KEPLER PROBLEM IN GENERAL RELATIVITY WITH LORENTZ-COVARIANT DEFORMED POISSON BRACKETS

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We consider a Lorentz-covariant deformed algebra, which in the nonrelativistic limit leads to an undeformed one. In the classical limit, this algebra leads to the Lorentz-covariant deformed Poisson brackets. Within covariant Hamiltonian mechanics, we consider a particle's motion in the Schwarzschild space-time with deformed Poisson brackets and obtain the precession angle of the orbit taking into account the deformation. As it turned out, the precession angle in the deformed case depends on the mass of the particle, which violates the weak equivalence principle. Assuming the mass-dependence of the deformation parameter, the equivalence principle can be recovered. Comparing our theoretical results with experimental data for Mercury's precession angle, we estimate the deformation parameter and the minimal length.

Key words: Kepler problem, Schwarzschild metric, precession of an orbit, Lorentz-covariant deformed algebra, minimal length

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I. INTRODUCTION

Recently a lot of attention has been devoted to the studies of different systems in a space with a deformed Heisenberg algebra with the minimal length. Historically, the first algebra of that kind in the relativistic case was considered by Snyder as a way of the regularization of UV divergences of quantum field theory [1]. However, his paper did not attract much attention for many years. Motivated by the studies in string theory and quantum gravity[2–4], the interest in the minimal length hypothesis resurged after several decades. These studies propose the Generalized Uncertainty Principle (GUP)

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta \Delta P\right) \tag{1}$$

leading to the existence of the fundamental minimal length $\Delta X_{\min} = \hbar\sqrt{\beta}$. The minimal length is supposed to be of the order of the Planck length $l_p = \sqrt{\hbar G/c^3} = 1.6 \cdot 10^{-35} m$.

Kempf *et al.* showed that the effect of minimal length as the minimal uncertainty for position operators can be obtained in the frame of a small quadratic modification (deformation) of usual canonical commutation relations [5–8]. According to Kempf, the deformed commutaion relation in one-dimensional space may read

$$[\hat{X}, \hat{P}] = i\hbar(1 + \beta \hat{P}^2). \tag{2}$$

In the case of higher dimensions, deformed algebra (3) can be generalized as [8]

$$\begin{aligned} &[\hat{X}_i, \hat{P}_j] = i\hbar[(1 + \beta \hat{P}^2)\delta_{ij} + \beta' \hat{P}_i \hat{P}_j], \\ &[\hat{X}_i, \hat{X}_j] = i\hbar \frac{2\beta - \beta' + (2\beta + \beta')\beta \hat{P}^2}{1 + \beta \hat{P}^2}(\hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i), \\ &[\hat{P}_i, \hat{P}_j] = 0, \end{aligned} \tag{3}$$

with β and β' being two small nonnegative parameters.

Deformed commutation relations suggested by Kempf are not Lorentz-covariant. The Lorentz-covariant version of that kind of commutation relations was proposed in [14], which also can be considered as a generalization of Snyder's algebra.

The framework of the minimal length hypothesis was applied to different quantum mechanical problems, such as harmonic oscillator [6, 9–12], Dirac oscillator [13, 14], hydrogen atom [15–19], gravitational quantum well [20, 21], a particle in delta potential [22, 23], one-dimensional Coulomb-like problem [22, 24, 25], particle in the singular inverse square potential [26, 27], the Casimir effect [28], particles scattering [29], et al.

The influence of the quantization of space has been studied at the classical level for the following problems: Keplerian orbits, statistical physics, composite systems, etc.[30–37].

Deformed Heisenberg algebra with minimal length allows of a phenomenological description of quantized space. But at the same time, it causes some problems of a fundamental nature, among them the problem of violation of the weak equivalence principle [36]. This problem occurs as a result of the assumption that the parameter of deformation is the same for elementary particles and macroscopic bodies. However, in [32, 36, 37] it was proposed to relate the parameter of deformed algebras with mass as

$$\beta = \frac{\gamma}{m^2}, \tag{4}$$

where γ is supposed to be some (fundamental) constant for all particles. This idea leads to recovering the weak equivalence principle in deformed space with minimal length and also preserves the additivity property of the kinetic energy of a composite system in deformed space and the independence of the kinetic energy from the system's composition.

Despite the large number of studies of the minimal length hypothesis, verification of the hypothesis is still needed. The Kepler problem is of particular interest for this purpose due to its accurate theoretical prediction as well as precise measurements. In paper [32] the Kepler problem in Special Relativity with the Lorentz-covariant deformed Poisson bracket is considered. It is known that the precession of the Keplerian orbit in Special Relativity differs from the correct one obtained in General Relativity. Therefore, the Kepler problem in General Relativity with the Lorentz-covariant deformed Poisson bracket deserves to be studied.

The paper is organized as follows. In Section II, we consider the Lorentz-covariant deformed algebra leading to the minimal length and its classical limit. In the nonrelativistic limit, the above algebra leads to an undeformed one. In order to set the covariant Hamiltonian formalism in Section III, we study the motion of a planet in the Schwarzschild spacetime in the undeformed case. The same problem with the Lorents-covariant deformed Poisson brackets is considered in Section IV. Finally, Section V contains a conclusion.

II. LORENTZ-COVARIANT DEFORMED ALGEBRA AND ITS CLASSICAL LIMIT

Let us consider a Lorentz-covariant deformed algebra generated by the coordinates $\hat{X}^\mu$ and the momenta $\hat{P}^\nu$, which satisfy the following commutation relations:

$$\begin{aligned} &[\hat{X}^\mu, \hat{P}^\nu] = -i\hbar[1-\beta(\hat{P}_\rho\hat{P}^\rho - m^2c^2)]\eta^{\mu\nu}, \\ &[\hat{X}^\mu, \hat{X}^\nu] = 2i\hbar\beta(\hat{X}^\nu\hat{P}^\mu - \hat{X}^\mu\hat{P}^\nu), \\ &[\hat{P}^\mu, \hat{P}^\nu] = 0, \end{aligned} \tag{5}$$

with $\mu, \nu = 0,1,2,3$, $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ being a metric tensor, β being a small nonnegative parameter, m being the mass of a particle, c being the speed of light. Deformed algebra (5) is the special case of the general one presented in [38] and can be considered as the generalization of the deformed algebra (3) with $\beta' = 0$ to the relativistic case.

The proposed algebra is the Lorentz-covariant one leading to (isotropic) minimal uncertainty in position

$$\Delta X_{\min} = \hbar\sqrt{3\beta\left(1-\beta\left[(p^0)^2 - m^2c^2\right]\right)}. \tag{6}$$

Algebra (5) possesses an interesting feature. If we assume that the parameter of deformation can be presented as follows

$$\beta = \frac{\delta}{m^2c^2}, \tag{7}$$

with δ being some dimensionless parameter not depending on the speed of light c or mass m of a particle, algebra (5) in the nonrelativistic limit ($c \to \infty$) leads to an undeformed one

$$[\hat{x}^j, \hat{p}^k] = -i\hbar\eta^{jk}, \quad [\hat{x}^j, \hat{x}^k] = 0, \quad [\hat{p}^j, \hat{p}^k] = 0.$$

Here j and k enumerate spatial dimensions. Due to this fact, deformed algebra (5) can be considered to be the effect which appears on the relativistic background only. Note that the proposed algebra is different from Snyder's algebra [1] or generalized to the relativistic case Kempf's one [14]; the nonrelativistic limits of those do not exist.

Dependence of parameter of the deformation β on m as in (7) was suggested for the recovery of the weak equivalence principle [32]. Dimensionless parameter δ can be considered as a new fundamental constant.

Moving from quantum to classical mechanics, one should make the replacement

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \Rightarrow \{A, B\}. \tag{8}$$

Thus, the Poisson brackets for canonical variables corresponding to commutation relations (5) read

$$\begin{aligned} &\{X^\mu, P^\nu\} = -[1-\beta(P_\rho P^\rho - m^2c^2)]\eta^{\mu\nu}, \\ &\{X^\mu, X^\nu\} = 2\beta(P^\mu X^\nu - P^\nu X^\mu), \\ &\{P^\mu, P^\nu\} = 0. \end{aligned} \tag{9}$$

The deformed Poisson brackets (9) possess the following representation

$$\begin{aligned} &X^\mu = [1-\beta(p_\rho p^\rho - m^2c^2)]x^\mu + i\hbar\gamma p^\mu, \\ &P^\mu = p^\mu, \end{aligned} \tag{10}$$

where x^μ and p^ν satisfy the undeformed Poisson brackets

$$\begin{aligned} &\{x^\mu, p^\nu\} = -i\hbar\eta^{\mu\nu}, \\ &\{x^\mu, x^\nu\} = 0, \\ &\{p^\mu, p^\nu\} = 0, \end{aligned} \tag{11}$$

and γ is an arbitrary real constant, which does not influence commutation relations (5). In other words, (10) presents the family of representations parametrized by γ. For simplicity γ is often taken to be zero.

III. KEPLER PROBLEM IN GENERAL RELATIVITY

In order to set the formalism, we start by considering the motion of a planet in the Schwarzschild spacetime with metric

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = \left(1-\frac{r_s}{r}\right)cdt \\ &- \left(1-\frac{r_s}{r}\right)^{-1}dr^2 - r^2d\Omega^2. \end{aligned} \tag{12}$$

Here $x^\mu = (ct, \mathbf{r})$, $r = |\mathbf{r}| = \sqrt{\sum_{i=1}^3 (x^i)^2}$ and Ω denotes the solid angle. We write the Hamiltonian of the system in the usual form

$$\begin{aligned} H_0 &= \lambda(g^{\mu\nu}p_\mu p_\nu - m^2c^2) \qquad (13) \\ &= \lambda\left[\left(1-\frac{r_s}{r}\right)^{-1}p_0^2 - (1-\frac{r_s}{r})p_r^2 - \frac{L^2}{r^2} - m^2c^2\right], \end{aligned}$$

where λ is the Lagrange multiplier and $p_\mu = (p_0, -\mathbf{p})$, $p^2 = \sum_{i=1}^{3} p_i^2$. Considering $p_r^2 = p^2 - \frac{L^2}{r^2}$, with $\mathbf{L} = [\mathbf{r}\times\mathbf{p}]$ being the orbital momentum vector, we rewrite the Hamiltonian as

$$\begin{aligned} H_0 &= \lambda\Big[\left(1-\frac{r_s}{r}\right)^{-1} p_0^2 \\ &- (1-\frac{r_s}{r})p^2 - \frac{r_s L^2}{r^3} - m^2c^2\Big]. \end{aligned} \tag{14}$$

We consider the effect of General Relativity as a perturbation to Keplerian orbits. This effect causes the precession of the perihelion of the elliptic orbit. It is convenient to calculate the precession rate using the Hamilton vector for which the precession rate coincides with that of perihelion [31]. The Hamilton vector has the form

$$\mathbf{u} = \frac{\mathbf{p}}{m} - \frac{\alpha}{L}\frac{[\mathbf{L}\times\mathbf{r}]}{Lr}. \tag{15}$$

The precession rate of the Hamilton vector is

$$\omega_\tau^0 = \frac{[\mathbf{u}\times\dot{\mathbf{u}}]}{u^2}, \tag{16}$$

with

$$\dot{\mathbf{u}} = \{\mathbf{u}, H_0\}. \tag{17}$$

The precession rate of the Hamilton vector in the linear approximation on $1/c^2$ can be written as

$$\begin{aligned} \omega_\tau^0 &= \frac{\lambda r_s L}{m r^3 u^2}\left(\frac{\alpha r}{L^2} - \frac{1}{m}\right)\left(\frac{2 r_s p_0^2}{r} + 3p^2 - \frac{3L^2}{r^2}\right) \\ &- \frac{2\alpha\lambda r_s}{m L u^2 r^2}\left(p^2 - \frac{L^2}{r^2}\right). \end{aligned} \tag{18}$$

Finally, the precession angle can be obtained by

$$\Delta\Theta_{\mathrm{GR}} = \int \omega_\tau^0\, d\tau, \tag{19}$$

where τ varies in the limits of one revolution. It is convenient to make a change of variables in the latter integral from the evolution parameter τ to time in the observational reference frame t and then to polar angle φ

$$\Delta\Theta_{\mathrm{GR}} = \int_0^T \omega_\tau^0 \frac{d\tau}{dt}\, dt = \int_0^{2\pi} \omega_\tau^0 \frac{d\tau}{dt}\left(\frac{d\varphi}{dt}\right)^{-1} d\varphi. \tag{20}$$

The derivative $\frac{d\tau}{dt}$ can be obtained with the required accuracy as:

$$\frac{d\tau}{dt} = -\frac{1}{2m\lambda} \tag{21}$$

and the angular velocity of motion equals:

$$\frac{d\varphi}{dt} = \frac{L}{mr^2}. \tag{22}$$

Integrating (20) up to the first order in the perturbation, we can use relations which are valid for an unperturbed Keplerian orbit. Calculations of the orbital precession angle of the system described by the Hamiltonian (14) yield:

$$\Delta\Theta_{\mathrm{GR}} = \frac{6\pi\alpha}{mc^2 R_0} = \frac{6\pi G M}{c^2 R_0}, \tag{23}$$

where the expression for coupling constant $\alpha = GmM$ is used. This result is the well known expression for the precesion angle in General Relativity.

IV. KEPLER PROBLEM WITH DEFORMED POISSON BRACKETS

Let us consider the motion of a planet in the Schwarzschild spacetime with the deformed Poisson brackets (9). We write the Hamiltonian of the system in the form

$$\begin{aligned} H = \lambda\Big[\left(1-\frac{r_s}{R}\right)^{-1} P_0^2 &- \left(1-\frac{r_s}{R}\right)P^2 \\ &- \frac{r_s\widetilde{L}^2}{R^3} - m^2c^2\Big]. \end{aligned} \tag{24}$$

Here α is the coupling constant, $R = \sqrt{\sum_{i=1}^{3}(X^i)^2}$, $P^2 = \sum_{i=1}^{3} P_i^2$ and $\mathbf{L} = [\mathbf{R}\times\mathbf{P}]$. Position X_μ and momenta P_ν satisfy the deformed Poisson algebra (9).

Using representation (10), we write the total Hamiltonian as follows

$$H = H_0 + \Delta H_\beta \tag{25}$$

with H_0 given by (14) and caused by the deformation term given by

$$\Delta H_\beta = \lambda\beta(p_0^2 - p^2 - m^2c^2)\frac{r_s p_0^2}{r}. \tag{26}$$

We consider the deformation and relativistic effects as a perturbation to Keplerian orbits. These effects cause the precession of the perihelion of the elliptic orbit. Similarly to the undeformed case, we calculate the precession rate of the Hamilton vector

$$\omega_\tau = \frac{[\mathbf{u}\times\dot{\mathbf{u}}]}{u^2}, \tag{27}$$

with

$$\dot{\mathbf{u}} = \{\mathbf{u}, H\}. \tag{28}$$

The precession rate of the Hamilton vector can be written as

$$\omega_\tau = \omega_\tau^0 + \omega_\tau^\beta \tag{29}$$

with ω_τ^0 is given by (18) and

$$\omega_\tau^\beta = -4\lambda\beta\alpha r_s m^2 c^2 \frac{L}{r^4 u^2}\left(\frac{\alpha r}{L^2} - \frac{1}{m}\right) \tag{30}$$

The precession angle can be obtained in the same way as in the undeformed case φ

$$\Delta\Theta = \int_0^{2\pi} \omega_\tau \frac{d\tau}{dt}\left(\frac{d\varphi}{dt}\right)^{-1} d\varphi. \tag{31}$$

The derivative $\frac{d\tau}{dt}$ and $\frac{d\phi}{dt}$ with the required accuracy are presented in (21) and (22). The integration of the orbital precession angle of the system described by the Hamiltonian (25) yields:

$$\Delta\Theta = \Delta\Theta_{\mathrm{GR}} + \Delta\Theta_\beta, \tag{32}$$

with $\Delta\Theta_{\mathrm{GR}}$ being the precession angle due to General Relativity and given by (23), and $\Delta\Theta_\beta$ being the precession angle caused by the deformation and given by the following formula

$$\Delta\Theta_\beta = -\frac{4\beta\pi m\alpha}{R_0} = \frac{4\pi\beta m^2 GM}{R_0}. \tag{33}$$

From (33) we see that in the case when parameter β does not depend on the mass, the weak equivalence principle is violated, because the angle of precession depends on the mass of the particle. But we can easily recover the equivalence principle if we assume that the parameter of deformation depends on the mass as in (7). Considering (7), the precession angle can be written in the form

$$\Delta\Theta = \frac{6\pi GM}{c^2 R_0}\left(1 - \frac{2\delta}{3}\right). \tag{34}$$

Comparing the experimental data of the precession angle of Mercury's perihelion [39] with the theoretical prediction, we can place a constraint on the deformation parameter. The observed advance of Mercury's perihelion that cannot be explained by Newtonian planetary perturbations, solar oblateness or the rotation of the Sun is [39]

$$\Delta\Theta_{\mathrm{obs}} = 42.9799 \pm 0.0009 \text{ arc-seconds/century} = 2\pi(7.98730 \pm 0.00017)\times 10^{-8} \text{ radians/revolution}. \tag{35}$$

This advance is usually explained by gravitoelectric effect [39]. The standard theory of General Relativity predicts [30]

$$\Delta\Theta_{\mathrm{GR}} = 2\pi\frac{3GM}{c^2R_0} = 2\pi(7.98744\times 10^{-8}) \text{ radians/revolution}. \tag{36}$$

In a manner as was done in [30, 32], we compare the perihelion shift caused by the minimal length (33) with

$$\Delta\Theta_{\mathrm{obs}} - \Delta\Theta_{\mathrm{GR}} = 2\pi(-0.00014 \pm 0.00017)\times 10^{-8} \text{ radians/revolution}, \tag{37}$$

and obtain a lower bound of $\Delta\Theta_\beta$, which at 3σ is

$$-2\pi(0.65\times 10^{-11}) \text{ radians/revolution} < \Delta\Theta_\beta = -2\pi\frac{2\delta GM}{c^2R_0}. \tag{38}$$

It is important to note that the Schwarzschild metric is an approximation of the distortion of space-time by the Sun's mass. A more accurate description of the gravitational field could reduce the discrepancy (37). Despite this, the constraint (38) remains correct.

From (38) we obtain the constraint on the dimensionless parameter

$$\delta < 1.3\cdot 10^{-4}. \tag{39}$$

Assuming that δ is the same for different particles, we calculate the constraint for minimal length for electron

$$\hbar\sqrt{\beta_e} < 4.3\cdot 10^{-15} \text{ m}, \tag{40}$$

which of course is weaker than the one of order 10^{-19} m obtained in [19], due to the high accuracy of measurements for the hydrogen atom spectrum. As a result, by taking into account the assumption of dependence (7) of the parameter of deformation on the particle's mass m, we arrive at the reconcilement of the estimations of the minimal length coming up from the studies of planetary motion and the hydrogen atom spectrum.

The constraint on the minimal length for Mercury is

$$\hbar\sqrt{\beta} < 1.2\cdot 10^{-68} \text{ m}. \tag{41}$$

We note that this result describes how Mercury senses the deformation of spacetime. A similar result was obtained in the nonrelativistic case [30] and in case of Special Relativity [32]. Therefore we can conclude that particles of different masses feel the deformation of spacetime in different ways.

V. CONCLUSION

We have considered the Lorentz-covariant deformed algebra (5) leading to the minimal length, with the parameter of deformation assumed to be $\beta = \delta/(m^2c^2)$.

With such an assumption, our algebra in the nonrelativistic limit leads to an undeformed one, and, thus, can be considered to be the effect which appears on the relativistic background only. To our knowledge, algebra (5) is unique with such a property. Dependence of the parameter of deformation β on the mass of particle $\beta = \delta/(m^2c^2)$ leads to the recovery of the equivalence principle in Special Relativity. From this point of view, the above algebra is of particular interest. Also, the above mentioned formula introduces dimensionless parameter δ, which can be considered a new fundamental constant.

In the classical limit, the Lorentz-covariant deformed algebra leads to the deformed Poisson brackets. We have considered the Kepler problem in General Relativity with the Lorentz-covariant deformed Poisson brackets leading to the minimal length. It is interesting that the angle of precession caused by the deformation of the Poisson brackets in General Relativity coincides with the one obtained in Special Relativity. In the case when parameter β does not depend on the mass, we have obtained that the angle of precession depends on the mass of a particle. This means that the weak equivalence principle is violated in the deformed space-time. But we can easily recover the equivalence principle if we assume that the parameter of deformation depends on the mass as in (4). Thus, particles of different masses have different perceptions of space quantization. This conclusion should be taken into account in studies of massive bodies within the frame of quantized space-time.

Comparing the experimental data of the precession angle of Mercury's perihelion with the theoretical prediction places a constraint on the value of the dimensionless parameter δ. The estimation of the value of parameter δ was obtained to be less then $1.3 \cdot 10^{-4}$. This constraint reconciles the estimations of the minimal length coming up from completely different measurements related to the hydrogen atom spectrum and planetary motion.

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ЗАДАЧА КЕПЛЕРА В ЗАГАЛЬНІЙ ТЕОРІЇ ВІДНОСНОСТІ З ЛОРЕНЦ-КОВАРІАНТНИМИ ДЕФОРМОВАНИМИ ДУЖКАМИ ПУАССОНА

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Ми вивчаємо лоренц-коваріантну деформовану алґебру, яка в нерелятивістській границі прямує до недеформованої. У класичній границі ця алґебра приводить до лоренц-коваріантних деформованих дужок Пуассона. У межах коваріантної гамільтонової механіки ми розглядаємо рух частинок у просторі–часі Шварцшильда з деформованими дужками Пуассона та отримуємо кут прецесії орбіти з урахуванням деформації. Як виявилося, кут прецесії в деформованому випадку залежить від маси частинки, що порушує слабкий принцип еквівалентності. Однак на основі припущення про залежність параметра деформації від маси вдається відновити цей принцип. Порівнюючи наші теоретичні результати з експериментальними даними для кута прецесії Меркурія, оцінили значення параметра деформації та мінімальної довжини.

Ключові слова: задача Кеплера, метрика Шварцшильда, прецесія орбіти, лоренц-коваріантна деформована алґебра, мінімальна довжина.