMPERATURE GRADIENT

The light scattering spectrum in equilibrium liquid is determined by the density–density correlation function. It can be found from a system of linear hydrodynamic equations for fluctuation perturbations of density, temperature, and velocity [32], or from hydrodynamic equations for fluctuation perturbations of pressure, entropy, and velocity [33].

Compared to the equations used in [32], the presence of a temperature gradient in the liquid leads to the appearance in the equation for the temperature perturbations of the term equal to the product of the fluctuation velocity and the temperature gradient and the term that takes into account buoyancy force in the equation of motion [21]. As a result, two pairs of sound and coupled viscous-thermal fluctuation waves are formed. Such a situation occurs when the temperature gradient is such that the value $\frac{\beta g \kappa^2}{\gamma k^2}\nabla T$ is of the order $(\nu-\chi)^2 k^4$ where β is the coefficient of thermal expansion, $\gamma = c_p/c_v$ is the ratio of specific heat, κ is the horizontal projection of $\mathbf{k}$, ν is the kinematic shear viscosity, χ is the thermal diffusivity. This is important for small values wave vector when considering the effects associated with convective instability. If, however, one is interested in the scattering of light for typical values of the wave vector $10^5\,\mathrm{cm}^{-1}$, then the connection between viscous and temperature modes is insignificant $\left((\nu-\chi)^2 k^4 \gg \frac{\beta g \kappa^2}{\gamma k^2}\nabla T\right)$ and the determining factor is the change in the simultaneous correlation functions by the temperature gradient.

We will use the well-known $\mathbf{k}$,t-representation of the density–density correlation function

$$\langle \rho_{\mathbf{k}}(t)\,\rho_{-\mathbf{k}}(0)\rangle^{\mathrm{eq}} = \frac{\rho_0 T_0}{c^2}\left[(\gamma-1)\,e^{-\chi k^2 t} + e^{-\Gamma k^2 t}\cos ckt\right], \quad (61)$$

where c is the sound speed, $\Gamma = \frac{1}{2}\left[D + (\gamma-1)\chi\right]$, D is generalized viscosity as above. Then, having performed the one-sided Fourier transform in time from formula (19) we obtain

$$S_{\rho,\rho}(\mathbf{k},\omega) = \frac{\rho_0 T_0}{c^2} \times \left\{ \begin{array}{l} (\gamma-1)\dfrac{2\chi k^2}{\omega^2 + (\chi k^2)^2}\left[1 - \dfrac{\mathbf{qk}}{k^2}\dfrac{2\omega\chi k^2}{\omega^2+(\chi k^2)^2}\right] \\ + \dfrac{4c^2k^2\Gamma k^2}{(\omega^2 - c^2k^2)^2 + 4\omega^2(\Gamma k^2)^2} \\ \times\left[1 - \dfrac{\mathbf{qk}}{k^2}\dfrac{2\omega^3\Gamma k^2}{(\omega^2-c^2k^2)^2 + 4\omega^2(\Gamma k^2)^2}\right] \end{array}\right\}. \quad (62)$$

Obviously, the changes in the Brillouin doublet also could be immediately written down using (26) and changing δ to Γk^2, and $\langle \xi_{\mathbf{k}}\xi_{\mathbf{k}'}\rangle^{\mathrm{eq}}$ to $\frac{\rho_0 T_0}{c^2}\delta_{\mathbf{k},-\mathbf{k}'}$. The asymmetry of the doublet is determined by the same formula (41).

As for the Rayleigh line, the integral over negative frequencies determines the contribution I_+ from fluctuation modes propagating in the direction of flux, and over the positive ones, the contribution I_- from fluctuation modes propagating in the direction opposite to flux, and we get

$$I_{\pm} = I_0\left(1 \pm \frac{2}{\pi}\frac{\mathbf{qk}}{k^2}\right), \quad (63)$$

where I_0 is the contribution when the gradient is absent. All features of the Rayleigh line in the presence of a temperature gradient are shown in the figure below.

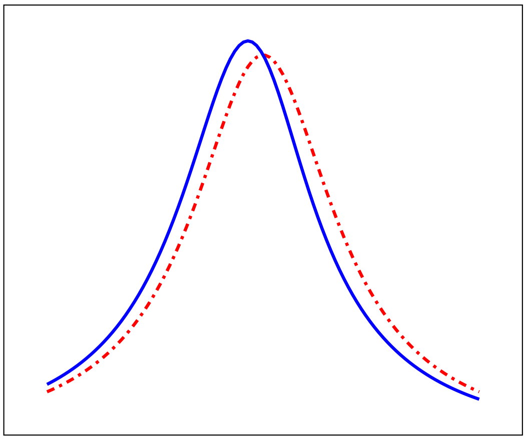

Fig. 2. Features of the Rayleigh line:1) Rayleigh line in equilibrium liquid (dash dot line), 2) Rayleigh line in the liquid under temperature gradient (solid line)

The asymmetry will be

$$\varepsilon_I = \frac{2}{\pi}\frac{\mathbf{qk}}{k^2}. \quad (64)$$

The maximum will be shifted to a point $\omega = -\chi\mathbf{qk}$ and the height of the maximum increases by factor $1+\left(\frac{\mathbf{qk}}{k^2}\right)^2$. All this, obviously, will not change the Landau–Placzek formula.

VIII. CONCLUSION

The FDT is a mathematical theorem that follows from the solution of the Cauchy problem and the subsequent averaging of the initial conditions. For this reason, the calculation of fluctuations using the FDT will give absolutely the same results for systems of linear homogeneous differential equations as the direct solution of the Cauchy problem and averaging over the initial conditions both in equilibrium systems and in the OHSS. The FDT like any mathematical theorem is not a subject of statistical physics.

Fluctuating hydrodynamics is erroneous because it uses fluctuating forces that contradict the FDT, and, therefore, is not related to the problems under consideration. For the OHSS, the statistical properties of hydrodynamic fluctuations are determined by the locally equilibrium distribution function according to the BBGKY hierarchy and the dynamics is determined by

Onsager's hypothesis. These two statements are sufficient to elucidate all the properties of hydrodynamic fluctuations, including reciprocal relations.

In the OHSS fluxes, changing the dynamics and initial values of fluctuations violates the detailed balance that takes place in equilibrium. As a result, Onsager's reciprocal relations are broken. The next criterion can be formulated. The fulfillment of Onsager's reciprocal relations indicates the absence of fluxes in the system under consideration, while their violation indicates the presence of fluxes.

The spatial heterogeneity causing flux in the OHSS leads to a violation of the symmetry of the correlation functions of fluctuations with respect to the time variable, resulting in an asymmetry of the light scattering spectra. The flux transfers the fluctuation energy from one frequency region to another. There is an increase of fluctuation modes propagating in the direction of flux, and a weakening in fluctuation modes propagating in the direction opposite to flux.

[1] L. Onsager, Phys. Rev. **15**, 405, 2265 (1931); `https://doi.org/10.1103/PhysRev.15.405`.

[2] L. I. Mandelstam, Dokl. Akad. Nauk SSSR **11**, 219 (1934); L. I. Mandelstam, *Complete Works* (Publishing House of the Acad. Sci. USSR, Moscow, 1948–1950).

[3] M. A. Leontovich, Zh. Eksp. Teor. Fiz. **9**, 1314 (1939); M. A. Leontovich, *Selected Works* (Nauka, Moscow, 1985).

[4] V. V. Vladimirskii, Dokl. Akad. Nauk SSSR **38**, 229 (1943).

[5] I. L. Fabelinskii, *Molecular Scattering of Light* (Vysshaja shkola, Moscow, 1965).

[6] M. A. Leontovich, Dokl. Akad. Nauk SSSR **1**, 97 (1935); M. A. Leontovich *Selected Works* (Nauka, Moscow, 1985).

[7] G. S. Landsberg, A. A. Shubin, Zh. Eksp. Teor. Fiz. **9**, 1309 (1939); G. S. Landsberg, Selected Works (Publishing House of the Acad. Sci. USSR, Moscow, 1958).

[8] D. Beysens, Y. Garrabos, G. Zalczer, Phys. Rev. Lett. **45**, 403 (1980); `https://doi.org/10.1103/PhysRevLett.45.403`.

[9] I. L. Fabelinskii, Phys.-Uspekhi **164**, 897, (1994); `https://doi.org/10.1070/PU1994v037n09ABEH000042`.

[10] G. E. Uhlenbeck, Usp. Fiz. Nauk **103**, 275 (1971); `https://doi.org/10.3367/UFNr.0103.197102c.0275`.

[11] R. F. Fox, G. E. Uhlenbeck, Phys. Fluids **13**, 1893 (1970); `https://doi.org/10.1063/1.1693183`.

[12] R. F. Fox, G. E. Uhlenbeck, Phys. Fluids **13**, 2881 (1970); `https://doi.org/10.1063/1.1692878`.

[13] M. Bixon, R. Zwanzig, Phys. Rev. **187**, 267 (1969); `https://doi.org/10.1103/PhysRev.187.267`.

[14] F. L. Hinton, Phys. Fluids **13**, 857 (1970); `https://doi.org/10.1063/1.1693027`.

[15] J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes* (Springer, New York, 1987); `https://doi.org/10.1007/978-1-4612-1054-2`.

[16] R. Zwanzig, *Nonequilibrium Statistical Mechanics* (Oxford University Press, 2001).

[17] L. D. Landau, E. M. Lifshitz, Sov. Phys. JETP **5**, 512 (1957).

[18] S. M. Rytov, *Theory of Electrical Fluctuations and Heat Radiation* (Publishing House of the Acad. Sci. USSR, Moscow, 1953).

[19] V. M. Zaitsev, M. I. Shliomis, Sov. Phys. JETP **32**, 866 (1971).

[20] J. M. Ortiz de Zarate, J. V. Sengers, *Hydrodynamic fluctuations in fluids and fluid mixtures* (Amsterdam, Elseveir, 2006); `https://doi.org/10.1007/s10765-008-0517-7`.

[21] V. P. Lesnikov, I. Z. Fisher, Sov. Phys. JETP **40**, 667 (1975).

[22] *Experimental Thermodynamics Vol. X: Non-equilibrium Thermodynamics with Applications*, edited by D. Bedeaux, S. Kjelstrup, J. Sengers (Royal Society of Chemistry, Cambridge, 2016); `https://doi.org/10.1039/9781782622543-00001`.

[23] N. N. Bogoliubov, *Problems of Dynamical Theory in Statistical Physics* (Gostekhizdat, Moscow, 1946); *Selected Works on Statistical Physics* (Moscow University Press, Moscow, 1979).

[24] L. D. Landau, E. M. Lifshitz, *Course of Theoretical Physics. Volume 5: Statistical Physics* (Addison-Wesley, Reading, MA, 1973).

[25] V. P. Lesnikov, J. Phys. Stud. **1**, 208 (1997); `https://doi.org/10.30970/jps.01.208`.

[26] V. P. Lesnikov, Ukr. J. Phys. **64**, 126 (2019); `https://doi.org/10.15407/ujpe64.2.126`.

[27] V. P. Lesnikov, Ukr. J. Phys. **49**, 279 (2004).

[28] A.-M. S. Tremblay, M. Arai, E. D. Siggia, Phys. Rev. A **23**, 1451 (1981); `https://doi.org/10.1103/PhysRevA.23.1451`.

[29] H. Kiefte, V. J. Clouter, R. Penney, Phys. Rev. B **30**, 4017 (1984); `https://doi.org/10.1103/PhysRevB.30.4017`.

[30] D. S. Chung, K. Y. Lee, E. Mazur, Phys. Lett. A **145**, 348 (1990); `https://doi.org/10.1016/0375-9601(90)90947-M`.

[31] L. D. Landau, E. M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1959).

[32] R. D. Mountain, Rev. Mod. Phys. **38**, 205 (1966); `https://doi.org/10.1103/RevModPhys.38.205`.

[33] D. McIntyre, J. V. Sengers, in *Physics of Simple Liquids*, edited by H. N. V. Tamperley, J. S. Rowlinson, G. S. Rushbrooke (North-Holland Publishing Company, Amsterdam, 1968).

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Теорію теплових флуктуацій у відкритих гідродинамічних стаціонарних станах (ВГСС) подано виключно в межах гідродинаміки. Описано історію вивчення флуктуацій у суцільному середовищі зі стаціонарним потоком. Показано, що застосування до ВГСС флуктуаційно-диссипаційної теореми з вимогою виконання співвідношень взаємності Онзаґера (флуктуаційна гідродинаміка) є помилковим. Причиною є те, що потік, змінюючи динаміку й початкові значення флуктуацій, порушує детальний баланс, що є в рівновазі. Це продемонстровано на прикладі задачі Мандельштама про флуктуації в середовищі з тепловим потоком. Для цієї задачі розраховано структурний динамічний фактор для ізотропного твердого тіла та рідини. Втрата часової симетрії кореляційними функціями флуктуацій та асиметрія їх спектральних зображень у цій задачі зумовлена просторовою зміною температури, яка визначає потік.

Щоб показати загальність цього результату для всіх ВГСС з просторовою неоднорідністю, також розглянуто задачу Кельвіна про теплові флуктуації зміщень поверхні розділу між двома рідинами. Потенціал швидкості верхньої рухомої рідини змінюється лінійно, так само як і температура в задачі Мандельштама.

Знайдено співвідношення взаємності як для задачі Мандельштама, так і Кельвіна.

Ключові слова: Мандельштам, відкриті гідродинамічні стаціонарні стани (ВГСС), флуктуації, потік, співвідношення взаємності, флуктуаційна гідродинаміка.