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THERMOMAGNETIC INSTABILITIES IN A NONUNIFORMLY ROTATING ELECTRICALLY CONDUCTIVE FLUID

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The paper investigates the stability of small axisymmetric disturbances in a nonuniformly rotating viscous electrically conductive fluid taking into account galvanometric and thermo-magnetic phenomena. In the local geometrical optics approximation, we obtained a dispersion equation taking into account the Hall, the Nernst, the Righi–Leduc effects and gradients of temperature ∇T_0 and thermo-electromotive force coefficient $\nabla\alpha$ in constant magnetic field $\mathbf{B}_0$ and gravitational field $\mathbf{g}$. The growth rates of thermomagnetic instability (TMI) in a nonuniformly rotating electrically conducting fluid without an external magnetic field ($\mathbf{B}_0 = 0$) are obtained for the case of “smooth” (a weakly inhomogeneous medium) gradients (∇T_0 and $\nabla\alpha$). The regions of the development of TMI are established depending on the profile of the angular velocity of rotation (Rossby number Ro) and the radial wave number k_R. The conditions under which the generation of a magnetic field with sharp gradients of temperature and thermo-electromotive force coefficient in the media with low ($\sigma \to 0$) and high ($\sigma \to \infty$) conductivity are found. The regions of the development of the Hall magnetorotational instability in an external magnetic field ($\mathbf{B}_0 \neq 0$) are established depending on the profile of the angular rotation velocity (Rossby number Ro) and the axial wavenumber k_z. The growth rates of TM instabilities for the propagation of perturbations with a wave vector $\mathbf{k}$ in the radial direction $\mathbf{k}\|\mathbf{e}_R$ are obtained taking into account the Nernst effect in an external magnetic field $\mathbf{B}_0$, the Righi–Leduc effect, the inhomogeneity of the equilibrium temperature and specific thermopower, and the buoyancy force in a temperature-stratified medium.

Key words: thermoelectromotive force, generation of magnetic fields, thermomagnetic instability, Boussinesq approximation, nonuniformly rotating electrically conductive fluid.

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I. INTRODUCTION

It is well known that instabilities caused by the temperature gradient play an important role in the dynamics of electrically conductive media (plasma, liquid metals, nanofluids, etc.). Thus, in an inhomogeneous plasma with a temperature gradient, the so-called thermomagnetic (TM) waves arise [1]. TM waves can grow [2] in the presence of an external magnetic field. In addition, as shown in [3], TM waves are possible in many metals and semimetals at sufficiently low temperatures. Even in a weak magnetic field, when the Larmor frequency Ω_L is much lower than the collision frequency $1/\tau$, TM waves in a solid can grow [3]. The instabilities of TM waves play an important role in the generation of magnetic fields. In the area of instability development, the plasma can become turbulent, which leads to an accelerated process of thermal energy transfer were investigated. In [4], the hydrodynamic instabilities of Alfven and thermomagnetic waves in an inhomogeneous plasma in the presence of an external magnetic field. It was shown in [4] that Alfven waves in a strong magnetic field can be strongly damping due to the anisotropic nature of thermal conductivity. Such damping may be important for the processes occurring in the solar corona.

Approaches to the generation of magnetic fields using TM effects can be found in earlier works. The idea of generating the Earth's magnetic field by thermoelectric currents was first developed in paper [5]. In [5], the thermoelectromotive force was supposed to arise directly in the liquid core of the Earth due to the temperature difference between ascending and descending convective flows. In this case, the resulting angular momentum of the currents around the Earth's axis must be asymmetric due to the prevailing influence of the Coriolis force on the convective motions to give a non-zero contribution. In another work [6] it was shown that the thermoelectric power arises due to the temperature difference between the cold mantle and the hot core. Indeed, the temperature gradient applied to the connection of two conductors leads to the occurrence of thermoelectric power (the Seebeck effect or the thermoelectric effect [7])

$$\mathscr{E}_T = \int_{T_1}^{T_2} \alpha\, dT, \tag{1}$$

where α is the thermo-electromotive force coefficient. As is known [7], the effect opposite to the Seebeck effect is called the Peltier effect: when current passes through a connection of two different conductors, the heat is being produced. On the basis of this effect, the work [8] concluded that if the mantle and core thermocurrents move oppositely, the Earth's crust cools down globally and an ice age comes, but if they move unidirectionally, then global warming comes. The calculations carried out in [8] showed that the Earth's surface can warm up by not more than 10°C. Compared to how the human factor affects the Earth's warming, this is an incomparably large value. The estimates made in [8] show that the power of the Earth's thermocurrents is enough to generate and maintain the Earth's magnetic field.

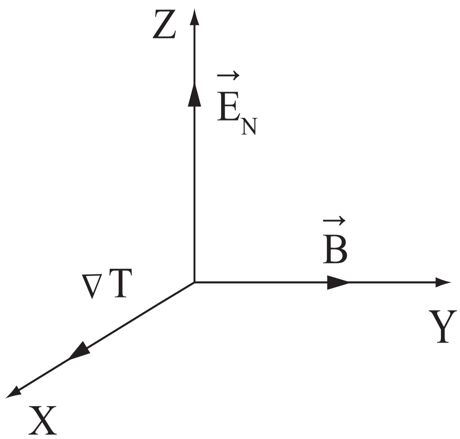

Fig. 1. The direction of the vectors $\mathbf{B}, \nabla T$, and $\mathbf{E}_N$ in the Nernst effect is shown

The theory of the Earth's magnetic field created by a thermoelectric current flowing in the Earth's core by the Nernst effect was developed in [9]. The Nernst effect is the effect of the appearance of an electric field $\mathbf{E}_N \sim [\mathbf{B} \times \nabla T]$ in a conductor that is directed perpendicular to the temperature gradient vector ∇T and the magnetic induction vector $\mathbf{B}$. For the geometry is shown in Fig. 1, the electric field strength is

$$E_{Nz} = -\mathcal{N} B_y \frac{dT}{dx}, \tag{2}$$

where $\mathcal{N}$ is the Nernst–Ettingshausen constant, which depends on the properties of the conductor and can take both positive and negative values. The Nernst effect occurs as a result of the deflection of a stream of charged particles by the Lorentz force. A directed flow of particles arises as a result of diffusion. The diffusion flow is always directed from the heated to the colder part of the conductor regardless of the sign of the charge of the particles. Naturally, the directions of the Lorentz force for positive and negative particles are mutually opposite. It leads to charge separation. Thermomagnetic e.m.f. created by the Nernst effect is

$$\mathscr{E}_N = \oint \mathbf{E}_{\text{eff}}\, d\mathbf{l} = \oint \mathcal{N}\,[\mathbf{B} \times \nabla T]\, d\mathbf{l} = \oint [\mathbf{G} \times \mathbf{B}]\, d\mathbf{l}, \tag{3}$$

where $\mathbf{G} = -\mathcal{N}\nabla T$ has the dimension of velocity [9]. The physical meaning of $\mathbf{G}$ will become clear after the magnetic field induction equation is obtained. Using Ohm's law taking into account the thermomagnetic field $\mathbf{E}_{\text{eff}}$ and Maxwell's equations, we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{G} \times \mathbf{B}] + \eta \nabla^2 \mathbf{B}, \tag{4}$$

where $\eta = 1/\mu\sigma$ is the coefficient of magnetic viscosity, μ is the coefficient of magnetic permeability. It can be seen from equation (4) that the magnetic field is strengthened by analogy with the effect of differential rotation of an electrically conductive liquid [10], i. e., due to the stretching of field lines with a thermal drift rate $\mathbf{G}$ when they are "partially" frozen. In [9], the thermomagnetic Reynolds number $Re_{\text{tm}} = GL/\eta$ (L is the characteristic scale of the field) was introduced, which is a measure of the influence of the induction term $\text{rot}[\mathbf{G} \times \mathbf{B}]$ over the diffusion term $\eta\nabla^2\mathbf{B}$. If the first term on the right side (4) is less than the second one for small $Re_{\text{tm}} \ll 1$, the magnetic field decays on the magnetic viscosity. When $Re_{\text{tm}} \gg 1$, the second term on the right side (4) is small compared to the first one, and the magnetic field lines are carried away at the speed $\mathbf{G}$. Thus, there is an analogy with the amplification of the magnetic field by the differential (inhomogeneous) rotation of an electrically conductive medium for which the Reynolds number $Re_\omega = \omega_0 L^2/\eta$ (ω_0 is the characteristic angular velocity of the medium rotation).

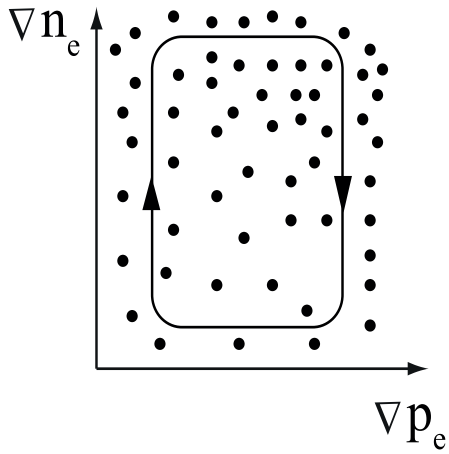

Fig. 2. A diagram of the Birman–Schlüter effect is shown. The magnetic field is excited as a result of charge separation due to non-collinear density gradients ∇n_e and pressure ∇p_e of the plasma

In [11] it was shown that the magnetic field of the Earth and planets can be created by thermoelectric currents $\mathbf{j} = \sigma(\mathbf{E} - \alpha\nabla T)$, where α is the specific thermoelectromotive force that depends on both the chemical composition of the medium and temperature.

Therefore, if the gradient of thermoelectromotive force coefficient $\nabla\alpha$ is caused by the inhomogeneity of the chemical composition of the medium, then the magnetic field $\partial\mathbf{B}/\partial t \approx [\nabla T \times \nabla\alpha]$ is excited due to the nonparallelism of the vectors ∇T and $\nabla\alpha$. This effect is similar to the Birman–Schlüter effect [12] (see Fig. 2). The main idea of this effect consists in the mechanism of self-excitation of magnetic fields by a thermoelectromotive force. As shown in [12], the inhomogeneity of the chemical composition of a space object can lead to nonparallelism of the electron pressure gradients ∇p_e and electron density ∇n_e. As a result, the electric field $\mathbf{E}^{(i)} = (1/en_e)\nabla p_e$ arises, leading to the excitation of magnetic fields $\partial\mathbf{B}/\partial t \approx \nabla\times\mathbf{E}^{(i)} = -\frac{1}{n_e^2}[\nabla n_e \times \nabla p_e]$. Such a mechanism is associated with the generation of initial magnetic fields, which at some point were absent. In [13], it was assumed that the temperature gradient ∇T_0 and the gradient of thermoelectromotive force coefficient $\nabla\alpha$ were parallel $[\nabla T_0 \times \nabla\alpha] = 0$; so it was concluded that the thermoelectric power in the Earth's core was insignificant.

TM instability was discovered in a number of papers [14–16], where the spontaneous generation of strong magnetic fields ($\sim 10^2$ T) was explained in various experiments on the interaction of laser radiation with matter in negligibly short times $\sim 10^{-9}$ s. A necessary condition for the development of TM instability is the inhomogeneity and non isothermality of the plasma. The physical mechanism of this instability is as follows. Let us consider a simplified equation for the magnetic field:

$$\frac{\partial\mathbf{B}}{\partial t} = \frac{c}{en_e}[\nabla T_e \times \nabla n_e] - \frac{c}{e}\nabla\times\left(\frac{\mathbf{R}_T}{n_e}\right) \qquad (5)$$

where T_e, n_e are the temperature and electron concentration, $\mathbf{R}_T$ is the thermal force [14]. Eq. (5) shows that a magnetic field cannot arise if $\nabla T_e \| \nabla n_e$. However, weak temperature perturbations T_1 acting in a direction different from the initial plasma inhomogeneity $(\nabla n_0, \nabla T_0)$ lead to the excitation of small fluctuations of the magnetic field due to the Birman-Schlüter effect [12] $\partial\mathbf{B}_1/\partial t \approx [\nabla T_1 \times \nabla n_0]$. Magnetic field fluctuations create an electron heat flux $\sim [\mathbf{B}_1 \times \nabla T_0]$ perpendicular to the main gradient fluxes (∇T_0). The Righi–Leduc effect appears when a conductor with a temperature gradient is placed in a magnetic field perpendicular to the heat flux. As a result, a secondary temperature difference arises, perpendicular to the magnetic field and the heat flux. The resulting heat flux brings energy to the area with elevated temperature. Thus, the additional temperature gradient ∇T_1 perpendicular to ∇n_0 contributes to the growth of the initial perturbations of the magnetic field in accordance with Eq. (5).

Astrophysical applications of TMI are discussed in detail in review [17], where an explanation of the appearance of strong magnetic fields in the cores of white dwarfs, binary systems, and neutron stars is given. In a recent paper [17], the generation of a magnetic field by TMI in the surface layers (hot plasma) of massive stars was considered. Such generation is possible in the upper layers of the atmosphere of hot stars, where deviations from local thermodynamic equilibrium form a region with an inverse temperature gradient. In [18], the case of generation of only small-scale magnetic fields with horizontal wavelengths $\lambda = 2\pi/k_x$, much smaller than the characteristic scale L ($\lambda \ll L$) of unperturbed quantities, was considered. In some recent works [19–21], the TM instability in nonuniformly rotating plasma media (hot galactic disks, accretion disks) in an external axial magnetic field was considered. These papers present an analysis of the linear stability of ionized hot disks with a temperature gradient and an external axial magnetic field. As shown in [19–21], the hydromagnetic and thermomagnetic effects associated with the Nernst effect can lead to the amplification of waves and make disks unstable. The regimes under which both thermomagnetic and magnetorotational instabilities (MRI) can operate were discussed. MRI arises when a weak axial magnetic field destabilizes the azimuthal differential rotation of the plasma, and when the condition $d\Omega^2/dR < 0$ for the case of a nondissipative plasma is satisfied [22]. Since this condition is also satisfied for Keplerian flows $\Omega \sim R^{-3/2}$, the MRI is the most likely source of turbulence in accretion disks [23]. It was noted in [21] that even in the absence of MRI, TM instability due to the Nernst effect is a good candidate for ensuring the onset of turbulence in disks.

In contrast to [19–21], where the approximation of non-dissipative magnetohydrodynamics was used, we considered TM effects in a nonuniformly rotating viscous electrically conductive fluid. In this work, we obtained some new effects that were not investigated in [19–22], i.e., the effects taking into account the inhomogeneity of the equilibrium temperature and the specific thermoelectromotive force coefficient, the Righi–Leduc effect, and the gravitational force. Furthermore, unlike papers [19–21], in this work we gave a rigorous justification of the local method of geometric optics for $\lambda \ll L$.

The results obtained in this work can find application in different problems of the magnetic geodynamo, as well as in laboratory studies into rotating magnetic convection taking into account thermomagnetic phenomena.

II. STATEMENT OF THE PROBLEM AND EQUATIONS OF EVOLUTION OF SMALL PERTURBATIONS

Let us suppose that a nonuniformly rotating electrically conducting fluid (for example, liquid metal or plasma) is in a constant gravitational $\mathbf{g}$ and magnetic field $\mathbf{B}_0$ at a constant vertical temperature gradient $\nabla T_0 = \text{const} = -A\mathbf{e}$ ($A > 0$ is a constant gradient, $\mathbf{e}$ is a unit vector directed vertically upward along the axis Z) and a gradient of the thermo-electromotive force coefficient $\nabla\alpha \parallel \mathbf{e}$. In this model, we assume that the gradient of the thermo-electromotive force coefficient $\nabla\alpha$ is associated with the inhomogeneity of the chemical composition of the conducting fluid. It is known that expressions for Ohm's law and heat flux $\mathbf{q}$ in the presence of a magnetic field $\mathbf{B}$ and a temperature gradient ∇T are modified taking into account thermomagnetic phenomena [24]:

$$\mathbf{E} + [\mathbf{V}\times\mathbf{B}] = \frac{\mathbf{j}}{\sigma} + \alpha\nabla T + \mathcal{R}[\mathbf{B}\times\mathbf{j}] + \mathcal{N}[\mathbf{B}\times\nabla T] \quad (6)$$

$$\mathbf{q} - \varphi\mathbf{j} = -\kappa\nabla T + \alpha T\mathbf{j} + \mathcal{N}T[\mathbf{B}\times\mathbf{j}] + \mathcal{L}[\mathbf{B}\times\nabla T], \quad (7)$$

where $\mathcal{R}, \mathcal{N}, \mathcal{L}$ are the Hall, Nernst, and Righi–Leduc coefficients, respectively; φ is the electrical potential. In expressions (6)–(7), we neglected the anisotropy of the coefficients of electrical conductivity $\sigma_\parallel \approx \sigma_\perp = \sigma$, thermal conductivity $\kappa_\parallel \approx \kappa_\perp = \kappa$, and thermo-electromotive force $\alpha_\parallel \approx \alpha_\perp = \alpha$ due to the weakness of the external magnetic field because the parameter is small $\beta = B_0^2/2\mu P_0 \ll 1$ (P_0 is the stationary pressure of the fluid, μ is the coefficient of magnetic permeability). By applying the operation $(\nabla\times)$ to Ohm's law (6), we obtain the equation for the magnetic field induction $\mathbf{B}$. After substitution of expression (7) into the heat balance equation

$$\rho_0 c_p \frac{dT}{dt} = -\nabla \mathbf{q},$$

let us write the equations of magnetohydrodynamics for a viscous incompressible fluid in the Boussinesq approximation taking into account thermomagnetic phenomena:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho_0}\nabla(P + \frac{\mathbf{B}^2}{2\mu}) + \frac{1}{\rho_0\mu}(\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{e}g\beta_T T + \nu\nabla^2\mathbf{V} \tag{8}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{V} = \eta\nabla^2\mathbf{B} - [\nabla\alpha\times\nabla T] - \frac{\mathcal{R}}{\mu}\nabla\times[\mathbf{B}\times\nabla\times\mathbf{B}] - \mathcal{N}\nabla\times[\mathbf{B}\times\nabla T] \tag{9}$$

$$\frac{\partial T}{\partial t} + (\mathbf{V}\cdot\nabla)T = -\frac{1}{\rho_0 c_p}\nabla\left(-\kappa\nabla T + \frac{\mathbf{j}^2}{\sigma} + \alpha T\mathbf{j} + \mathcal{N}T[\mathbf{B}\times\mathbf{j}] + \mathcal{L}[\mathbf{B}\times\nabla T]\right) \tag{10}$$

$$\nabla\mathbf{B} = 0, \quad \nabla\mathbf{V} = 0, \tag{11}$$

where β_T is the coefficient of thermal expansion, $\rho_0 = \mathrm{const}$ is the density of the medium, ν is the coefficient of kinematic viscosity, $\eta = 1/\mu\sigma$ is the coefficient of magnetic viscosity. Eq. (9) contains a source of excitation of a magnetic field $[\nabla\alpha\times\nabla T]$, which is an analog of the "battery Birmann-Schluter effect in the plasma. The drift of the lines of force of the magnetic field in Eq. (9) is associated not only with the movement of the fluid $\mathbf{V}$ but also with the heat flux where the rate of thermal drift: $\mathbf{V}_T = \mathcal{N}\nabla T$. The drift of the magnetic field due to the Nernst effect contributes to its penetration to a large area of the medium. Let us estimate the excited magnetic field in the stationary regime without taking into account the drift of the field and the Hall effect. Then from Eq. (9) for the ϕ-component of the (toroidal) magnetic field, we obtain: $B_\phi^{\max} \approx \alpha T\mu\sigma(L_B/L_\alpha)$, where L_B is the characteristic scale of the excited magnetic field, L_α is the characteristic scale of the medium inhomogeneity. Replacing the values of the parameters for the fluid Earth's core: $\alpha T \cong 10^{-2}$V (at temperature $T \cong 1000$K), $\mu = 4\pi\cdot 10^{-7}$ V·s/A·m, $\sigma = 3\cdot 10^5$ (Ohm·m)$^{-1}$ [25] with the ratio of scales $(L_B/L_\alpha) = 10^2$, we obtain an estimated value of the toroidal magnetic field of the Earth's core $B_\phi^{\max} \cong 10^{-1}$T, which coincides in the order of magnitude with the data from monograph [25].

Let us represent all quantities in Eqs. (8)–(11) as the sum of the stationary and perturbed parts $\mathbf{V} = \mathbf{V}_0 + \mathbf{u}$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $P = P_0 + p$, $T = T_0 + \theta$. Here we assume that the stationary rotation velocity of the fluid has an azimuthal direction $\mathbf{V}_0 = R\Omega(R)\mathbf{e}_\phi$, where the angular velocity of rotation $\Omega(R)$ is directed vertically upward along the axis OZ. We simulate the stationary flow of a nonuniformly rotating fluid by the Couette–Taylor flow enclosed between two rotating cylinders with an angular velocity of rotation $\Omega(R)$ (see Fig. 3):

$$\Omega(R) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} + \frac{(\Omega_1 - \Omega_2)R_1^2 R_2^2}{R^2(R_2^2 - R_1^2)},$$

where $R_1 = R_{\mathrm{in}}, R_2 = R_{\mathrm{out}}, \Omega_1 = \Omega_{\mathrm{in}}, \Omega_2 = \Omega_{\mathrm{out}}$ are the radius and angular velocity of rotation of the inner and outer cylinders, respectively. The constant magnetic field $\mathbf{B}_0 \| \mathbf{\Omega}$ is also directed along the axis OZ: $\mathbf{B}_0 = (0, 0, B_0)$. Further, the magnetic field $\mathbf{B}_0$ will be called axial in the cylindrical coordinate system (R, ϕ, z). The stationary state of the system of equations (8)–(11) is described by the following equations

$$\Omega^2 R = \frac{1}{\rho_0}\frac{dP_0}{dR}, \quad \frac{1}{\rho_0}\frac{dP_0}{dz} = g\beta_T T_0, \quad \frac{d^2 T_0}{dz^2} = 0, \tag{12}$$

$$B_0 \frac{d}{dz}\Omega(R)R = [\nabla\alpha\times\nabla T_0]_\phi = 0. \tag{13}$$

Equations (12) shows that centrifugal equilibrium is established in the radial direction and hydrostatic equilibrium in the vertical direction. From Eq. (13) it follows that the thermo-electromotive coefficient α has a constant value in the radial direction: $d\alpha/dR = 0$; then α can have a dependence on the coordinates (ϕ, z). If we consider the distribution of the chemical composition of the medium to be axisymmetric, then the condition is satisfied: $d\alpha/dz \neq 0$. In this case, the condition of collinearity of vectors is also satisfied $[\nabla\alpha\times\nabla T_0] = 0$, and the gradients $\nabla\alpha$ and ∇T_0 can be both parallel to each other $\nabla\alpha \uparrow\uparrow \nabla T_0$ and antiparallel: $\nabla\alpha \uparrow\downarrow \nabla T_0$.

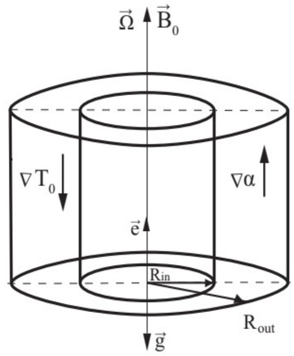

Fig. 3. The geometry of the problem: the nonuniform rotation is modeled by the Couette-Taylor flow enclosed between two concentric cylinders of radii $R_{\rm in}$ and $R_{\rm out}$, which rotate at velocities $\Omega_{\rm in}$ and $\Omega_{\rm out}$. The angular velocity of rotation $\mathbf{\Omega}(R)$ and the external magnetic field $\mathbf{B}_0$ are directed vertically upward along the axis OZ. Vertical gradients of temperature ∇T_0 and thermo-electromotive force coefficient $\nabla\alpha$ are antiparallel to each other

Then, we consider the evolution of small perturbations $(\mathbf{u},\mathbf{b},p,\theta)$ against the background of a stationary state Eqs. (12)–(13) to clarify the physical mechanism of the generation of TM-waves and instabilities in a nonuniformly rotating magnetized fluid. As long as the medium is stratified by temperature and rotates at an inhomogeneous angular velocity, a justification for the applicability of the local geometrical optics approximation should be provided.

III. LOCAL GEOMETRICAL OPTICS APPROXIMATION AND DISPERSION EQUATION

Let us consider the limit of a weakly inhomogeneous medium when the spatial scale of the medium inhomogeneity $(L_\alpha,\ L_T)$ in the z-coordinate is much larger than the characteristic scale of perturbations (wavelength) λ: $L\gg\lambda$. The approximation of geometric optics is fulfilled in the short-wavelength limit $(L\gg\lambda)$, and, therefore, all perturbed quantities in the formulas (8)-(11) can be represented by a dependence of the form $\exp(i\mathbf{kr}+\gamma t)$, where $\mathbf{k}$ is the wave vector, γ is the amplification (or damping) factor of the perturbations [26].

Following papers [27, 28], we present a more rigorous justification of the short-wave approximation using the asymptotic WKB method. For this purpose, we represent the solutions of the linearized system of equations (8)–(11) in the form of an asymptotic series in the small parameter ε ($0<\varepsilon\ll1$):

$$\begin{pmatrix}\mathbf{u}\\ \mathbf{b}\\ p\\ \theta\end{pmatrix}=e^{i\Phi(\mathbf{x},t)/\epsilon}\begin{pmatrix}\mathbf{u}^{(0)}(\mathbf{x},t)+\epsilon\mathbf{u}^{(1)}(\mathbf{x},t)+\cdots\\ \mathbf{b}^{(0)}(\mathbf{x},t)+\epsilon\mathbf{b}^{(1)}(\mathbf{x},t)+\cdots\\ \theta^{(0)}(\mathbf{x},t)+\epsilon\theta^{(1)}(\mathbf{x},t)+\cdots\\ p^{(0)}(\mathbf{x},t)+\epsilon p^{(1)}(\mathbf{x},t)+\cdots\end{pmatrix},\tag{14}$$

where $\mathbf{x}=(R,\phi,z)$ are the cylindrical coordinates recorded in the vector form; $\Phi(\mathbf{x},t)$ is a scalar function, called the phase of the perturbed quantities oscillations; $\mathbf{u}^{(n)}$, $\mathbf{b}^{(n)}$, $\theta^{(n)}$, $p^{(n)}$ $(n=0,1,\ldots)$ are the amplitudes of disturbances. Using the WKB representation (14), the diffusion term in the equations (8)–(11) is written as

$$\nabla^2\begin{pmatrix}\mathbf{u}\\ \mathbf{b}\\ \theta\end{pmatrix}=e^{i\Phi(\mathbf{x},t)/\epsilon}\left(\nabla^2+i\frac{2}{\epsilon}(\nabla\Phi\cdot\nabla)+\frac{i}{\epsilon}\nabla^2\Phi-\frac{(\nabla\Phi)^2}{\epsilon^2}\right)\begin{pmatrix}\mathbf{u}^{(0)}(\mathbf{x},t)+\epsilon\mathbf{u}^{(1)}(\mathbf{x},t)+\cdots\\ \mathbf{b}^{(0)}(\mathbf{x},t)+\epsilon\mathbf{b}^{(1)}(\mathbf{x},t)+\cdots\\ \theta^{(0)}(\mathbf{x},t)+\epsilon\theta^{(1)}(\mathbf{x},t)+\cdots\end{pmatrix}.\tag{15}$$

It is clear that the WKB ansatz (14) quickly dies out because of diffusion unless ε has a quadratic dependence on diffusion coefficients. We rescale the dimensional diffusivities (ν,η,χ) as

$$\nu=\epsilon^2\widetilde{\nu},\quad \eta=\epsilon^2\widetilde{\eta},\quad \chi=\epsilon^2\widetilde{\chi}.$$

Similar reasoning leads to the rescaling of the coefficients

$$\alpha=\epsilon\widetilde{\alpha},\quad \mathcal{R}=\epsilon^2\widetilde{\mathcal{R}},\quad \mathcal{N}=\epsilon^2\widetilde{\mathcal{N}},\quad u_T=\epsilon\widetilde{u}_T.$$

Substituting expansions (14) into the system of equations (8)–(11), we obtain the system of local differential equations for ε^{-1} and ε^0 orders.

According to papers [27, 28], the basic equations are the equations for the order ϵ^0 that have the form

$$\frac{\partial\mathbf{u}^{(0)}}{\partial t}+(\mathbf{V}_0\cdot\nabla)\mathbf{u}^{(0)}+\mathcal{U}\mathbf{u}^{(0)}+\widetilde{\nu}(\nabla\Phi)^2\mathbf{u}^{(0)}$$
$$=-\frac{i\nabla\Phi}{\rho_0}\left(p^{(1)}+\frac{\mathbf{B}_0\cdot\mathbf{b}^{(1)}}{\mu}\right)$$
$$+\frac{1}{\rho_0\mu}(\mathbf{B}_0\cdot\nabla)\mathbf{b}^{(0)}+\mathbf{e}g\beta_T\theta^{(0)},$$

$$\frac{\partial \mathbf{b}^{(0)}}{\partial t} + (\mathbf{V}_0 \cdot \nabla)\mathbf{b}^{(0)} + i(\widetilde{\mathbf{u}}_T \cdot \nabla\Phi)\mathbf{b}^{(0)} - \mathcal{U}\mathbf{b}^{(0)} + \widetilde{\eta}(\nabla\Phi)^2\mathbf{b}^{(0)} \tag{16}$$
$$= (\mathbf{B}_0 \cdot \nabla)\mathbf{u}^{(0)} - i\widetilde{\alpha}[\mathbf{K}_\alpha \times \nabla\Phi]\theta^{(0)} - \frac{\mathcal{R}}{\mu}(\mathbf{B}_0 \cdot \nabla\Phi)(\nabla\Phi \times \mathbf{b}^{(0)})$$
$$+\mathcal{N}(\mathbf{B}_0(\nabla\Phi)^2\theta^{(0)} - \theta^{(0)}(\mathbf{B}_0 \cdot \nabla\Phi)\nabla\Phi)\frac{\partial \theta^{(0)}}{\partial t} + (\mathbf{V}_0 \cdot \nabla)\theta^{(0)} - K_T T_0 u_z^{(0)} + \widetilde{\chi}(\nabla\Phi)^2\theta^{(0)}$$
$$= \chi_\wedge T_0 \mathbf{K}_T \cdot \nabla \times \mathbf{b}^{(0)} + i\mathbf{s}'(\nabla\Phi \times \mathbf{b}^{(0)}) + \frac{\widetilde{\mathcal{N}} T_0}{\rho_0 c_p \mu}(\nabla\Phi)^2 \mathbf{B}_0 \cdot \mathbf{b}^{(0)}$$

where the matrix of the non-uniform rotation $\mathcal{U}$ is

$$\mathcal{U} = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega + R\frac{d\Omega}{dR} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The perturbations $\mathbf{u}^{(0)}$, $\mathbf{b}^{(0)}$, $\theta^{(0)}$, $p^{(1)}$ in Eq. (16) can be represented in the form of plane waves

$$\begin{pmatrix} \mathbf{u}^{(0)} \\ \mathbf{b}^{(0)} \\ \theta^{(0)} \\ p^{(1)} \end{pmatrix} = \begin{pmatrix} \mathbf{U} \\ \mathbf{H} \\ \Theta \\ P \end{pmatrix} \exp(\gamma t + ik_R R + ik_z z). \tag{17}$$

After substituting (17) into the system of equations (16), we obtain the dispersion equation:

$$\begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix} = v_{11}v_{22}v_{33} + v_{21}v_{32}v_{13} + v_{12}v_{23}v_{31} - v_{13}v_{22}v_{31} - v_{32}v_{23}v_{11} - v_{21}v_{12}v_{33} = 0. \tag{18}$$

Here the tensor components v_{ij} are as follows:

$$v_{11} = \gamma + \omega_\nu + a_3 m_1 + a_4 n_1 + a_5 l_1,\ v_{12} = a_1 + a_3 m_2 + a_4 n_2 + a_5 l_2,$$

$$v_{13} = a_2 + a_3 m_3 + a_4 n_3 + a_5 l_3,$$

$$v_{21} = a_6 + a_7 m_1,\ v_{22} = \gamma + \omega_\nu + a_7 m_2,\ v_{23} = a_7 m_3,$$

$$v_{31} = a_{10} m_1 + a_{11} n_1,\ v_{32} = a_9 + a_{10} m_2 + a_{11} n_2,\ v_{33} = a_8 + a_{10} m_3 + a_{11} n_3,$$

where

$$m_1 = -\frac{b_4 b_6}{c_1},\ m_2 = \frac{b_8}{c_1}(\gamma + i\omega_T + \omega_\eta),\ m_3 = -\frac{c_2}{c_1},$$

$$c_1 = b_6\left(b_1 - \frac{b_2 b_{11}}{b_{10}}\right) - \left(b_5 - \frac{b_7 b_{11}}{b_{10}}\right)(\gamma + i\omega_T + \omega_\eta),$$

$$c_2 = b_6\left(b_3 - \frac{b_2 b_{12}}{b_{10}}\right) - \left(b_9 - \frac{b_7 b_{12}}{b_{10}}\right)(\gamma + i\omega_T + \omega_\eta),$$

$$n_1 = -\frac{b_{11}}{b_{10}} m_1, \quad n_2 = -\frac{b_{11}}{b_{10}} m_2, \quad n_3 = -\frac{b_{11}}{b_{10}} m_3 - \frac{b_{12}}{b_{10}},$$

$$l_1 = \left(b_5 - \frac{b_7 b_{11}}{b_{10}}\right)\frac{b_4}{c_1}, \quad l_2 = -\left(b_1 - \frac{b_2 b_{11}}{b_{10}}\right)\frac{b_8}{c_1},$$

$$l_3 = \frac{1}{c_1}\left[\left(b_3 - \frac{b_2 b_{12}}{b_{10}}\right)\left(b_5 - \frac{b_7 b_{11}}{b_{10}}\right) - \left(b_9 - \frac{b_7 b_{12}}{b_{10}}\right)\left(b_1 - \frac{b_2 b_{11}}{b_{10}}\right)\right].$$

Further, we give expressions for the coefficients $a_1, a_2, \dots a_{11}$ and $b_1, b_2, \dots b_{12}$:

$$a_1 = -2\Omega\frac{k_z^2}{k^2}, \quad a_2 = g\beta_T\frac{k_R k_z}{k^2}\frac{K_T T_0}{\gamma+\omega_\chi}, \quad a_3 = g\beta_T\frac{k_R k_z}{k^2}\frac{ik_R s}{\gamma+\omega_\chi},$$

$$a_4 = g\beta_T\frac{k_R k_z}{k^2}\frac{\omega_\mathcal{N} V_A^2}{c_p B_0(\gamma+\omega_\chi)},$$

$$a_5 = -\frac{i(\mathbf{k}\cdot\mathbf{B}_0)}{\rho_0\mu}, \quad a_6 = 2\Omega(1+\mathrm{Ro}), \quad a_7 = -\frac{i(\mathbf{k}\cdot\mathbf{B}_0)}{\rho_0\mu},$$

$$a_8 = \gamma+\omega_\nu - g\beta_T\frac{k_R^2}{k^2}\frac{K_T T_0}{\gamma+\omega_\chi},$$

$$a_9 = 2\Omega\frac{k_R k_z}{k^2}, \quad a_{10} = -g\beta_T\frac{ik_R^3 s}{k^2(\gamma+\omega_\chi)}, \quad a_{11} = -g\beta_T\frac{k_R^2}{k^2}\frac{\omega_\mathcal{N} V_A^2}{c_p B_0(\gamma+\omega_\chi)} - \frac{i(\mathbf{k}\cdot\mathbf{B}_0)}{\rho_0\mu},$$

$$b_1 = \frac{ik_R^2 s\mathcal{N}(\mathbf{k}\cdot\mathbf{B}_0)}{\gamma+\omega_\chi} - \frac{\mathcal{R}}{\mu}(\mathbf{k}\cdot\mathbf{B}_0)k_z, \quad b_2 = \mathcal{N}\frac{\omega_\mathcal{N} V_A^2}{c_p B_0(\gamma+\omega_\chi)}k_R(\mathbf{k}\cdot\mathbf{B}_0),$$

$$b_3 = \mathcal{N}(\mathbf{k}\cdot\mathbf{B}_0)\frac{k_R K_T T_0}{\gamma+\omega_\chi},$$

$$b_4 = -i(\mathbf{k}\cdot\mathbf{B}_0), \quad b_5 = (\gamma+i\omega_T+\omega_\eta) - \frac{\alpha s K_\alpha k_R^2}{\gamma+\omega_\chi}, \quad b_6 = -\left(2\Omega\mathrm{Ro} - \frac{\mathcal{R}}{\mu}k_z(\mathbf{k}\cdot\mathbf{B}_0)\right),$$

$$b_7 = \frac{i\alpha K_\alpha k_R}{\gamma+\omega_\chi}\frac{\omega_\mathcal{N} V_A^2}{c_p B_0} - \frac{\mathcal{R}}{\mu}k_R(\mathbf{k}\cdot\mathbf{B}_0), \quad b_8 = -i(\mathbf{k}\cdot\mathbf{B}_0), \quad b_9 = \frac{i\alpha T_0 K_\alpha K_T k_R}{\gamma+\omega_\chi},$$

$$b_{10} = (\gamma+i\omega_T+\omega_\eta) - \frac{\mathcal{N}\omega_\mathcal{N} V_A^2 k_R}{c_p(\gamma+\omega_\chi)}, \quad b_{11} = \frac{\mathcal{R}}{\mu}k_R(\mathbf{k}\cdot\mathbf{B}_0) - \frac{i\mathcal{N}B_0 s K_R^3}{\gamma+\omega_\chi},$$

$$b_{12} = -\left(i(\mathbf{k}\cdot\mathbf{B}_0) + \frac{\mathcal{N}B_0 K_T T_0 k_R^2}{\gamma+\omega_\chi}\right).$$

Thus, the problem of stability of a nonuniformly rotating electrically conductive fluid with collinear gradients of temperature and thermo-electromotive force coefficient leads to the problem of finding eigenvalues γ from Eq. (18). Equation (18) is a rather cumbersome expression in its expanded form. Therefore, we will analyze this equation in some limiting cases. Without taking into account TM effects, Eq. (18) completely coincides with the results of the paper [29].

IV. ANALYSIS OF THE DISPERSION EQUATION FOR THE CASE $\mathbf{B}_0 = 0$

In this section, we investigate the possibility of spontaneous generation of a magnetic field by TMI in an inhomogeneous electrically conducting fluid without an external magnetic field $\mathbf{B}_0 = 0$. Let us consider the inhomogeneity of an electrically conductive medium within two limits: 1) a weakly inhomogeneous medium, when the values of $T_0(z)$ and $\alpha(z)$ change smoothly in z, where the geometric optics approximation is locally fulfilled; and 2) a strongly inhomogeneous medium with a jump-like dependence on $T_0(z)$ and $\alpha(z)$, i. e., when the geometric optics approximation is not applicable.

A. Generation of magnetic fields in a weakly inhomogeneous medium (a short-wavelength limit $L \gg \lambda$)

Let us consider the influence of TM effects on the stability of small perturbations in a nonuniformly rotating electrically conducting fluid without an external magnetic field $\mathbf{B}_0 = 0$. In this case, the dispersion equation (18) is simplified

$$\left[(\gamma+\omega_\nu)^2 + \kappa^2\xi^2\right]\left[(\gamma+i\omega_T+\omega_\eta)(\gamma+\omega_\chi) - \omega_{TM}\right]$$
$$-\frac{k_R^2}{k^2}\omega_{VB}^2(\gamma+i\omega_T+\omega_\eta)(\gamma+\omega_\nu) = 0, \qquad (19)$$

where $\omega_{VB} = \sqrt{g\beta_T K_T T_0}$ is the Visel-Brent frequency depending on the temperature gradient, $\kappa = 2\Omega\sqrt{1+\mathrm{Ro}}$ is the epicyclic frequency, $\xi = k_z/k$, $\omega_{TM} = \sqrt{\alpha s k_R^2 K_\alpha}$ is the thermomagnetic frequency. Under the condition of $\mathbf{B}_0 = 0$, we exclude the occurrence of MRI, since a necessary criterion for its development is the presence of a weak external magnetic field in a nonuniformly rotating fluid.

We represent the dispersion equation (19) in the form of a fourth-order polynomial with complex coefficients :

$$\mathscr{P}(\gamma) \equiv A_0\gamma^4 + (A_1 + iB_1)\gamma^3 + (A_2 + iB_2)\gamma^2 + (A_3 + iB_3)\gamma + A_4 + iB_4 = 0, \tag{20}$$

where

$$A_0 = 1,$$

$$A_1 = 2\omega_\nu + \omega_\chi + \omega_\eta, \quad B_1 = \omega_T,$$

$$A_2 = \omega_\eta\omega_\chi + 2\omega_\nu(\omega_\eta + \omega_\chi) + \omega_\nu^2 + \kappa^2\xi^2 - \omega_{TM}^2 - \omega_{VB}^2\frac{k_R^2}{k^2}, \quad B_2 = \omega_T(2\omega_\nu + \omega_\chi),$$

$$A_3 = 2\omega_\nu\omega_\eta\omega_\chi + (\omega_\nu^2 + \kappa^2\xi^2)(\omega_\chi + \omega_\eta) - 2\omega_{TM}^2\omega_\nu - \omega_{VB}^2\frac{k_R^2}{k^2}(\omega_\nu + \omega_\eta), \quad B_3 = \omega_T\left[2\omega_\nu\omega_\chi + \omega_\nu^2 + \kappa^2\xi^2 - \omega_{VB}^2\frac{k_R^2}{k^2}\right],$$

$$A_4 = (\omega_\eta\omega_\chi - \omega_{TM}^2)(\omega_\nu^2 + \kappa^2\xi^2) - \omega_{VB}^2\frac{k_R^2}{k^2}\omega_\nu\omega_\eta, \quad B_4 = \omega_T\left[\omega_\chi(\omega_\nu^2 + \kappa^2\xi^2) - \omega_{VB}^2\frac{k_R^2}{k^2}\omega_\nu\right].$$

The analytical solution of Eq. (20) is not possible. However, the conclusion about the stability of the disturbances described by the equation (20) with complex coefficients can be made by analyzing its coefficients using the Bilgarz-Frank criterion [28].

The imaginary part of the coefficients in Eq. (20) is associated with the Nernst effect, due to which there is a thermal drift of disturbances of speed $\omega_T/k_z = -\mathcal{N}dT_0/dz$. If the Nernst coefficient is small $\mathcal{N} \to 0$, then the classical Rauss–Hurwitz stability criterion [30] applies to the dispersion equation (20) but already with real coefficients. Applying this criterion, we find a necessary and sufficient condition for stability according to axisymmetric perturbations

$$\mathrm{Ro} > -\frac{(\omega_{TM}^2 - \omega_\eta\omega_\chi)(\omega_\nu^2 + 4\Omega^2\xi^2) + \omega_{VB}^2\frac{k_R^2}{k^2}\omega_\nu\omega_\eta}{4\Omega^2\xi^2(\omega_{TM}^2 - \omega_\eta\omega_\chi)}$$

$$= -1 - \frac{\omega_\nu^2}{4\Omega^2\xi^2} - \frac{\omega_{VB}^2\frac{k_R^2}{k^2}\omega_\nu\omega_\eta}{4\Omega^2\xi^2(\omega_{TM}^2 - \omega_\eta\omega_\chi)} = \mathrm{Ro}_{\mathrm{cr}}, \tag{21}$$

where the critical Rossby number $\mathrm{Ro}_{\mathrm{cr}}$ corresponds to the profile of inhomogeneous rotation for the neutral state at the stability boundary.

It can be seen from Eq. (21) that for a non-viscous medium $\nu = 0$, the critical Rossby number coincides with the Rayleigh rotation profile $\mathrm{Ro}_{\mathrm{cr}} = -1$ [31]. If the temperature is constant $T_0 = \mathrm{const}$ ($\omega_{VB} = 0$) and $\nu \neq 0$, then we obtain the well-known stability criterion for the “non-magnetized” Couette flow [33]. In the general case, the formula (21) shows that the critical rotation profile can have both positive $\mathrm{Ro}_{\mathrm{cr}} > 0$ and negative $\mathrm{Ro}_{\mathrm{cr}} < 0$ values depending on the direction of the temperature gradient.

Let us consider the question of the development of TMI at Rossby numbers using the inequality (21) in dimensionless variables:

$$\mathrm{Ro} > -1 - \frac{a^6}{\pi^2\mathrm{Ta}} - \frac{\mathrm{k}^2\mathrm{Ra}}{\pi^2\mathrm{Ta}(f-1)} = \mathrm{Ro}_{\mathrm{cr}} \tag{22}$$

In Eq. (22) we have performed the transition from dimensional wave numbers to dimensionless: $k^2 \to a^2/L^2$, $k_R^2 \to \mathrm{k}^2/L^2$, $k_z^2 \to \pi^2/L^2$. Dimensionless parameters are the Taylor number $\mathrm{Ta} = 4\Omega^2L^4/\nu^2$, the Rayleigh number $\mathrm{Ra} = g\beta_T K_T T_0 L^4/\nu\chi$ on the characteristic scale of the inhomogeneity L, the generation parameter

$$f = \frac{\omega_{TM}^2}{\omega_\eta\omega_\chi} = \frac{\alpha^2 T_0 k_R^2}{\rho_0 c_p \mu\chi\eta k^4}\left[\left(1 + \frac{\mu\mathcal{L}}{\alpha}\right)K_T K_\alpha - K_\alpha^2\right].$$

Let us find out how the area of development of TMI in the plane $(\mathrm{k}, \mathrm{Ro})$ changes for different values of the generation parameter f at fixed numbers $\mathrm{Ta} = 2000$ and $\mathrm{Ra} = 5000$. Figures 4,a–b show the area of the development of TMI for negative values of the generation parameters: $f = -2$ and $f = -1/2$, respectively. Negative values f are possible with small temperature gradients $K_\alpha > K_T(1 + \mu\mathcal{L}/\alpha)$ or with parallel gradients $\mathbf{K}_\alpha \| \mathbf{K}_T$. From Fig. 4,b it can be seen that with an increase in the generation parameter, the instability region exists for both negative ($\mathrm{Ro} < 0$) and positive ($\mathrm{Ro} > 0$) Rossby numbers. Figure 4,c shows the case $f = 0$ that is possible with a uniform spatial distribution of the specific thermoelectric power: $\mathbf{K}_\alpha = 0$. The instability region shifts here towards an increase in positive Rossby numbers. On the contrary, when the generation parameter $f > 1$, the instability region shifts towards an increase in positive Rossby numbers (see Fig. 4,d). Obviously, such a situation is possible at large temperature gradients: $K_T > K_\alpha$.

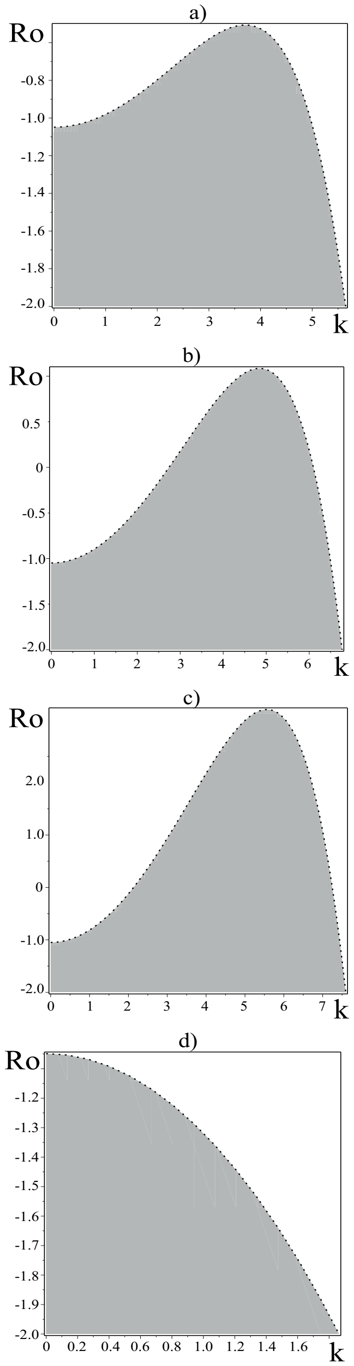

Fig. 4. The plots show the region of the TMI without an external magnetic field $\mathbf{B}_0 = 0$ for Ta = 2000, Ra = 5000 and the generation parameters: a) $f = -2$; b)$f = -1/2$; c) $f = 0$; d) $f = 2$

It can be noted that the equation (19) gives known results in some limiting cases.

1. Let us consider the case of a non-conductive medium homogeneous in temperature ($K_T = 0$) and chemical composition ($K_\alpha = 0$), without dissipation, and nonuniformly rotating medium at angular velocity $\Omega = \Omega(R)$. Then from Eq. (19) we get:

$$\gamma^2 + \xi^2\kappa^2 = 0. \qquad (23)$$

From the above, it follows that the necessary and sufficient condition for the stability of the rotating shear flow [31] is the reality of the epicyclic frequency $\kappa^2 > 0$ or the realization of the inequality $\mathrm{Ro} > -1$. For a flow with a Rossby profile $\mathrm{Ro} = -1$, the axisymmetric disturbances in this extreme case are neutrally stable $\gamma = 0$.

2. From Eq. (19), taking into account the temperature stratification ($K_T \neq 0$) and $\nu = \chi = \eta = 0$, we find

$$\gamma^2 + \xi^2\kappa^2 - \omega_{VB}^2 \frac{k_R^2}{k^2} = 0 \qquad (24)$$

It can be seen from Eq. (23) that the temperature stratification can both stabilize ($\omega_{VB}^2 < 0$) and destabilize ($\omega_{VB}^2 > 0$) the stable Couette flow ($\kappa^2 > 0$), depending on the direction of the temperature gradient.

3. If the rotation and thermomagnetic effects are absent, then we obtain the dispersion equation for a non-conducting medium from Eq. (19):

$$\gamma^2 + \gamma(\omega_\nu + \omega_\chi) + \omega_\nu\omega_\chi - \omega_{VB}^2(1 - \xi^2) = 0 \qquad (25)$$

When passing to dimensionless variables $\gamma \to \frac{\nu}{h^2}\gamma$, $k_R h \to k$, $k_z h \to \pi n$ in the equation (25), we obtain the Rayleigh equation describing free convection in a liquid layer with a thickness h. Its solution looks like [32]:

$$\gamma_n = -\frac{(1+\mathrm{Pr})}{2\,\mathrm{Pr}}(k^2 + \pi^2 n^2) \qquad (26)$$

$$\pm\sqrt{\left(\frac{(\mathrm{Pr}-1)}{2\,\mathrm{Pr}}\right)^2 (k^2+\pi^2 n^2)^2 + \frac{\mathrm{Ra}k^2}{\mathrm{Pr}(k^2+\pi^2 n^2)}},$$

where n is an integer characterizing the vertical scale. The magnitude of the instability growth rate γ_n depends on the dimensionless Rayleigh $\mathrm{Ra} = g\beta A L^4/\nu\chi$, Prandtl $\mathrm{Pr} = \nu/\chi$ numbers, and the wavenumber $K = \sqrt{k^2 + \pi^2 n^2}$. The condition for the stability of small perturbations is the positiveness of the radical expression, which corresponds to the Rayleigh numbers $\mathrm{Ra} > 0$.

Let us analyze the dispersion equation (19) for the case of very small oscillations of a fluid element in a stratified medium $\omega_{VB} \to 0$, which is possible if the change in the density of the medium does not depend on temperature (the Boussinesq approximation is violated), or with a small Archimedean force $\mathbf{g} \to 0$, for example, under conditions of weightlessness. As a result, the equation (19) splits into two equations. The first equation generalizes the result (23) taking into account the viscosity of the medium:

$$\gamma + \omega_\nu = \pm\sqrt{-\xi^2\kappa^2}.$$

We see that the perturbations are damping at $\kappa^2 > 0$. Therefore, the perturbations decay at the decrement

$\gamma = -\nu k^2$ for a Rayleigh profile Ro $= -1$, taking into account the hydrodynamic viscosity. The perturbations can be unstable for negative Rossby numbers Ro < -1, for example. Then, for the profile of nonuniform rotation Ro $= -2$, the criterion for the development of instability has the form:

$$\frac{2\Omega k_z}{\nu k^3} > 1.$$

The second dispersion equation contains thermomagnetic effects:

$$\gamma^2 + \gamma\left(\omega_\eta + \omega_\chi + i\omega_{\mathcal{N}}\frac{k_z K_T}{k^2}\right) + \omega_\eta\omega_\chi - \omega_{TM}^2 + i\omega_{\mathcal{N}}\omega_\chi\frac{k_z K_T}{k^2} = 0 \tag{27}$$

We represent the solutions of the quadratic equation (27) in the form where the expressions for the growth rate $\Gamma_{1,2}$ and frequency $\omega_{01,02}$ of TM perturbations are

$$\Gamma_1 \approx -\frac{\alpha^2 T_0 k_R^2}{\rho_0 c_p \mu\chi k^2(1+\widetilde{\mathrm{Pm}})}\left[\left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha - K_\alpha^2\right] - \frac{(\chi+\widetilde{\mathrm{Pm}}(\eta+\chi))k^2}{1+\widetilde{\mathrm{Pm}}}, \quad \omega_{01} \approx -\frac{\mathcal{N}\widetilde{\mathrm{Pm}}}{1+\widetilde{\mathrm{Pm}}}k_z\frac{dT_0}{dz}, \tag{28}$$

$$\Gamma_2 \approx \frac{\alpha^2 T_0 k_R^2}{\rho_0 c_p \mu\chi k^2(1+\widetilde{\mathrm{Pm}})}\left[\left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha - K_\alpha^2\right] - \frac{\eta k^2}{1+\widetilde{\mathrm{Pm}}}, \quad \omega_{02} \approx -\frac{\mathcal{N}k_z\frac{dT_0}{dz}}{1+\widetilde{\mathrm{Pm}}}, \tag{29}$$

respectively. The presence of an oscillating frequency of oscillations in Eqs. (28)–(29) shows that small perturbations can move in a conducting medium. As can be seen from Eqs. (28)–(29), the generation of a magnetic field (GMF) can be caused only by the inhomogeneity of the Seebeck coefficient $K_\alpha \neq 0$ (see (28) at $K_T = 0$). The Righi–Leduc effect makes a significant contribution to the TMI if the condition is met: $\mu\mathcal{L}/\alpha \gg 1$. For the parameters of the Earth's core, this condition gives a numerical estimate: $\mathcal{L} \gg 8$ A·m/s·K. In practice, the Righi–Leduc coefficient S [33] is used, which is connected with $\mathcal{L}$ the as $\mathcal{L} = \kappa|S|$. If we take the thermal conductivity coefficient, for example, for iron in a molten state $\kappa = 39$W/m· K [33], then the value should be $|S| \gg 0.2$ m^2/V·s.

Let us define the criteria for the development of TMI, and as a consequence of the GMF, in two cases: a) a low-conductive medium, when the magneto-thermal Prandtl number is large $\widetilde{\mathrm{Pm}} = \eta/\chi \gg 1$, and b) a highly conductive medium when $\widetilde{\mathrm{Pm}} = \eta/\chi \ll 1$ is small. The case a) is very well fulfilled for the parameters of the Earth's core. Taking the density of molten iron $\rho_0 \approx 7\cdot 10^3$kg/m^3, specific heat $c_p \approx 835$ J/kg·K [25], we find the value of the thermal diffusivity $\chi = \kappa/\rho_0 c_p \approx 6.7\cdot 10^{-6}$ m^2/s, which turns out to be much smaller than the coefficient of magnetic viscosity $\eta = 1/\mu\sigma \approx 2.65$ m^2/s: $\eta \gg \chi$. Then from the expressions (28)–(29) we have:

$$\Gamma_1^{(l)} \approx \frac{\alpha^2 T_0 k_R^2}{\rho_0 c_p \mu\eta k^2}\left[K_\alpha^2 - \left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha\right] - \eta k^2,$$

$$\omega_{01} \approx -\mathcal{N}k_z\frac{dT_0}{dz}, \tag{30}$$

$$\Gamma_2^{(l)} \approx \frac{\alpha^2 T_0 k_R^2}{\rho_0 c_p \mu\eta k^2}\left[\left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha - K_\alpha^2\right] - \chi k^2,$$

$$\omega_{02} \approx -\frac{\mathcal{N}k_z}{\widetilde{\mathrm{Pm}}}\frac{dT_0}{dz} \tag{31}$$

If the temperature gradient is small $K_T \ll K_\alpha$ or equal to zero $K_T = 0$, then the second root of the quadratic equation (27) is negative $\Gamma_2^{(l)} < 0$, i. e., the TM perturbations disappear. Therefore, consider the criteria of feasibility $\Gamma_1^{(l)} > 0$:

$$\frac{\alpha^2 T_0}{\rho_0 c_p \mu\eta^2} > \frac{k^4}{k_R^2 K_\alpha^2} \cong \left(\frac{L_\alpha}{\lambda}\right)^2 \tag{32}$$

For the parameters of the Earth's core the inequality (32) does not hold since the left-hand side (32) turns out to be much smaller than the right-hand side for a weakly inhomogeneous medium $L_\alpha \gg \lambda$:

$$10^{-9} > (L_\alpha/\lambda)^2.$$

If the gradients of temperature and thermo-electromotive force coefficient are approximately equal $K_T \approx K_\alpha$ or if $K_T \gg K_\alpha$ and $\mu\mathcal{L}/\alpha \gg 1$ are satisfied, then the GMF can arise due to the Righi–Leduc effect (see the formula (31))

$$\frac{\alpha T_0}{\eta}|S| > \left(\frac{L_T L_\alpha}{\lambda^2}\right) \tag{33}$$

Substituting the values of the parameters (α, η, T_0) for the liquid Earth's core into (33), we obtain an estimate for the coefficient S when the GMF is possible:

$|S| > 2.65 \cdot 10^2 (L_T L_\alpha/\lambda^2)$. It should be noted that for a weakly inhomogeneous medium $L_\alpha, L_T \gg \lambda$, the generation of a magnetic field by the TMI is possible only at very high values of S. Thus, from the results (32)-(33) it follows that the GMF is not efficient in a weakly inhomogeneous weakly conductive fluid with the TMI. The damping of the TM perturbations is associated with the thermal and electrical conductivity of the medium. Similar conclusions hold for parallel gradients of temperature and specific thermoelectric power: $\mathbf{K}_T \| \mathbf{K}_\alpha$.

Let consider case b) of a highly conductive medium $\widetilde{\mathrm{Pm}} = \eta/\chi \ll 1$. Then from Eqs. (28)–(29) we find:

$$\Gamma_1^{(h)} \approx \frac{\alpha^2 T_0 k_R^2}{\rho_0 c_p \mu \chi k^2}\left[K_\alpha^2 - \left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha\right] - \chi k^2,$$

$$\omega_{01} \approx -\mathcal{N}\widetilde{\mathrm{Pm}} k_z \frac{dT_0}{dz}, \tag{34}$$

$$\Gamma_2^{(h)} \approx \frac{\alpha^2 T_0 k_R^2}{\rho_0 c_p \mu \chi k^2}\left[\left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha - K_\alpha^2\right] - \eta k^2,$$

$$\omega_{02} \approx -\mathcal{N} k_z \frac{dT_0}{dz}. \tag{35}$$

If the temperature gradient is small $K_T \ll K_\alpha$ or $K_T = 0$, then the second root $\Gamma_2^{(h)} < 0$ is negative (a damped solution). Therefore, we will be interested in the instability criterion $\Gamma_1^{(h)} > 0$:

$$\frac{\alpha^2 T_0}{\rho_0 c_p \mu \chi^2} > \frac{k^4}{k_R^2 K_\alpha^2} \cong \left(\frac{L_\alpha}{\lambda}\right)^2 \tag{36}$$

Since the value of the thermal diffusivity χ is large, the fulfillment of the criterion (36) is unlikely for a weakly inhomogeneous medium $L_\alpha \gg \lambda$. If we assume $K_T \gg K_\alpha$ or $K_T \approx K_\alpha$, then we have the damped solution $\Gamma_1^{(h)} < 0$. The instability criterion takes the form ($\Gamma_2^{(h)} > 0$):

$$|S| > \frac{\eta}{\alpha T_0}\left(\frac{L_T L_\alpha}{\lambda^2}\right) \tag{37}$$

As can be seen from Eq. (37), magnetic fields are most efficiently generated in the case of the high conductivity of the medium due to the development of TMI.

V. ANALYSIS OF THE DISPERSION EQUATION FOR THE CASE $\mathbf{B}_0 \neq 0$

Let us begin the analysis of the dispersion equation (18) for the case when there is an external magnetic field $\mathbf{B}_0 \neq 0$. To describe the physical mechanism of the TM effects, we consider disturbances propagating in the axial ($\mathbf{k} = \mathbf{e}_z k_z$) direction and radial ($\mathbf{k} = \mathbf{e}_R k_R$) direction in a nonuniformly rotating electrically conducting medium with gradients of the temperature and thermo-electromotive force coefficient.

A. The instability for the waves propagating in the axial direction ($\mathbf{k} = \mathbf{e}_z k_z$)

If the disturbances propagate only in the axial direction ($\mathbf{k} = \mathbf{e}_z k_z$), then the equation (18) splits into two dispersion equations of the following form

$$(\gamma + \omega_\nu)(\gamma + i\omega_T + \omega_\eta) + \omega_A^2 = 0, \tag{38}$$

$$\begin{aligned} &[(\gamma + \omega_\nu)((\gamma + i\omega_T + \omega_\eta)^2 + \omega_{\mathcal{R}}(\omega_{\mathcal{R}} - 2\Omega\mathrm{Ro})) \\ +\ &\omega_A^2(\gamma + i\omega_T + \omega_\eta)]^2 + [2\Omega(1+\mathrm{Ro})((\gamma + i\omega_T + \omega_\eta)^2 \\ +\ &\omega_{\mathcal{R}}(\omega_{\mathcal{R}} - 2\Omega\mathrm{Ro})) + \omega_A^2(2\Omega\mathrm{Ro} - \omega_{\mathcal{R}})] \\ \times\ &[2\Omega((\gamma + i\omega_T + \omega_\eta)^2 + \omega_{\mathcal{R}}(\omega_{\mathcal{R}} - 2\Omega\mathrm{Ro})) - \omega_A^2\omega_{\mathcal{R}}] \\ =\ &0, \end{aligned} \tag{39}$$

where $\omega_A = \sqrt{k_z^2 B_0^2/\rho_0\mu}$ is the Alfven frequency.
The dispersion equation (38) describes the damping of frequency-oscillating Alfven waves in a plasma with viscous and ohmic dissipation. In this equation, there is no influence of rotation on the growth rate of perturbations, so we will start analyzing the dispersion equation (39). In this equation, some TM effects associated with the inhomogeneity of the thermo-electromotive force coefficient, the effect of "magnetization" of thermal conductivity (the Righi–Leduc effect) and the action of the Archimedean force dropped out, since we consider wave disturbances only in the axial direction. In Eq. (39), the Nernst effect leads to oscillations of disturbances with frequency ω_T. Therefore, we will focus on studying the influence of the Hall effect on the development of a standard MRI. At first, we present some well-known results obtained in various limiting cases.

Let the conditions $\omega_{\mathcal{R}} \gg 2\Omega\mathrm{Ro}$ and $\omega_T = 0$ be satisfied in Eq. (39). The first condition is satisfied for the perturbation wavelength $\lambda \ll 2\pi\sqrt{\mathcal{R}B_0/2\Omega\mathrm{Ro}\mu}$, the second condition is satisfied at $T_0 = \mathrm{const}$. Under these conditions, for the convenience of analyzing the asymptotic stability of disturbances, we write Eq. (39) in the form of a polynomial of the sixth degree:

$$\mathscr{P}(\gamma) \equiv A_0\gamma^6 + A_1\gamma^5 + A_2\gamma^4 + A_3\gamma^3 + A_4\gamma^2 + A_5\gamma + A_6 = 0, \tag{40}$$

with real coefficients

$$A_0 = 1,$$

$$A_1 = 2(2\omega_\eta + \omega_\nu),\ A_2 = (2\omega_\eta + \omega_\nu)^2 + 2(\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2) + 4\Omega^2(1 + \mathrm{Ro}),$$

$$A_3 = 2(2\omega_\eta + \omega_\nu)(\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2) + 2(\omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + \omega_\eta\omega_A^2) + 16\Omega^2(1 + \mathrm{Ro})\omega_\eta,$$

$$A_4 = 2(2\omega_\eta + \omega_\nu)(\omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + \omega_\eta\omega_A^2) + (\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2)^2$$

$$+8\Omega^2(1 + \mathrm{Ro})(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + 16\Omega^2(1 + \mathrm{Ro})\omega_\eta^2 - 2\Omega(2 + \mathrm{Ro})\omega_A^2\omega_{\mathcal{R}},$$

$$A_5 = 2(\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2)(\omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + \omega_\eta\omega_A^2) + 16\Omega^2(1 + \mathrm{Ro})\omega_\eta(\omega_\eta^2 + \omega_{\mathcal{R}}^2) - 4\Omega(2 + \mathrm{Ro})\omega_A^2\omega_{\mathcal{R}}\omega_\eta,$$

$$A_6 = (\omega_\eta\omega_A^2 + \omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2))^2 + 4\Omega^2(1 + \mathrm{Ro})(\omega_\eta^2 + \omega_{\mathcal{R}}^2)^2 - 2\Omega(2 + \mathrm{Ro})(\omega_{\mathcal{R}}^2 + \omega_\eta^2)\omega_A^2\omega_{\mathcal{R}} + \omega_A^4\omega_{\mathcal{R}}^2.$$

The analytical solution of Eq. (40) in the general case is not possible. However, the conclusion about the stability of the disturbances described by Eq. (40) with real coefficients can be made without solving it, but only by analyzing its coefficients using the Routh-Hurwitz or Lienard–Shipard criteria [31]. For the Lienard–Shipard criterion, the number of determinant inequalities is approximately half that in the Routh–Hurwitz conditions, so it is advisable to use it. The Lienard-Shipard criterion for the asymptotic stability of perturbations described by the algebraic equation (40) is as follows. If a polynomial $\mathscr{P}(\gamma)$ has all roots with negative real parts, it is necessary and sufficient that a) all the coefficients of the polynomial $\mathscr{P}(\gamma)$ be positive: $A_j > 0$, $j = 0\dots 6$; b) there should be inequalities for the Hurwitz determinants: $\Delta_{j-1} > 0$, $\Delta_{j-3} > 0\ \dots$, where Δ_m — denotes the Hurwitz determinant m-order:

$$\Delta_m = \begin{vmatrix} A_1 & A_3 & A_5 & \cdot\ \cdot \\ A_0 & A_2 & A_4 & \cdot\ \cdot \\ 0 & A_1 & A_3 & \cdot\ \cdot \\ 0 & A_0 & A_2 & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ A_m \end{vmatrix}.$$

Using the Lienard-Shipard algorithm, we obtain the necessary and sufficient stability conditions:

$$A_j > 0, \quad j = 0\dots 6, \quad \Delta_3 > 0, \quad \Delta_5 > 0. \qquad (41)$$

From conditions (41) using the explicit form of the coefficients A_j, we find the following inequalities:

1. $A_0 = 1 > 0$, $A_1 = 2(2\omega_\eta + \omega_\nu) > 0$. These inequalities are fulfilled automatically.
2. $A_2 = (2\omega_\eta + \omega_\nu)^2 + 2(\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2) + 4\Omega^2(1 + \mathrm{Ro}) > 0$.
   It can be seen that dissipative processes naturally lead to the stabilization of the stability of perturbations. The stabilizing factors are also a uniform magnetic field, the Hall effect, and nonuniform rotation if the profile of the angular velocity of rotation corresponds to positive Rossby numbers ($\mathrm{Ro} > 0$).
3. $A_3 = 2(2\omega_\eta + \omega_\nu)(\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2) + 2(\omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + \omega_\eta\omega_A^2) + 16\Omega^2(1 + \mathrm{Ro})\omega_\eta > 0$.
   We see that in addition to dissipative processes the stabilizing factors are also a homogeneous magnetic field, the Hall effect, and inhomogeneous rotation if the profile of the angular velocity of rotation corresponds to positive Rossby numbers ($\mathrm{Ro} > 0$).
4. $A_4 > 0 \Rightarrow 2(2\omega_\eta + \omega_\nu)(\omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + \omega_\eta\omega_A^2) + (\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2)^2 + 8\Omega^2(1 + \mathrm{Ro})(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + 16\Omega^2(1 + \mathrm{Ro})\omega_\eta^2 > 2\Omega(2 + \mathrm{Ro})\omega_A^2\omega_{\mathcal{R}}$,
   $A_5 > 0 \Rightarrow 2(\omega_\eta^2 + \omega_{\mathcal{R}}^2 + 2\omega_\nu\omega_\eta + \omega_A^2)(\omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2) + \omega_\eta\omega_A^2) + 16\Omega^2(1 + \mathrm{Ro})\omega_\eta(\omega_\eta^2 + \omega_{\mathcal{R}}^2) > 4\Omega(2 + \mathrm{Ro})\omega_A^2\omega_{\mathcal{R}}\omega_\eta$.
   We also see that the Hall effect and external magnetic field can have a destabilizing effect on positive Rossby numbers.
5. The criterion of asymptotic stability ($A_6 > 0$) shows that axisymmetric disturbances are stable for profiles of nonuniform rotation $\mathrm{Ro} > \mathrm{Ro}_{\mathrm{cr}}$:

$$\mathrm{Ro} > -\frac{1}{4\Omega^2(\omega_{\mathcal{R}}^2 + \omega_\eta^2)^2 - 2\Omega(\omega_{\mathcal{R}}^2 + \omega_\eta^2)\omega_A^2\omega_{\mathcal{R}}}$$
$$\times[(\omega_\eta\omega_A^2 + \omega_\nu(\omega_\eta^2 + \omega_{\mathcal{R}}^2))^2 + 4\Omega^2(\omega_\eta^2 + \omega_{\mathcal{R}}^2)^2$$
$$+\omega_A^4\omega_{\mathcal{R}}^2 - 4\Omega\omega_A^2\omega_{\mathcal{R}}(\omega_\eta^2 + \omega_{\mathcal{R}}^2)] = \mathrm{Ro}_{\mathrm{cr}}, \qquad (42)$$

   where $\mathrm{Ro}_{\mathrm{cr}}$ is the critical Rossby number.

We now turn to stability conditions consisting of inequalities with the Hurwitz determinants (41). For the determinant Δ_3 we have:

$$\Delta_3 = \begin{vmatrix} A_1 & A_3 & A_5 \\ A_0 & A_2 & A_4 \\ 0 & A_1 & A_3 \end{vmatrix} = A_1A_2A_3 + A_0A_1A_5 - A_1^2A_4 - A_0A_3^2$$

and the stability criterion has the form:

$$A_1A_2A_3 + A_0A_1A_5 > A_1^2A_4 + A_0A_3^2. \qquad (43)$$

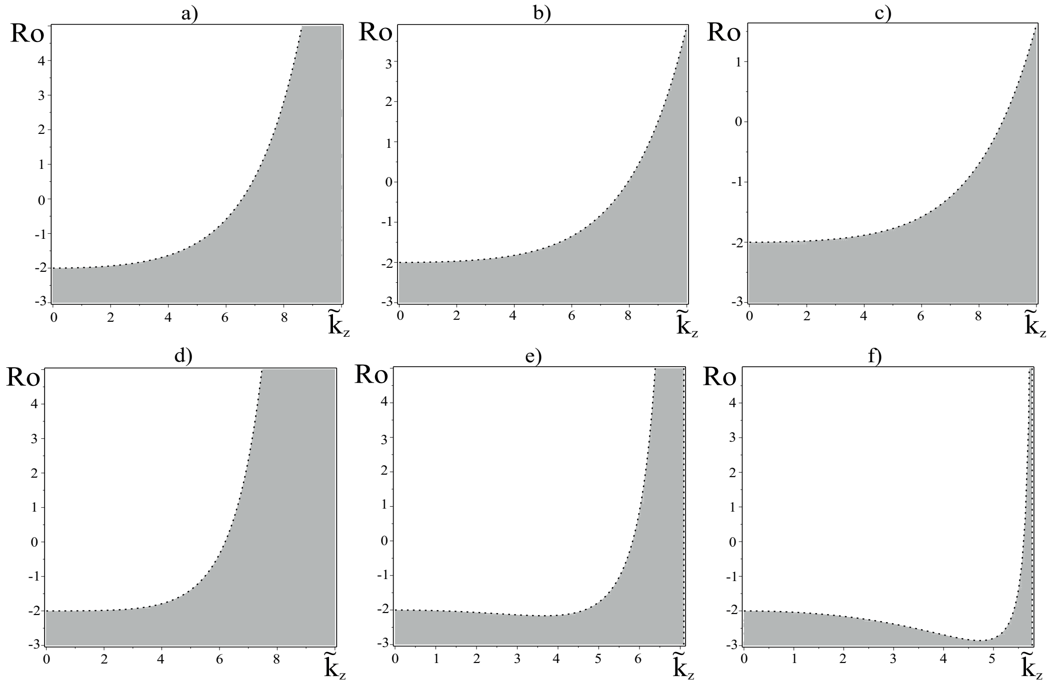

Fig. 5. The area where the Hall MRI appears for various Hall parameters is shown in gray: a) $r = 10$; b)$r = 20$; c) $r = 30$ and Hall numbers: d) $R_{\mathrm{H}} = 2$; e) $R_{\mathrm{H}} = 3$; f) $R_{\mathrm{H}} = 3.7$. Graphs a)–f) are plotted at fixed parameters $\mathrm{Ta} = 1000, \mathrm{Q} = 100, \mathrm{Pm} = 1$

For the second Hurwitz determinant from condition (41)

$$\Delta_5 = \begin{vmatrix} A_1 & A_3 & A_5 & 0 & 0 \\ A_0 & A_2 & A_4 & A_6 & 0 \\ 0 & A_1 & A_3 & A_5 & 0 \\ 0 & A_0 & A_2 & A_4 & 0 \\ 0 & 0 & A_1 & A_3 & A_5 \end{vmatrix} = A_1A_2(A_3A_4A_5 - A_2A_5^2)$$

$$-A_1A_4(A_1A_4A_5 - A_0A_5^2) + A_1A_6(A_1A_2A_5 - A_0A_3A_5)$$

$$-A_3A_0(A_3A_4A_5 - A_2A_5^2) + A_5A_0(A_1A_4A_5 - A_0A_5^2),$$

we obtain the following stability criterion

$$A_4A_5(A_1A_2A_3 + A_0A_1A_5 - A_1^2A_4 - A_0A_3^2)$$
$$+A_1^2A_2A_5A_6 + A_0A_5^2(A_2A_3 + A_1A_4) \tag{44}$$
$$> A_0A_1A_3A_5A_6 + A_5^2(A_1A_2^2 + A_0^2A_5).$$

It follows from expressions (43)–(44) that the Hall effect in an external magnetic field can have both a stabilizing and a destabilizing effect depending on the profile of nonuniform rotation, i. e., the sign of the Rossby number.

Next, we discuss the influence of the Hall effect on the area of development of the standard MRI. For convenience, we write the expression for $\mathrm{Ro_{cr}}$ in dimensionless variables

$$\mathrm{Ro_{cr}} = -\frac{1}{\mathrm{Ta}(\widetilde{k}_z^2(1+R_{\mathrm{H}}^2)^2 - (1+R_{\mathrm{H}}^2)\mathrm{QPm}(r/2))}$$
$$\times[\widetilde{k}_z^2(\widetilde{k}_z^2(1+R_{\mathrm{H}}^2) + \mathrm{Q})^2 + \widetilde{k}_z^2\mathrm{Ta}(1+R_{\mathrm{H}}^2) + \widetilde{k}_z^2\mathrm{Q}^2R_{\mathrm{H}}^2$$
$$-\mathrm{TaQPm}(1+R_{\mathrm{H}}^2)r] \tag{45}$$

where $\widetilde{k}_z = k_zL$ is the dimensionless axial wave number, $\mathrm{Q} = B_0^2L^2/\mu\rho_0\nu\eta$ is the Chandrasekhar number, $\mathrm{Pm} = \nu/\eta$ is the magnetic Prandtl number, $R_{\mathrm{H}} = \mathcal{R}B_0/\mu\eta$ is the Hall number, $r = \omega_{\mathcal{R}}/\Omega$ is the Hall parameter.

Figures 5,a–c it show in gray the area of development of the standard MRI taking into account the Hall effect for fixed parameters $R_{\mathrm{H}} = 1, \mathrm{Ta} = 1000, \mathrm{Q} = 100, \mathrm{Pm} = 1$. From Fig. 5,a–c can be seen that with an increase in the Hall parameter, the region of instability decreases towards negative Rossby numbers. Then, by fixing the Hall parameter to $r = 10$, we change the Hall number R_{H} at constant values $\mathrm{Ta} = 1000, \mathrm{Q} = 100, \mathrm{Pm} = 1$. It follows from the graphs in Fig. 5,a and Fig. 5,d that with an increase in the number R_{H} from 1 to 2, the region of instability increases. However, with a further increase in the number R_{H}, the region of instability decreases (see Fig. 5,e–f). Consequently, the Hall effect can both stabilize and destabilize the growth of disturbances depending on the profile of inhomogeneous rotation (Rossby number).

B. The instability for the waves propagating in the radial direction ($\mathbf{k} = \mathbf{e}_Rk_R$)

Let us consider a situation when disturbances propagate only in the radial direction ($\mathbf{k} = \mathbf{e}_Rk_R$). Under these conditions, the standard MRI does not arise, since $\mathbf{k}\cdot\mathbf{B}_0 = 0$, and the general dispersion equation (18) is reduced to the following form:

$$(\gamma+\omega_\nu)[(\gamma+\omega_\eta)(\gamma+\omega_\chi)-\zeta\omega_{AR}^2-\omega_{TM}^2]-\omega_{VB}^2(\gamma+\omega_\eta) = 0, \tag{46}$$

(85) where $\zeta = \mathcal{N}\omega_{\mathcal{N}}/c_p$ is the dimensionless parameter due to the influence of the Nernst effect, $\omega_{AR} = k_RB_0/\sqrt{\mu\rho_0}$ is the radial Alfven frequency. At first, let

us consider the situation when $\omega_{VB} \to 0$, i.e., the change in the density of the medium does not depend on temperature (the Boussinesq approximation is violated) or at a small Archimedean force $\mathbf{g} \to 0$, for example, under conditions of weightlessness. As a result, Eq. (46) splits into two equations:

$$\gamma + \omega_\nu = 0, \quad (\gamma + \omega_\eta)(\gamma + \omega_\chi) - \zeta\omega_{AR}^2 - \omega_{TM}^2 = 0.$$

The first equation shows that the perturbations decay at the hydrodynamic viscosity $\gamma = -\nu k_R^2$. Solutions of the second quadratic equation have positive ($\gamma_+ > 0$) and negative ($\gamma_- < 0$) roots. Let us write down a positive root that contributes to the development of instability:

$$\gamma_+ = \frac{\zeta\omega_{AR}^2 + \omega_{TM}^2 - \omega_\eta\omega_\chi}{\omega_\eta + \omega_\chi} \tag{47}$$

Let us define the criteria for the development of instability in two limiting cases: a) a low-conductivity medium, when the Prandtl magneto-thermal number is large $\widetilde{\mathrm{Pm}} = \eta/\chi \gg 1$, and b) a high-conductivity medium when $\widetilde{\mathrm{Pm}} = \eta/\chi \ll 1$ it is small. For case a), the instability increment has the form:

$$\begin{aligned}\gamma_+^{(a)} &= \frac{\alpha^2 T_0}{\rho_0 c_p \mu\eta}\left[\left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha - K_\alpha^2\right] \\ &+ \left(\frac{\mathcal{N}^2 T_0 B_0^2}{\rho_0 c_p \mu\eta} - \chi\right)k_R^2.\end{aligned} \tag{48}$$

Hence, we see that the thermal conductivity of the medium leads to the stabilization of the growth of disturbances. The instability is possible due to the Nernst effect in a homogeneous medium $\mathbf{k} = \mathbf{e}_R k_R$ if the following inequality is satisfied:

$$\frac{\mathcal{N}^2 T_0 B_0^2}{\rho_0 c_p \mu\eta} > \chi.$$

Let us consider the possible variants of the development of TMI in an inhomogeneous medium at $\mathbf{k} = \mathbf{e}_R k_R$ and ($K_T \neq 0, K_\alpha \neq 0$).

1. If in an inhomogeneous medium the temperature gradient is small $K_T \ll K_\alpha$ or equals to zero $K_T = 0$, then instability arises under the condition

$$\frac{\mathcal{N}^2 T_0 B_0^2}{\rho_0 c_p \mu\eta} > \frac{\alpha^2 T_0}{\rho_0 c_p \mu\eta}K_\alpha^2 + \chi k_R^2.$$

Let us compare the terms on the right side of the inequality with each other, substituting the numerical values of the physical quantities ($\rho_0, c_p, \mu, \eta, \chi$) for the parameters of the earth's core at $T_0 = 2000$K and $\alpha \approx 10^{-3}$V/K:

$$\frac{\alpha^2 T_0}{\rho_0 c_p \mu\eta\chi}\frac{K_\alpha^2}{k_R^2} \approx \frac{16}{(L_\alpha/\lambda)^2}.$$

Thus, for a weakly inhomogeneous medium ($L_\alpha \gg \lambda$) is quite possible $L_\alpha/\lambda \cong 4$. Then the terms on the right-hand side of the inequality are of the same order. In this case, the condition for the development of instability takes the form of a simple inequality:

$$\frac{\mathcal{N}^2 B_0}{\alpha^2} > \frac{1}{L_\alpha^2}.$$

2. If the gradients of temperature and thermo-electromotive force coefficient are approximately equal $K_T \approx K_\alpha$, then the instability arises when the inequality is fulfilled:

$$\frac{\mathcal{N}^2 T_0 B_0^2}{\rho_0 c_p \mu\eta}k_R^2 + \frac{\alpha T_0 \mathcal{L}}{\rho_0 c_p \eta}K_\alpha^2 > \chi k_R^2.$$

It can be seen that the instability arises due to the Nernst and Righi–Leduc effects. If the Righi–Leduc effect becomes prevailing, then the criterion for the occurrence of instability is the condition that coincides with the expression (34) for $L_\alpha = L_T$. As noted above, TMI for the parameters of the Earth's core is possible only at very large values of the coefficient S.

3. Finally, if $K_T \gg K_\alpha$, then instability is possible when the inequality is satisfied:

$$\frac{\mathcal{N}^2 T_0 B_0^2}{\rho_0 c_p \mu\eta}k_R^2 + \frac{\alpha^2 T_0}{\rho_0 c_p \mu\eta}\left[\left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha\right] > \chi k_R^2.$$

For a weak Nernst effect and $\mu\mathcal{L}/\alpha \gg 1$, the criterion of the onset instability also coincides with the expression (34).

It follows from the results obtained above that in a weakly inhomogeneous low-conductive electrically conductive medium ($k_R^2 \gg K_T K_\alpha, K_\alpha^2$) in an axial magnetic field, TMI arises mainly due to the Nernst effect. For the case of a highly conductive medium (b), the growth rate coefficient takes the form:

$$\begin{aligned}\gamma_+^{(b)} &= \frac{\alpha^2 T_0}{\rho_0 c_p \mu\chi}\left[\left(\frac{\mu\mathcal{L}}{\alpha}+1\right)K_T K_\alpha - K_\alpha^2\right] \\ &+ \left(\frac{\mathcal{N}^2 T_0 B_0^2}{\rho_0 c_p \mu\chi} - \eta\right)k_R^2.\end{aligned} \tag{49}$$

We see that the stabilization of the growth of disturbances is caused by the electrical conductivity of the medium. As in the previous case a) , the instability is possible in a homogeneous medium ($K_T = K_\alpha = 0$) due to the Nernst effect if the following inequality is satisfied:

$$\frac{\mathcal{N}^2 T_0 B_0^2}{\rho_0 c_p \mu\chi} > \eta.$$

We now turn to the study of the stability of perturbations in Eq. (46) at $\omega_{VB} \neq 0$. Then, for an ideal ($\omega_\nu = \omega_\eta = 0$) and heat-conducting ($\omega_\chi \neq 0$) fluid, the positive growth rate of the TMI takes the form:

$$\gamma_+ = \frac{1}{\omega_\chi}(\zeta\omega_{AR}^2 + \omega_{TM}^2 + \omega_{VB}^2). \tag{50}$$

The first term in the instability increment (50) is associated with the Nernst effect in an external magnetic field; the second term is due to the influence of the Righi–Leduc effect and the inhomogeneity of the equilibrium temperature and specific thermopower; the third term describes the effect of the Archimedean force or the buoyancy force in a temperature-stratified medium. To study the stability of disturbances in the general case, we represent Eq. (46) in the form of a polynomial of the third degree in γ

$$a_0\gamma^3 + a_1\gamma^2 + a_2\gamma + a_3 = 0. \tag{51}$$

In this equation, all coefficients $(a_i, i = 0, 1, 2, 3)$ are real, so we can apply to Eq. (51) the criterion of asymptotic stability of Lienard–Shipard. According to this criterion

1) the coefficients $(a_i, i = 0, 1, 2, 3)$ of the polynomial must be positive

$$a_0 = 1 > 0,\ a_1 = \omega_\nu + \omega_\eta + \omega_\chi > 0,$$

$$a_2 = \omega_\nu\omega_\eta + \omega_\nu\omega_\chi + \omega_\eta\omega_\chi - (\zeta\omega_{AR}^2 + \omega_{TM}^2 + \omega_{VB}^2) > 0,$$

$$a_3 = \omega_\nu\omega_\eta\omega_\chi - \omega_\nu(\zeta\omega_{AR}^2 + \omega_{TM}^2) - \omega_\eta\omega_{VB}^2 > 0;$$

2) the condition for the Hurwitz determinant $\Delta_2 > 0$ holds:

$$\omega_\eta^2(\omega_\nu + \omega_\chi) + \omega_\chi^2(\omega_\nu + \omega_\eta) + \omega_\nu^2(\omega_\chi + \omega_\eta) + 2\omega_\nu\omega_\eta\omega_\chi$$

$$> (\omega_\chi + \omega_\eta)(\zeta\omega_{AR}^2 + \omega_{TM}^2 + \omega_{VB}^2) + \omega_\nu\omega_{VB}^2.$$

The first two conditions in 1) are fulfilled automatically. The third and fourth stability conditions in 1) show that the Nernst effect in an external magnetic field $\mathbf{B}_0$ $(\zeta\omega_{AR})$, the Righi–Leduc effect, and the inhomogeneity of the equilibrium temperature and thermo-electromotive force coefficient, as well as the effect of the Archimedean force in a temperature-stratified medium lead to destabilization of axisymmetric disturbances in a dissipative medium. A similar conclusion follows from the stability criterion 2).

VI. CONCLUSION

In this work, we investigated the mechanism of magnetic field generation in an inhomogeneously rotating electrically conductive fluid by TMI, which arises at collinear temperature gradients ∇T_0 and thermo-electromotive force coefficient $\nabla\alpha$: $[\nabla\alpha \times \nabla T_0] = 0$. The gradient of the thermo-electromotive force coefficient $\nabla\alpha$ is caused by the inhomogeneity of the chemical composition of the electrically conductive fluid. The dispersion equation in the local geometrical optics approximation is obtained for small TM perturbations in a nonuniformly rotating electrically conducting fluid in an external axial magnetic field $\mathbf{B}_0 \| OZ$. In the absence of in external magnetic field $\mathbf{B}_0 = 0$, criteria for the stability of TM disturbances in a weakly inhomogeneous unbounded medium are obtained depending on the profile of nonuniform rotation Ro. In the case of "smooth" gradients ∇T_0 and $\nabla\alpha$ (a weakly inhomogeneous medium), the growth rates of TMI were obtained and it was found that for a low-conductive medium ($\mathrm{Pm} = \eta/\chi \gg 1$), the generation of a magnetic field is ineffective because of losses associated with the thermal conductivity and viscosity of the fluid. It is shown that in a weakly inhomogeneous medium the generation of magnetic fields is effective for a highly conductive ($\mathrm{Pm} = \eta/\chi \ll 1$) fluid, if the effect of "magnetization" of thermal conductivity (the Righi–Leduc effect) is prevailing in comparison with the convection of heat caused by the gradients ∇T_0 and $\nabla\alpha$.

In the presence of an external magnetic field $\mathbf{B}_0 \neq 0$, the dispersion equation is investigated taking into account thermogalvanomagnetic effects in two limiting cases, i. e., when perturbations propagate in the axial ($\mathbf{k} = \mathbf{e}_z k_z$) and radial ($\mathbf{k} = \mathbf{e}_R k_R$) directions. For disturbances propagating in the axial direction, it was found that the Hall effect can both stabilize and destabilize the growth of disturbances depending on the profile of nonuniform rotation Ro (Rossby number). For perturbations propagating in the radial direction, it was found that the Nernst effect in an external magnetic field $\mathbf{B}_0$ $(\zeta\omega_{AR})$, the Righi–Leduc effect and the inhomogeneity of the equilibrium temperature and thermo-electromotive force coefficient (ω_{TM}), as well as the effect of the Archimedean force in a medium stratified in temperature (ω_{VB}) lead to destabilization of axisymmetric perturbations or to generating a magnetic field. In this case, the rotation does not affect the development of TMI, and the usual (or standard) MRI does not arise.

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ТЕРМОМАГНІТНІ НЕСТІЙКОСТІ В ЕЛЕКТРОПРОВІДНІЙ РІДИНІ, ЩО НЕОДНОРІДНО ОБЕРТАЄТЬСЯ

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Досліджено стійкість малих осесиметричних збурень у в'язкій електропровідній рідині, що неоднорідно обертається, з урахуванням термогальваномагнітних явищ. У наближенні локальної геометричної оптики отримано дисперсійне рівняння, що містить ефекти Холла, Нернста, Ледюка–Риги, ґрадієнти рівноважної температури ∇T_0 та питомої термоерс $\nabla\alpha$ у сталих магнітному $\mathbf{B}_0$ і ґравітаційному $\mathbf{g}$ полях. Отримано інкременти термомагнітної (ТМ) нестійкості в електропровідному середовищі, що неоднорідно обертається, без зовнішнього магнітного поля $\mathbf{B}_0 = 0$ для "плавних" (слабонеоднорідне середовище) ґрадієнтів ∇T_0 і $\nabla\alpha$. ТМ-нестійкість виникає через ґрадієнт температури ∇T_0 і ґрадієнт питомої термоерс $\nabla\alpha$. Необхідною умовою виникнення ТМ-нестійкості є колінеарність ґрадієнтів температури ∇T_0 і питомої термоерс $\nabla\alpha$. За наявності зовнішнього магнітного поля $\mathbf{B}_0 \neq 0$ установлено ділянки розвитку холлівської магнітообертальної нестійкості залежно від профілю кутової швидкості обертання (числа Росбі Ro) та аксіального хвильового числа k_z. Під час поширення збурень із хвильовим вектором $\mathbf{k}$ у радіальному $\mathbf{k}\|\mathbf{e}_R$-напрямку отримано інкременти ТМ-нестійкостей з урахуванням ефектів Нернста в зовнішньому магнітному полі $\mathbf{B}_0$, Ледюка-Риги, неоднорідності рівноважної температури та питомої термоерс, сили плавучості в стратифікованому за температурою середовищі. Установлено, що ефект Холла може як стабілізувати, так і дестабілізувати зростання збурень залежно від профілю неоднорідного обертання (числа Росбі Ro). Для збурень, що поширюються в радіальному напрямку, встановлено, що ефект Нернста в зовнішньому магнітному полі $\mathbf{B}_0$ ($\zeta\omega_{AR}$), ефект Ледюка-Риги та неоднорідності рівноважної температури й питомої термоерс (ω_{TM}), а також ефект архімедової сили в стратифікованому за температурою середовищі (ω_{VB}) призводять до дестабілізації осесиметричних збурень, тобто до ґенерації магнітного поля.

Ключові слова: термоелекторушійна сила, ґенерація магнітних полів, термомагнітна нестійкість, наближення Буссинеска, нерівномірно обертова електропровідна рідина