DEFORMED HEISENBERG ALGEBRAS OF DIFFERENT TYPES WITH PRESERVED WEAK EQUIVALENCE PRINCIPLE

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In the paper, a review of the results for recovering the weak equivalence principle in a space with deformed commutation relations for operators of coordinates and momenta is presented. Different types of deformed algebras leading to a space quantization are considered, among them noncommutative algebra of a canonical type, algebra of the Lie type, the Snyder algebra, the Kempf algebra and nonlinear deformed algebra with an arbitrary function of deformation depending on momenta. The motion of a particle and a composite system in a gravitational field is examined and the implementation of the weak equivalence principle is studied. We conclude that the Eötvös parameter is not equal to zero even in the case when the gravitational mass is equal to the inertial mass. The principle is preserved in a quantized space if we consider parameters of deformed algebras to be dependent on mass. It is also shown that the dependencies of parameters of deformed algebras on mass lead to preserving the properties of the kinetic energy in quantized spaces and solving the problem of the significant effect of space quantization on the motion of macroscopic bodies (the problem is known as the soccer-ball problem).

Key words: quantum space, minimal length, deformed Heisenberg algebra, weak equivalence principle, macroscopic body, soccer-ball problem, kinetic energy.

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I. INTRODUCTION

Deformed commutation relations for coordinates and momenta were firstly proposed by Heisenberg. The author of the first paper with the formalization of the idea of deformed commutation relations is Snyder [1]. It is worth noting that a great interest in studies of different types of deformed algebras leading to the minimal length follows from investigations in the String Theory and Quantum Gravity (see, for instance, [2, 3]).

Snyder's algebra is well known and studied (see, for example, [4–8]). The algebra in a nonrelativistic case reads

$$[X_i, X_j] = i\hbar\beta(X_iP_j - X_jP_i), \tag{1}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \beta P_iP_j), \tag{2}$$

$$[P_i, P_j] = 0. \tag{3}$$

Also, a well studied algebra leading to a minimal length is the deformed algebra proposed by Kempf (see, for instance, [9–15])

$$[X_i, X_j] = i\hbar\frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2} \times(P_iX_j - P_jX_i), \tag{4}$$

$$[X_i, P_j] = i\hbar(\delta_{ij}(1 + \beta P^2) + \beta' P_iP_j), \tag{5}$$

$$[P_i, P_j] = 0, \tag{6}$$

where β, β' are constants. In the space, the minimal length is defined by the parameters of deformations and it reads $\hbar\sqrt{\beta + \beta'}$.

It is worth noting that algebras (1)-(3), (4)-(6) are not invariant under translations in the configuration space. The deformed algebra characterized by the following commutation relation

$$[X_i, X_j] = 0, \tag{7}$$

$$[X_i, P_j] = i\hbar(\delta_{ij}(1 + \beta P^2) + 2\beta P_iP_j), \tag{8}$$

$$[P_i, P_j] = 0, \tag{9}$$

describes a uniform space. This algebra can be obtained from (4)-(6) up to the first order in the parameter of deformation, considering particular case $\beta' = 2\beta$. We can also write a deformed algebra

$$[X_i, X_j] = 0, \tag{10}$$

$$[X_i, P_j] = i\hbar\sqrt{1 + \beta P^2}(\delta_{ij} + \beta P_iP_j), \tag{11}$$

$$[P_i, P_j] = 0, \tag{12}$$

which is invariant upon translations in the configuration space and leads to the minimal length (see [16]).

In a more general case, one can consider the following commutation relations for coordinates and momenta

$$[X_i, P_j] = i\hbar F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3), \tag{13}$$

where $F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3)$ are deformation functions. For preserving the time-reversal symmetry and for invariance upon the parity transformations, the functions have to be even

$$F_{ij}(-\sqrt{\beta}P_1, -\sqrt{\beta}P_2, -\sqrt{\beta}P_3) = F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3). \tag{14}$$

Algebra (13) with

$$F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3) = \delta_{ij} - \sqrt{\beta}\left(P\delta_{ij} + \frac{P_iP_j}{P}\right) + \beta(P^2\delta_{ij} + 3P_iP_j), \quad (15)$$

was considered in the literature to describe a space with minimal length and maximal momentum [17]. Also one-dimensional algebras

$$[X, P] = i\hbar F(\sqrt{\beta}|P|), \quad (16)$$

were examined [18, 19]. In (16) $F(\sqrt{\beta}|P|)$ is a deformation function, β is a parameter, $\beta \geq 0$, $F(0) = 1$.

In the case of $F(\sqrt{\beta}|P|) = 1 + \beta P^2$, from deformed commutation relation (16) follows the well known generalized uncertainty principle (GUP)

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta\Delta P\right), \quad (17)$$

leading to the minimal length $X_{min} = \hbar\sqrt{\beta}$.

Also, other cases of the deformation functions leading to a minimal length and to a minimal momentum have been studied. Namely, in [20, 21] the authors proposed to consider $F(\sqrt{\beta}|P|) = 1/(1 - \beta P^2)$. In the paper [22] $F(\sqrt{\beta}|P|)$ was chosen to be $F(\sqrt{\beta}|P|) = (1-\sqrt{\beta}|P|)^2$. In [23] the case of $F(\sqrt{\beta}|P|) = 1/(1-\sqrt{\beta}|P|)$ was examined. The minimal length and the minimal momentum are defined by the parameter of deformation and are proportional to $\hbar\sqrt{\beta}$ and $1/\sqrt{\beta}$, respectively [17, 20, 22, 23].

Algebras it which commutators for operators of coordinates and momenta are modified and give constants are known as noncommutative algebras of a canonical type. In a general case these algebra read

$$[X_i, X_j] = i\hbar\theta_{ij}, \quad (18)$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \sigma_{ij}), \quad (19)$$

$$[P_i, P_j] = i\hbar\eta_{ij}, \quad (20)$$

where θ_{ij} are parameters of coordinate noncommutativity, η_{ij} are parameters of momentum noncommutativity and σ_{ij} are constants. Noncommutativity of coordinates leads to a minimal length. From the noncommutativity of momenta follows existence of the minimal momentum [24]. Because of the simplicity of the algebra, it has received much attention [25–34]. More complicated types of noncommutative algebras are those of the Lie type

$$[X_i, X_j] = i\hbar\theta^k_{ij}X_k, \quad (21)$$

where θ^k_{ij} are constants [35–38].

An important problem is the construction of deformed algebra which leads to a space quantization and does not cause violation of fundamental physical laws and principles. For instance, a well known problem within the frame of deformed algebras of different types is the violation of the weak equivalence principle or the Galilean equivalence principle or universality of the free fall principle. The deformation of a commutation relation for coordinates and momenta leads to the dependence of the velocity and the position of a point mass in a gravitational field on mass. In the case of algebras with noncommutativity of coordinates of a canonical type, the equivalence principle was considered in [39–44]. A more general case of noncommutativity of coordinates and noncommutativity of momenta was examined in [39, 40, 43, 44] and the problem of the ununiversality of free fall in the space was studied. In paper [40] it was concluded that the equivalence principle holds in the quantized space in the sense that an accelerated frame of reference is locally equivalent to a gravitational field, unless parameters of noncommutativity are anisotropic ($\eta_{xy} \neq \eta_{xz}$). Generalized uncertainty relations preserving of the equivalence principle were studied in [45].

In the paper, we present a way to recover the weak equivalence principle in spaces characterized by different types of deformed algebras, including noncommutative algebra of a canonical type, noncommutative algebra of the Lie type, the case of a nonlinear deformed algebra with an arbitrary deformation function depending on momentum. The solution of the problem is besed on the idea of dependence of the parameters of deformed algebras on mass. It is important to stress the the idea leads also to recovering the properties of a kinetic energy and solving the well known soccer-ball problem (the problem of description of the motion of a macroscopic body) in a space with the minimal length.

The structure of the paper is as follows. In Section II, a space with GUP is considered (16), (13) and the implementation of the weak equivalence principle in the case of nonlinear deformed algebras is recovered. In Section III, a noncommutative algebra of a canonical type is examined. The influence of noncommutativity of coordinates and noncommutativity of momenta on the Eötvös parameter for the Sun–Earth–Moon system is found. Relations for the parameters of nocommutativity with mass for preserving the weak equivalence principle are found. The noncommutative algebra which is rotationally- and time-reversal invariant and does not lead to a violation of the weak equivalence principle is studied in Section IV. Implementation of the Galilean equivalence principle in a space with noncommutative algebra of the Lie type is considered in Section V. Section VI is devoted to conclusions.

II. PRESERVING OF THE WEAK EQUIVALENCE PRINCIPLE IN A SPACE WITH GUP

A. Motion in a gravitational field in a space with nonlinear deformed algebras

As a first step of studying the weak equivalence principle in spaces with nonlinear deformed algebras, let us consider a one-dimensional case of algebra with an arbitrary function of deformation dependent on momenta (16). Relation (16) corresponds to the following deformed

Poisson bracket

$$\{X, P\} = F(\sqrt{\beta}|P|). \tag{22}$$

For a particle with mass m in gravitational field $V(X)$, writing Hamiltonian

$$H = \frac{P^2}{2m} + mV(X), \tag{23}$$

and taking into account the deformation of the Poisson brackets, we find equations of motion as

$$\dot{X} = \{X, H\} = \frac{P}{m}F(\sqrt{\beta}|P|), \tag{24}$$

$$\dot{P} = \{P, H\} = -m\frac{\partial V(X)}{\partial X}F(\sqrt{\beta}|P|). \tag{25}$$

On the basis of the obtained expressions, we can conclude that even if we consider in (23) the inertial mass, to be equal to the gravitational mass the motion of a particle in a gravitation field in a space with GUP depends on its mass and the weak equivalence principle is violated.

From equations (24), (25) follows that the motion of a particle in a gravitational field in the space (16) depends on its mass. So, deformation of commutation relation (16) leads to a violation of the weak equivalence principle.

One faces the same problem in the three-dimensional case of deformed algebra (13) and deformed Poisson brackets

$$\{X_i, P_j\} = F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3), \tag{26}$$

$$\{X_i, X_j\} = \{P_i, P_j\} = 0. \tag{27}$$

Here we would like to note that we use the ordinary Poisson brackets $\{X_i, X_j\}$ and $\{P_i, P_j\}$ (27) because in this case, the deformed algebra (26), (27) is invariant with respect to translations in the configuration space. Similarly as in the one-dimensional case we study a particle with Hamiltonian $H = \sum_i P_i^2/2m + mV(\mathbf{X})$. Using (26), (27), the equations of motion of the particle in the gravitational field read

$$\dot{X}_i = \sum_i \frac{P_j}{m}F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3), \tag{28}$$

$$\dot{P}_i = -m\sum_j \frac{\partial V(\mathbf{X})}{\partial X_j}F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3). \tag{29}$$

On the basis of the obtained results, we conclude that the weak equivalence principle is violated.

It is important to stress that the deformation of the commutation relations causes great corrections to the Eötvös parameter and a great violation of the weak equivalence principle. For instance, in the case of uniform field $V(X) = -gX$ (g is the gravitational acceleration) equations of motions (24), (25) transform to

$$\dot{X} = \frac{P}{m}F(\sqrt{\beta}|P|), \tag{30}$$

$$\dot{P} = mgF(\sqrt{\beta}|P|), \tag{31}$$

and the expression for the acceleration written up to the first order in the parameter of deformation is as follows

$$\ddot{X} = g + 3F'(0)g\sqrt{\beta}m|v| + (2F''(0) - (F'(0))^2)g\beta m^2 v^2, \tag{32}$$

where $F'(x) = dF/dx$, $F''(x) = d^2F/dx^2$ and v is a velocity of motion in gravitational field $V(X) = -gX$ in the case of $\beta = 0$. On the basis of (32), for particles with masses m_1, m_2 the Eötvös parameter reads

$$\frac{\Delta a}{a} = \frac{2(\ddot{X}^{(1)} - \ddot{X}^{(2)})}{\ddot{X}^{(1)} + \ddot{X}^{(2)}} = 3F'(0)|v|\sqrt{\beta}(m_1 - m_2) + (2F''(0) - (F'(0))^2)v^2\beta(m_1^2 - m_2^2). \tag{33}$$

If we consider the minimal length to be equal to the Planck length $\hbar\sqrt{\beta} = l_{\mathrm{P}}$, we obtain

$$\frac{\Delta a}{a} = 3F'(0)\frac{|v|}{c}\frac{(m_1 - m_2)}{m_{\mathrm{P}}} + (2F''(0) - (F'(0))^2)\frac{v^2}{c^2}\frac{(m_1^2 - m_2^2)}{m_{\mathrm{P}}^2}, \tag{34}$$

where c is the speed of light, G is the gravitational constant, m_{P} is the Planck mass [46].

For bodies with masses $m_1 = 1$ kg, $m_2 = 0.1$ kg and $F(\sqrt{\beta}|P|) = 1 + \beta P^2$, the Eötvös parameter has large value $\Delta a/a \approx 0.1$. Such a violation of the weak equivalence principle could be easily seen in an experiment. But we know that the equivalence principle holds with hight precision; for instance, from the Lunar Laser ranging experiment follows $\Delta a/a = (-0.8 \pm 1.3) \cdot 10^{-13}$ [47].

The problem is solved if the parameter of deformation β is considered to be dependent on mass as follows

$$\sqrt{\beta_a}m_a = \gamma = \mathrm{const}. \tag{35}$$

Here constant γ which is the same for different particles, is introduced [46, 48, 49].

If relation (35) holds the Eötvös parameter (33) is equal to zero and equations (24), (25) transform to

$$\dot{X} = \frac{P}{m}F\left(\gamma\frac{|P|}{m}\right), \tag{36}$$

$$\frac{\dot{P}}{m} = -\frac{\partial V(X)}{\partial X}F\left(\gamma\frac{|P|}{m}\right). \tag{37}$$

In (36), (37) we have that the mass is present only in expression P/m. So, $X(t)$, $P(t)/m$ do not depend on mass and the problem of violation of the weak equivalence principle is solved [46, 48].

The same conclusion can be made in the three-dimensional case. In the case of preserving the condition (35), introducing $P_i' = P_i/m$ from (28), (29) we have

$$\dot{X}_i = \sum_j P_j' F_{ij}\left(\gamma P_1', \gamma P_2', \gamma P_3'\right), \tag{38}$$

$$\dot{P}_i' = -\sum_j \frac{\partial V(\mathbf{X})}{\partial X_j}F_{ij}\left(\gamma P_1', \gamma P_2', \gamma P_3'\right). \tag{39}$$

So, the mass is canceled in (38), (39), and the motion in a gravitational field does not depend on mass, the weak equivalence principle is preserved.

Let us recall that we considered deformed algebra with ordinary relations for $\{X_i, X_j\}$, $\{P_i, P_j\}$ (27). But even in the case of a more complicated deformed algebra, the idea of dependence of parameters of deformation on mass gives a possibility to recover the weak equivalence principle. For instance, in the case of the following commutation relations

$$[X_i, X_j] = G(P^2)(X_i P_j - X_j P_i), \tag{40}$$

$$[X_i, P_j] = f(P^2)\delta_{ij} + F(P^2) P_i P_j, \tag{41}$$

$$[P_i, P_j] = 0. \tag{42}$$

Algebra (40)–(42) is a generalization of the well known Snyder (1)–(3) and Kempf (4)–(6) algebras. Functions $G(P^2)$, $F(P^2)$, $f(P^2)$ in (40)–(42) cannot be chosen independently [50]. From the Jacobi identity follows the following relation

$$f(F - G) - 2\frac{\partial f}{\partial P}(f + FP^2) = 0. \tag{43}$$

Let us study the weak equivalence principle in a quantized space (40)–(42). Considering a particle in a gravitational field with Hamiltonian $H = \sum_i \frac{P_i^2}{2m} + mV(\mathbf{X})$ in a space with deformed algebra (40)–(42) and the parameter of deformation satisfying condition (35), we can write the equations of motion as follows

$$\dot{X}_i = P'_i \tilde{f}(\gamma^2 (P')^2) + \gamma^2 \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} \tilde{G}(\gamma^2 (P')^2) \left(X_i P'_j - X_j P'_i \right), \tag{44}$$

$$\dot{P}'_i = -\frac{\partial V(\mathbf{X})}{\partial X_i} \tilde{f}(\gamma^2 (P')^2) - \gamma^2 \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} \tilde{F}(\gamma^2 (P')^2) P'_i P'_j, \tag{45}$$

where $\tilde{f}(\beta P^2)$, $\tilde{F}(\beta P^2)$, $\tilde{G}(\beta P^2)$ are dimensionless functions corresponding to $f(P^2)$, $F(P^2)$, $G(P^2)$ respectively. On the basis of equations (44), (45) we have that the weak equivalence principle is preserved in the general case of the deformed algebra (40)-(42) due to condition (35) [46].

In the next subsection, in addition we will show that with the help of relation (35) the properties of the kinetic energy can be preserved within the frame of the deformed algebra.

B. Properties of kinetic energy in a space with GUP and dependence of the parameter of deformation on mass

Using the relation of momenta with velocity (36), the kinetic energy of a free particle (a body) of mass m in the space with GUP (22) up to the first order in β reads

$$T = \frac{P^2}{2m} = \frac{m\dot{X}^2}{2} - F'(0)\sqrt{\beta} m^2 |\dot{X}| \dot{X}^2 + (5(F'(0))^2 - F''(0)) \frac{\beta m^3 \dot{X}^4}{2}. \tag{46}$$

On the other hand, from the additivity property for a system of N particles with masses m_a that move with the same velocities, we can write

$$T = \sum_a T_a = \frac{m\dot{X}^2}{2} - F'(0)\sqrt{\beta} |\dot{X}| \dot{X}^2 \sum_a m_a^2 + (5(F'(0))^2 - F''(0)) \frac{\beta \dot{X}^4}{2} \sum_a m_a^3, \tag{47}$$

where $m = \sum_a m_a$

$$T_a = \frac{m_a \dot{X}_a^2}{2} - F'(0)\sqrt{\beta} m_a^2 |\dot{X}_a| \dot{X}_a^2 + (5(F'(0))^2 - F''(0)) \frac{\beta m_a^3 \dot{X}_a^4}{2}, \tag{48}$$

and we take into account $\dot{X}_a = \dot{X}$. The obtained results (47), (46) are not the same. Note that $m^2 = (\sum_a m_a)^2 > \sum_a m_a^2$ and $m^3 = (\sum_a m_a)^3 > \sum_a m_a^3$. Therefore, absolute values of the corrections to the kinetic energy (46) of the first and the second order are bigger than absolute values of the corrections in (47).

It is worth noting that for a system made of N particles with the same masses, we have

$$T = N\frac{m_a \dot{X}^2}{2} - N^2 F'(0)\sqrt{\beta} m_a^2 |\dot{X}| \dot{X}^2 + N^3 (5(F'(0))^2 - F''(0)) \frac{\beta m_a^3 \dot{X}^4}{2}, \tag{49}$$

$$T = NT_a = N\frac{m_a \dot{X}^2}{2} - N\left(F'(0)\sqrt{\beta} m_a^2 |\dot{X}| \dot{X}^2 - (5(F'(0))^2 - F''(0)) \frac{\beta m_a^3 \dot{X}^4}{2} \right), \tag{50}$$

here we take into account that $m = N m_a$.

The dependencies of corrections to the kinetic energy on the number of particles N caused by the deformation (22) are different in (49) and (50). Analyzing (49) we have that corrections of the first and the second order in $\sqrt{\beta}$ are proportional to N^2 and N^3, respectively. The zero order term in (49) is proportional to N. So, with an increase in the number of particles in a system (in a macroscopic body), corrections to the kinetic energy caused by GUP increase faster than the zero order term. From this follows that the effect of space quantization on the motion of a macroscopic body is significant [46].

The problem is similar to the problem of macroscopic bodies in Double Special Relativity, which is well known as the soccer-ball problem [51–53].

If condition (35) is satisfied for the parameters of deformation of particles $\sqrt{\beta_a}m_a = \gamma = \text{const}$ and for the parameter of deformation of a composite system (macroscopic body) $\sqrt{\beta}m = \gamma = \text{const}$, the kinetic energy has additivity property, does not depend on the composition; besides it is proportional to the mass. On the basis of (49) and (50) we obtain

$$T = \frac{m\dot{X}^2}{2} - F'(0)\gamma m|\dot{X}|\dot{X}^2 + (5(F'(0))^2 - F''(0))\frac{\gamma^2 m\dot{X}^4}{2}. \quad (51)$$

So, the problem of the violation of the properties of the kinetic energy and the soccer-ball problem are solved due to relation (35).

The same conclusion can be made in all orders in the parameter of deformation. If condition (35) is satisfied, we can rewrite (36) as

$$\dot{X} = \frac{P}{m}F\left(\gamma\frac{|P|}{m}\right). \quad (52)$$

From this equation, we have that P/m is a function of velocity $\dot{X}$ and γ

$$\frac{P}{m} = f(\dot{X}, \gamma), \quad (53)$$

So, P is proportional to mass m. Using relation (53), we can rewrite the kinetic energy of the particle in the following form

$$T = \frac{P^2}{2m} = \frac{m(f(\dot{X},\gamma))^2}{2}. \quad (54)$$

Let us consider a system of N particles which move with the same velocities. This is equivalent to the case of a body divided into N parts that can be considered as particles. The kinetic energy of the system according to the additivity property can be written as

$$T = \sum_a T_a = \sum_a \frac{m_a(f(\dot{X},\gamma))^2}{2} = \frac{m(f(\dot{X},\gamma))^2}{2}. \quad (55)$$

Here we use notation m for the total mass of the system $m = \sum_a m_a$. Note that we obtain the same result (55) on the basis of expression (54), substituting $m = \sum_a m_a$.

Another property of kinetic energy, its independence of composition, is also preserved due to relation (35). According to (55), the kinetic energy of a system is proportional to its total mass and does not depend on its composition as it is in the ordinary space (space with $\beta = 0$). So, besides recovering the weak equivalence principle, relation (35) gives a possibility to preserve the properties of kinetic energy in the space with GUP [46, 48].

Similarly, in a three-dimensional space (26) the kinetic energy has additivity property and is independent of the composition if relation (35) is satisfied. For a free particle $H = \sum_i P_i^2/2m$ in the space (26) we have

$$\dot{X}_i = \{X, H\} = \sum_j \frac{P_j}{m}F_{ij}(\sqrt{\beta}P_1, \sqrt{\beta}P_2, \sqrt{\beta}P_3) = \sum_j \frac{P_j}{m}F_{ij}\left(\gamma\frac{P_1}{m}, \gamma\frac{P_2}{m}, \gamma\frac{P_3}{m}\right). \quad (56)$$

Therefore, if relation (35) holds, the values P_i/m depend on velocities $\dot{X}_i$ and γ and do not depend on mass

$$\frac{P_i}{m} = f_i(\dot{X}_1, \dot{X}_2, \dot{X}_3, \gamma). \quad (57)$$

So, the kinetic energy of a particle with mass m can be written as

$$T = \sum_i \frac{m(f_i(\dot{X}_1, \dot{X}_2, \dot{X}_3, \gamma))^2}{2}. \quad (58)$$

For a system of particles which move with the same velocities according to the additivity property, we can write

$$T = \sum_a T_a = \sum_a \sum_i \frac{m_a(f_i(\dot{X}_1, \dot{X}_2, \dot{X}_3, \gamma))^2}{2} = \sum_i \frac{m(f_i(\dot{X}_1, \dot{X}_2, \dot{X}_3, \gamma))^2}{2}, \quad (59)$$

here $m = \sum_a m_a$. Result (59) corresponds to (58). So, the properties of kinetic energy are satisfied in all orders in the parameter of deformation if one considers the dependence of the parameters of deformation corresponding to particles and macroscopic bodies on their masses (35) [46].

According to condition (35), parameters of deformation of macroscopic bodies are lower than those corresponding to elementary particles. From (35) the parameter of deformation of a macroscopic body reads

$$\beta = \beta_{\mathrm{E}}\frac{m_{\mathrm{E}}^2}{m^2}, \quad (60)$$

where m_{E}, β_{E} are the mass and the parameter of deformation of an elementary particle. On the basis of (60), we can conclude that there is a reduction by the factor m_{E}^2/m^2 of the parameter of macroscopic body β with respect to the parameter of deformation β_{E} corresponding to an elementary particle. Because of this reduction, the problem of macroscopic bodies does not appear.

At the end of this section, we would like to note that if relation (35) is satisfied for the parameter of deformation of a macroscopic body, the motion of the body in a gravitational field in a space with GUP does not depend on its mass and composition and the weak equivalence principle is satisfied.

III. MOTION IN A GRAVITATIONAL FIELD IN A NONCOMMUTATIVE PHASE SPACE

A. Recovering the weak equivalence principle in a space with noncommutativity of coordinates and noncommutativity of momenta

In a two-dimensional space with noncommutativity of coordinates and noncommutativity of momenta of a canonical type, the commutation relations for operators of coordinates and operators of momenta are as follows

$$[X_1, X_2] = i\hbar\theta, \tag{61}$$

$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{62}$$

$$[P_1, P_2] = i\hbar\eta, \tag{63}$$

where θ, η are parameters of noncommutativity $i, j = (1, 2)$.

Let us consider the influence of noncommutativity of coordinates and noncommutativity of momenta on thea motion of a particle in a uniform gravitational field with Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - mgX_1, \tag{64}$$

and examine the weak equivalence principle [41, 43, 44].

The Poisson brackets that correspond to relations of the deformed algebra (61)-(63) read

$$\{X_1, X_2\} = \theta, \tag{65}$$

$$\{X_i, P_j\} = \delta_{ij}, \tag{66}$$

$$\{P_1, P_2\} = \eta. \tag{67}$$

The definition of the deformed Poisson brackets is as follows

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial X_i}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial X_i}\right) + \theta\left(\frac{\partial f}{\partial X_1}\frac{\partial g}{\partial X_2} - \frac{\partial f}{\partial X_2}\frac{\partial g}{\partial X_1}\right) + \eta\left(\frac{\partial f}{\partial P_1}\frac{\partial g}{\partial P_2} - \frac{\partial f}{\partial P_2}\frac{\partial g}{\partial P_1}\right). \tag{68}$$

One obtains the following equations of motion and expressions for the trajectory of the particle in the gravitational field in a noncommutative phase space of a canonical type

$$\dot{X}_1 = \{X_1, H\} = \frac{P_1}{m}, \tag{69}$$

$$\dot{X}_2 = \{X_2, H\} = \frac{P_2}{m} + mg\theta, \tag{70}$$

$$\dot{P}_1 = \{P_1, H\} = mg + \eta\frac{P_2}{m}, \tag{71}$$

$$\dot{P}_2 = \{P_2, H\} = -\eta\frac{P_1}{m}, \tag{72}$$

$$X_1(t) = \frac{m\upsilon_{01}}{\eta}\sin\frac{\eta}{m}t + \left(\frac{m^2g}{\eta^2} - \frac{m^2g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\left(1 - \cos\frac{\eta}{m}t\right) + X_{01}, \tag{73}$$

$$X_2(t) = \left(\frac{m^2g}{\eta^2} - \frac{m^2g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\sin\frac{\eta}{m}t - \frac{m\upsilon_{01}}{\eta}\left(1 - \cos\frac{\eta}{m}t\right) - \frac{mg}{\eta}t + mg\theta t + X_{02}. \tag{74}$$

Here we use notations X_{01}, X_{02}, υ_{01}, υ_{02} for the initial coordinates and velocities of the particle, $X_1(0) = X_{01}$, $X_2(0) = X_{02}$, $\dot{X}_1(0) = \upsilon_{01}$, $\dot{X}_2(0) = \upsilon_{02}$.

From the obtained results we can conclude that the motion of a particle in a gravitational field depends on its mass. So, in a noncommutative phase space of a canonical type we also face a problem of violation of the weak equivalence principle. It can be solved in the case when parameters of noncommutativity depend on mass as

$$\theta m = \gamma = \text{const}, \tag{75}$$

$$\frac{\eta}{m} = \alpha = \text{const}, \tag{76}$$

where γ, α are constants that have the same values for different particles [43]. Using (75), (76), (73), (74), we have that the mass is canceled in the expressions for the trajectory of a particle in a gravitational field in a noncommutative phase space

$$X_1(t) = \frac{\upsilon_{01}}{\alpha}\sin\alpha t + \left(\frac{g}{\alpha^2} - \frac{g\gamma}{\alpha} + \frac{\upsilon_{02}}{\alpha}\right)(1-\cos\alpha t) + X_{01}, \tag{77}$$

$$X_2(t) = \left(\frac{g}{\alpha^2} - \frac{g\gamma}{\alpha} + \frac{\upsilon_{02}}{\alpha}\right)\sin\alpha t - \frac{\upsilon_{01}}{\alpha}(1-\cos\alpha t) - \frac{g}{\alpha}t + \gamma g t + X_{02}. \tag{78}$$

and the problem of violation of the weak equivalence principle is solved [43].

Here it is worth adding that in the case of a space with noncommutativity of coordinates $\theta \neq 0$, $\eta \to 0$ on the basis of (73), (74), we have that the trajectory of a particle in a uniform field is not affected by noncommutativity $X_1(t) = gt^2/2 + \upsilon_{01}t + X_{01}$, $X_2(t) = \upsilon_{02}t + X_{02}$, but for the momenta we have the following expressions $P_1 = m\dot{X}_1$, $P_2 = m(\dot{X}_2 + mg\theta)$. Note, that the momentum P_2 is not proportional to mass. It is also worth mentioning that for $\eta \to 0$ expressions (73), (74) transform to the well known result for the trajectory of a particle in a uniform gravitational field in the ordinary space, $X_1(t) = gt^2/2 + \upsilon_{01}t + X_{01}$, $X_2(t) = \upsilon_{02}t + X_{02}$. At the same time, if relation (75) is satisfied, the proportionality of momentum to mass is recovered $P_2 = m(\dot{X}_2 + \gamma g)$.

Let us consider a more general case. Let us study the motion of a composite system in a nonuniform gravitational field in a noncommutative phase space and examine the weak equivalence principle. For this purpose, we need to generalize relations of noncommutative algebra for coordinates and momenta for different particles. We have

$$\{X_1^{(a)}, X_2^{(b)}\} = \delta^{ab}\theta_a, \tag{79}$$

$$\{X_i^{(a)}, P_j^{(b)}\} = \delta^{ab}\delta_{ij}, \tag{80}$$

$$\{P_1^{(a)}, P_2^{(b)}\} = \delta^{ab}\eta_a, \tag{81}$$

where indices a, b label the particles, $X_i^{(a)}$, $P_i^{(a)}$ are coordinates and momenta of the particle with index a, $i = (1,2)$, $j = (1,2)$. In (79)–(81) we consider a general case when coordinates and momenta of different particles satisfy a noncommutative algebra with different parameters of noncommutativity. We use notations θ_a, η_a for the parameters of noncommutativity corresponding to a particle with index a. Also, in (79)–(81) we assume that the Poisson brackets for coordinates and momenta corresponding to different particles are equal to zero.

Let us consider a composite system made of N particles with masses m_a. Defining the coordinates and momenta of the center-of-mass, coordinates and momenta of the relative motion as in the ordinary space

$$\tilde{\mathbf{P}} = \sum_a \mathbf{P}^{(a)}, \quad \tilde{\mathbf{X}} = \sum_a \mu_a \mathbf{X}^{(a)}, \tag{82}$$

$$\Delta\mathbf{P}^a = \mathbf{P}^{(a)} - \mu_a\tilde{\mathbf{P}}, \quad \Delta\mathbf{X}^{(a)} = \mathbf{X}^{(a)} - \tilde{\mathbf{X}}, \tag{83}$$

(here $\mathbf{X}^{(a)} = (X_1^{(a)}, X_2^{(a)})$, $\mathbf{P}^{(a)} = (P_1^{(a)}, P_2^{(a)})$, $\mu_a = m_a/\sum_b m_b$) and using (79)–(81), one obtains the following relations

$$\{\tilde{X}_1, \tilde{X}_2\} = \tilde{\theta}, \quad \{\tilde{P}_1, \tilde{P}_2\} = \tilde{\eta}, \tag{84}$$

$$\{\tilde{X}_i, \tilde{P}_j\} = \{\Delta X_i, \Delta P_j\} = \delta_{ij}, \tag{85}$$

$$\{\Delta X_1^{(a)}, \Delta X_2^{(b)}\} = -\{\Delta X_2^{(a)}, \Delta X_1^{(b)}\} = \delta^{ab}\theta_a - \mu_a\theta_a - \mu_b\theta_b + \tilde{\theta}, \tag{86}$$

$$\{\Delta P_1^{(a)}, \Delta P_2^{(b)}\} = -\{\Delta P_2^{(a)}, \Delta P_1^{(b)}\} = \delta^{ab}\eta_a - \mu_b\eta_a - \mu_a\eta_b + \mu_a\mu_b\tilde{\eta}. \tag{87}$$

Parameters $\tilde{\theta}$, $\tilde{\eta}$ are defined as

$$\tilde{\theta} = \frac{\sum_a m_a^2\theta_a}{(\sum_b m_b)^2}, \tag{88}$$

$$\tilde{\eta} = \sum_a \eta_a, \tag{89}$$

and are called effective parameters of noncommutativity. So, coordinates and momenta of the center-of-mass of a composite system satisfy a noncommutative algebra with effective parameters which depend on the masses of particles forming it and on parameters of noncommutativity θ_a, η_a [43]. It is important that the motion of the center-of-mass is not independent of the relative motion because of relations

$$\{\tilde{X}_1, \Delta X_2^{(a)}\} = -\{\tilde{X}_2, \Delta X_1^{(a)}\} = \mu_a\theta_a - \tilde{\theta}, \tag{90}$$

$$\{\tilde{P}_1, \Delta P_2^a\} = -\{\tilde{P}_2, \Delta P_1^a\} = \eta_a - \mu_a\sum_b \eta_b. \tag{91}$$

The situation changes if we consider conditions on the parameters of noncommutativity (75), (76). In this case

$$\{\tilde{X}_1, \Delta X_2^{(a)}\} = -\{\tilde{X}_2, \Delta X_1^{(a)}\} = 0, \tag{92}$$

$$\{\tilde{P}_1, \Delta P_2^a\} = -\{\tilde{P}_2, \Delta P_1^a\} = 0, \tag{93}$$

and we have that the motion of the center-of-mass is independent of the relative motion. Also due to relations (75), (76), the effective parameters of noncommutativity do not depend on the masses of particles forming the system or its composition. Using (88), (89) and considering conditions (75), (76), we obtain that the effective parameter of coordinate noncommutativity is proportional inversely to the total mass of the system

$$\tilde{\theta} = \frac{\gamma}{M}. \tag{94}$$

The effective parameter of momentum noncommutativity is proportional to the total mass of the system

$$\tilde{\eta} = \alpha M. \tag{95}$$

So, conditions (75), (76) are also satisfied for effective parameters of noncommutativity [43].

Let us examine the motion of a composite system in a gravitational field in a noncommutative phase space of a canonical type taking into account the obtained results and conclusions about features of noncommutative algebra for coordinates and momenta of the center-of-mass and relative motion. We study the following Hamiltonian

$$H = \frac{\tilde{\mathbf{P}}^2}{2M} + MV(\tilde{X}_1, \tilde{X}_2) + H_{\text{rel}}. \tag{96}$$

Coordinates and momenta of the center-of-mass $\tilde{X}_i$, $\tilde{P}_i$ (82) satisfy noncommutative algebra (84), (85) with parameters $\tilde{\theta}$, $\tilde{\eta}$ given by (88), (89), M is the total mass of the system, the term H_{rel} corresponds to the relative motion.

If parameters of noncommutativity are related with mass (75), (76), the Poisson brackets for coordinates and momenta of the center-of-mass and relative motion are equal to zero (92), (93), therefore,

$$\left\{ \frac{\tilde{\mathbf{P}}^2}{2M} + MV(\tilde{X}_1, \tilde{X}_2), H_{\text{rel}} \right\} = 0. \tag{97}$$

So, the equations of motion for the center-of-mass of a composite system in a gravitational field read

$$\dot{\tilde{X}}_1 = \frac{P_1}{M} + M\tilde{\theta}\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_2} = \frac{P_1}{M} + \gamma\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_2}, \tag{98}$$

$$\dot{\tilde{X}}_2 = \frac{P_2}{M} - M\tilde{\theta}\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_1} = \frac{P_2}{M} - \gamma\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_1}, \tag{99}$$

$$\dot{\tilde{P}}_1 = -M\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_1} + \tilde{\eta}\frac{P_2}{M} = -M\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_1} + \alpha P_2, \tag{100}$$

$$\dot{\tilde{P}}_2 = -M\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_2} - \tilde{\eta}\frac{P_1}{M} = -M\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_2} - \alpha P_1. \tag{101}$$

Note that if conditions (75), (76) are not satisfied, the equations of motion of a composite system in a gravitational field depend on effective parameters of noncommutativity (88), (89), within are determined by the masses and parameters of noncommutativity of particles forming the system and depend on its composition. This causes a violation of the weak equivalence principle. If relations (75), (76) are preserved, the weak equivalence principle holds, the motion of a composite system (a body) in a gravitational field depends on the constants γ, α and does not depend on its mass or composition [43].

Also, due to conditions (75), (76) the properties of the kinetic energy are preserved in a noncommutative phase space of a canonical type. This will be shown in the next Subsection.

B. Motion of a composite system in a gravitational filed and the properties of kinetic energy

Let us consider a composite system which is made of N particles that move with the same velocities. On the basis of (69)–(72), considering the case when the influence of relative motion on the motion of the center-of-mass of the system is small, for the composite system in a uniform gravitational field we can write

$$\tilde{P}_1 = M\tilde{\upsilon}_{01}\cos\frac{\tilde{\eta}}{M}t + (M\tilde{\upsilon}_{02} + \frac{M^2 g}{\tilde{\eta}} - M^2 g\tilde{\theta})\sin\frac{\tilde{\eta}}{M}t, \tag{102}$$

$$\tilde{P}_2 = -M\tilde{\upsilon}_{01}\sin\frac{\tilde{\eta}}{M}t + (M\tilde{\upsilon}_{02} + \frac{M^2 g}{\tilde{\eta}} - M^2 g\tilde{\theta})\cos\frac{\tilde{\eta}}{M}t - \frac{M^2 g}{\tilde{\eta}}, \tag{103}$$

where M is the total mass of the system, $\tilde{\theta}$, $\tilde{\eta}$ are effective parameters of noncommutativity corresponding to the system (88), (89), $\tilde{\upsilon}_{01}$, $\tilde{\upsilon}_{02}$ are initial velocities of the center-of-mass of the system, $\dot{\tilde{X}}_1(0) = \dot{\tilde{\upsilon}}_{01}$, $\tilde{X}_2(0) = \tilde{\upsilon}_{02}$. Using

(102)–(103), the kinetic energy of the system can be written in the following form

$$T = \frac{\tilde{P}_1^2}{2M} + \frac{\tilde{P}_2^2}{2M} = T_0 + g^2 M^3 \left(\frac{1}{\tilde{\eta}^2} + \frac{\tilde{\theta}^2}{2} - \frac{\tilde{\theta}}{\tilde{\eta}} \right) + M^2 g \tilde{\upsilon}_{02} \left(\frac{1}{\tilde{\eta}} - \tilde{\theta} \right)$$
$$+ \frac{M^2 g}{\tilde{\eta}} \left(\tilde{\upsilon}_{01} \sin \frac{\tilde{\eta}}{M} t + \left(\frac{Mg}{\tilde{\eta}} - Mg\tilde{\theta} + \tilde{\upsilon}_{02} \right) \cos \frac{\tilde{\eta}}{M} t \right). \quad (104)$$

According to the additivity property, taking into account that the velocities of particles are the same, we can write

$$T = \sum_a T_a = \sum_a \frac{(P_1^{(a)})^2}{2m_a} + \frac{(P_2^{(a)})^2}{2m_a}$$
$$= \sum_a \left[T_{0a} + g^2 m_a^3 \left(\frac{1}{\eta_a^2} + \frac{\theta_a^2}{2} - \frac{\theta_a}{\eta_a} \right) + m_a^2 g \tilde{\upsilon}_{02} \left(\frac{1}{\eta_a} - \theta_a \right) \right.$$
$$\left. + \frac{m_a^2 g}{\eta_a} \left(\tilde{\upsilon}_{01} \sin \frac{\eta_a}{m_a} t + \left(\frac{m_a g}{\eta_a} - m_a g \theta_a + \tilde{\upsilon}_{02} \right) \cos \frac{\eta_a}{m_a} t \right) \right]. \quad (105)$$

Expression (105) does not correspond to (104). The properties of kinetic energy are violated in a noncommutative phase space. Namely, if parameters of noncommutativity are considered to be the same for different particles, one faces a problem of nonadditivity of the kinetic energy and its dependence on composition. Considering conditions (75), (76), we can rewrite (104), (105) as

$$T = T_0 + \sum_a m_a \left[g^2 \left(\frac{1}{\alpha^2} + \frac{\gamma^2}{2} - \frac{\gamma}{\alpha} \right) + g \tilde{\upsilon}_{02} \left(\frac{1}{\alpha} - \gamma \right) . + \frac{g}{\alpha} \left(\tilde{\upsilon}_{01} \sin \alpha t + \left(\frac{g}{\alpha} - g\gamma + \tilde{\upsilon}_{02} \right) \cos \alpha t \right) \right]. \quad (106)$$

On the basis of (106), we can conclude that the additivity property of kinetic energy is preserved and the kinetic energy of a composite system does not depend on its composition [43].

So, besides preserving the weak equivalence principle in a noncommutative phase space of a canonical type, conditions (75), (76) give a possibility to recover the properties of kinetic energy, to consider the motion of the center-of-mass independently of the relative motion [41, 43, 54].

In the next Subsection, using the obtained results we study the effect of noncommutativity of coordinates and noncommutativity of momenta on the weak equivalence principle considering the Sun–Earth–Moon system.

C. Effect of noncommutativity on the Eötvös parameter

According to the Lunar laser ranging experiment, the weak equivalence principle holds with accuracy

$$\frac{\Delta a}{a} = \frac{2(a_{\rm E} - a_{\rm M})}{a_{\rm E} + a_{\rm M}} = (-0.8 \pm 1.3) \cdot 10^{-13}, \quad (107)$$

(see [47]). In (107) $a_{\rm E}$, $a_{\rm M}$ are the free fall accelerations of Earth and the Moon toward the Sun when the bodies are at the same distance from the source of gravity. On the basis of this result, one can examine conditions for the parameters of coordinates and momentum noncommutativity (75), (76) proposed for preserving the weak equivalence principle. For this purpose, we study the influence of noncommutativity of coordinates and noncommutativity of momenta on the motion of Earth and the Moon in the gravitational field of the Sun.

We consider the following Hamiltonian

$$H = \frac{(\mathbf{P}^{\rm E})^2}{2m_{\rm E}} + \frac{(\mathbf{P}^{\rm M})^2}{2m_{\rm M}} - G\frac{m_{\rm E} m_{\rm S}}{R_{\rm ES}}$$
$$- G\frac{m_{\rm M} m_{\rm S}}{R_{\rm MS}} - G\frac{m_{\rm M} m_{\rm E}}{R_{\rm EM}}. \quad (108)$$

The distances between bodies $R_{\rm ES}$, $R_{\rm MS}$, $R_{\rm EM}$ in the case when the Sun is considered to be at the origin of the coordinate system read

$$R_{\rm ES} = \sqrt{(X_1^{\rm E})^2 + (X_2^{\rm E})^2},$$
$$R_{\rm MS} = \sqrt{(X_1^{\rm M})^2 + (X_2^{\rm M})^2}, \quad (109)$$

$$R_{\rm EM} = \sqrt{(X_1^{\rm E} - X_1^{\rm M})^2 + (X_2^{\rm E} - X_2^{\rm M})^2}. \quad (110)$$

Coordinates and momenta $X_i^{\rm E}$, $X_i^{\rm M}$, $P_i^{\rm E}$, $P_i^{\rm M}$ correspond to Earth and the Moon, G is the gravitational constant, $m_{\rm S}$, $m_{\rm E}$, $m_{\rm M}$ are the masses of the Sun, Earth and the Moon, respectively. It is worth noting that in (108) we consider the case when the inertial mass off Earth (mass in the first term) is equal to its gravitational mass (mass in the third and the fifth terms), also the inertial mass of the Moon (mass in the second term) is equal to its gravitational mass (mass in the fourth and the fifth terms).

In noncommutative phase space of canonical type we have the following Poisson brackets

$$\{X_1^{\rm E},X_2^{\rm E}\}=\theta_{\rm E},\quad \{P_1^{\rm E},P_2^{\rm E}\}=\eta_{\rm E},\quad \{X_i^{\rm E},P_j^{\rm E}\}=\delta_{ij}, \tag{111}$$

$$\{X_1^{\rm M},X_2^{\rm M}\}=\theta_{\rm M},\quad \{P_1^{\rm M},P_2^{\rm M}\}=\eta_{\rm M},\quad \{X_i^{\rm M},P_j^{\rm M}\}=\delta_{ij}, \tag{112}$$

$$\{X_i^{\rm M},X_j^{\rm E}\}=\{P_i^{\rm M},P_j^{\rm E}\}=0, \tag{113}$$

$\theta_{\rm E}$, $\theta_{\rm M}$, $\eta_{\rm E}$, $\eta_{\rm M}$ are parameters of coordinates and momentum noncommutativity corresponding to Earth and the Moon. Taking this into account we can write equations of motion [44]

$$\dot X_1^{\rm E}=\frac{P_1^{\rm E}}{m_{\rm E}}+\theta_{\rm E}\frac{Gm_{\rm E}m_{\rm S}X_2^{\rm E}}{R_{\rm ES}^3}+\theta_{\rm E}\frac{Gm_{\rm E}m_{\rm M}(X_2^{\rm E}-X_2^{\rm M})}{R_{\rm EM}^3}, \tag{114}$$

$$\dot X_2^{\rm E}=\frac{P_2^{\rm E}}{m_{\rm E}}-\theta_{\rm E}\frac{Gm_{\rm E}m_{\rm S}X_1^{\rm E}}{R_{\rm ES}^3}-\theta_{\rm E}\frac{Gm_{\rm E}m_{\rm M}(X_1^{\rm E}-X_1^{\rm M})}{R_{\rm EM}^3}, \tag{115}$$

$$\dot P_1^{\rm E}=\eta_{\rm E}\frac{P_2^{\rm E}}{m_{\rm E}}-\frac{Gm_{\rm E}m_{\rm S}X_1^{\rm E}}{R_{\rm ES}^3}-\frac{Gm_{\rm E}m_{\rm M}(X_1^{\rm E}-X_1^{\rm M})}{R_{\rm EM}^3}, \tag{116}$$

$$\dot P_2^{\rm E}=-\eta_{\rm E}\frac{P_1^{\rm E}}{m_{\rm E}}-\frac{Gm_{\rm E}m_{\rm S}X_2^{\rm E}}{R_{\rm ES}^3}-\frac{Gm_{\rm E}m_{\rm M}(X_2^{\rm E}-X_2^{\rm M})}{R_{\rm EM}^3}, \tag{117}$$

$$\dot X_1^{\rm M}=\frac{P_1^{\rm M}}{m_{\rm M}}+\theta_{\rm M}\frac{Gm_{\rm M}m_{\rm S}X_2^{\rm M}}{R_{\rm MS}^3}-\theta_{\rm M}\frac{Gm_{\rm E}m_{\rm M}(X_2^{\rm E}-X_2^{\rm M})}{R_{\rm EM}^3}, \tag{118}$$

$$\dot X_2^{\rm M}=\frac{P_2^{\rm M}}{m_{\rm M}}-\theta_{\rm M}\frac{Gm_{\rm M}m_{\rm S}X_1^{\rm E}}{R_{\rm MS}^3}+\theta_{\rm M}\frac{Gm_{\rm E}m_{\rm M}(X_1^{\rm E}-X_1^{\rm M})}{R_{\rm EM}^3}, \tag{119}$$

$$\dot P_1^{\rm M}=\eta_{\rm M}\frac{P_2^{\rm M}}{m_{\rm M}}-\frac{Gm_{\rm M}m_{\rm S}X_1^{\rm M}}{R_{\rm MS}^3}+\frac{Gm_{\rm E}m_{\rm M}(X_1^{\rm E}-X_1^{\rm M})}{R_{\rm EM}^3}, \tag{120}$$

$$\dot P_2^{\rm M}=-\eta_{\rm M}\frac{P_1^{\rm M}}{m_{\rm M}}-\frac{Gm_{\rm M}m_{\rm S}X_2^{\rm M}}{R_{\rm MS}^3}+\frac{Gm_{\rm E}m_{\rm M}(X_2^{\rm E}-X_2^{\rm M})}{R_{\rm EM}^3}. \tag{121}$$

On the basis of these equations accelerations of Earth and the Moon can be found. Up to the first order in the parameters of coordinate and momentum noncommutativity we obtain

$$\begin{aligned}\ddot X_1^{\rm E} &= -\frac{Gm_{\rm S}X_1^{\rm E}}{R_{\rm ES}^3}-\frac{Gm_{\rm M}(X_1^{\rm E}-X_1^{\rm M})}{R_{\rm EM}^3}+\eta_{\rm E}\frac{\dot X_2^{\rm E}}{m_{\rm E}}+\theta_{\rm E}\frac{Gm_Sm_{\rm E}\dot X_2^{\rm E}}{R_{\rm ES}^3}\\ &+ \theta_{\rm E}\frac{Gm_Mm_{\rm E}}{R_{\rm EM}^3}(\dot X_2^{\rm E}-\dot X_2^{\rm M})-\theta_{\rm E}\frac{3Gm_{\rm S}m_{\rm E}}{R_{\rm ES}^5}(\mathbf{R}_{\rm ES}\cdot\dot{\mathbf{R}}_{\rm ES})X_2^{\rm E}\\ &- \theta_{\rm E}\frac{3Gm_{\rm M}m_{\rm E}}{R_{\rm EM}^5}(\mathbf{R}_{\rm EM}\cdot\dot{\mathbf{R}}_{\rm EM})(X_2^{\rm E}-X_2^{\rm M}),\end{aligned} \tag{122}$$

$$\begin{aligned}\ddot X_1^{\rm M} &= -\frac{Gm_{\rm S}X_1^{\rm M}}{R_{\rm MS}^3}+\frac{Gm_{\rm E}(X_1^{\rm E}-X_1^{\rm M})}{R_{\rm EM}^3}+\eta_{\rm M}\frac{\dot X_2^{\rm M}}{m_{\rm M}}+\theta_{\rm M}\frac{Gm_Sm_{\rm M}\dot X_2^{\rm M}}{R_{\rm MS}^3}\\ &- \theta_{\rm M}\frac{Gm_Mm_{\rm E}}{R_{\rm EM}^3}(\dot X_2^{\rm E}-\dot X_2^{\rm M})-\theta_{\rm M}\frac{3Gm_{\rm S}m_{\rm M}}{R_{\rm MS}^5}(\mathbf{R}_{\rm MS}\cdot\dot{\mathbf{R}}_{\rm MS})X_2^{\rm M}\\ &+ \theta_{\rm M}\frac{3Gm_{\rm M}m_{\rm E}}{R_{\rm EM}^5}(\mathbf{R}_{\rm EM}\cdot\dot{\mathbf{R}}_{\rm EM})(X_2^{\rm E}-X_2^{\rm M}),\end{aligned} \tag{123}$$

where $\mathbf{R}_{\rm ES}(X_1^{\rm E},X_2^{\rm E})$, $\mathbf{R}_{\rm MS}(X_1^{\rm M},X_2^{\rm M})$, $\mathbf{R}_{\rm EM}(X_1^{\rm E}-X_1^{\rm M},X_2^{\rm E}-X_2^{\rm M})$ [44].

In the case when the distance from the bodies to the Sun is the same, we can write $R_{\rm MS} = R_{\rm ES} = R$. For convenience we consider the X_1 axis to pass through the middle of $\mathbf{R}_{\rm EM}$ and to be perpendicular to $\mathbf{R}_{\rm EM}$, the X_2 axis to be parallel to the $\mathbf{R}_{\rm EM}$. Let us remind that we have chosen the origin of the frame of references to be at the Sun's center. So, taking into account that $R_{\rm EM}/R \sim 10^{-3}$, one obtains

$$X_1^{\rm E} = X_1^{\rm M} = R\sqrt{1 - \frac{R_{\rm EM}^2}{4R^2}} \simeq R, \quad X_2^{\rm E} = -X_2^{\rm M} = \frac{R_{\rm EM}}{2}. \tag{124}$$

Note that

$$\dot{X}_1^{\rm E} = 0, \quad \dot{X}_1^{\rm M} = \upsilon_{\rm M}, \quad \dot{X}_2^{\rm E} = \dot{X}_2^{\rm M} = \upsilon_{\rm E}, \tag{125}$$

where $\upsilon_{\rm M}$, $\upsilon_{\rm E}$ are the orbital velocities of the Moon and Earth. So, the free fall accelerations of the Moon and Earth toward the Sun in the case when the bodies are at the same distance to it read

$$a_{\rm E} = \ddot{X}_1^{\rm E} = -\frac{Gm_{\rm S}}{R^2} + \eta_{\rm E}\frac{\upsilon_{\rm E}}{m_{\rm E}} + \theta_{\rm E}\frac{Gm_S m_{\rm E}\upsilon_{\rm E}}{R^3}\left(1 - \frac{3R_{\rm EM}}{2\upsilon_{\rm E}R^2}(\mathbf{R}_{\rm ES}\cdot\dot{\mathbf{R}}_{\rm ES})\right), \tag{126}$$

$$a_{\rm M} = \ddot{X}_1^{\rm M} = -\frac{Gm_{\rm S}}{R^2} + \eta_{\rm M}\frac{\upsilon_{\rm E}}{m_{\rm M}} + \theta_{\rm M}\frac{Gm_S m_{\rm M}\upsilon_{\rm E}}{R^3}\left(1 + \frac{3R_{\rm EM}}{2\upsilon_{\rm E}R^2}(\mathbf{R}_{\rm MS}\cdot\dot{\mathbf{R}}_{\rm MS})\right). \tag{127}$$

We have $R_{\rm EM}/R \sim 10^{-3}$, $\upsilon_{\rm M}/\upsilon_{\rm E} \sim 10^{-2}$, therefore

$$\frac{3R_{\rm EM}(\mathbf{R}_{\rm ES}\cdot\dot{\mathbf{R}}_{\rm ES})}{2\upsilon_{\rm E}R^2} \sim 10^{-6}, \quad \frac{3R_{\rm EM}(\mathbf{R}_{\rm MS}\cdot\dot{\mathbf{R}}_{\rm MS})}{2\upsilon_{\rm E}R^2} \sim 10^{-5}, \tag{128}$$

and the last terms in the expressions for the accelerations (126), (127) can be neglected. So, for the Eötvös parameter for Earth and the Moon in a noncommutative phase space we obtain the following result

$$\begin{aligned}\frac{\Delta a}{a} &= \frac{\upsilon_{\rm E}R^2}{Gm_{\rm S}}\left(\frac{\eta_{\rm E}}{m_{\rm E}} - \frac{\eta_{\rm M}}{m_{\rm M}}\right) + \frac{\upsilon_{\rm E}}{R}\left(\theta_{\rm E}m_{\rm E} - \theta_{\rm M}m_{\rm M}\right)\\ &= \frac{\Delta a^\eta}{a} + \frac{\Delta a^\theta}{a},\end{aligned} \tag{129}$$

where $\Delta a^\eta/a$, $\Delta a^\theta/a$ are corrections to the Eötvös parameter caused by the coordinate noncommutativity and momentum noncommutativity

$$\frac{\Delta a^\eta}{a} = \frac{\upsilon_{\rm E}R^2}{Gm_{\rm S}}\left(\frac{\eta_{\rm E}}{m_{\rm E}} - \frac{\eta_{\rm M}}{m_{\rm M}}\right), \tag{130}$$

$$\frac{\Delta a^\theta}{a} = \frac{\upsilon_{\rm E}}{R}\left(\theta_{\rm E}m_{\rm E} - \theta_{\rm M}m_{\rm M}\right), \tag{131}$$

respectively.

It is important to stress that even if we consider the inertial masses of the bodies to be equal to their gravitational masses [see (108)], the Eötvös parameter is not equal to zero. Noncommutativity of coordinates and noncommutativity of momenta causes the violation of the weak equivalence principle. In addition, it is worth emphasizing that parameters $\theta_{\rm E}$, $\eta_{\rm E}$, $\theta_{\rm M}$, $\eta_{\rm M}$ correspond to macroscopic bodies; they are effective parameters of noncommutativity which depend on the composition of the bodies and are defined as (88), (89). So, even for two bodies with the same masses but different compositions the Eötvös-parameter is not equal to zero [44].

Let us introduce constants

$$\alpha_{\rm E} = \frac{\eta_{\rm E}}{m_{\rm E}}, \quad \alpha_{\rm M} = \frac{\eta_{\rm M}}{m_{\rm M}}, \qquad \gamma_{\rm E} = \theta_{\rm E}m_{\rm E}, \quad \gamma_{\rm M} = \theta_{\rm M}m_{\rm M}, \tag{132}$$

and estimate the values $|\alpha_{\rm E} - \alpha_{\rm M}|$, $|\gamma_{\rm E} - \gamma_{\rm M}|$ on the basis of the Lunar laser ranging experiment results [47]. We assume that the following inequality is satisfied

$$\left|\frac{\Delta a^\theta + \Delta a^\eta}{a}\right| \leq 2.1\cdot 10^{-13}. \tag{133}$$

Here $2.1\cdot 10^{-13}$ is the largest value in (107) [47]. To estimate the orders of the values $|\alpha_{\rm E} - \alpha_{\rm M}|$, $|\gamma_{\rm E} - \gamma_{\rm M}|$ we consider inequalities

$$\left|\frac{\Delta a^\theta}{a}\right| \leq 10^{-13}, \quad \left|\frac{\Delta a^\eta}{a}\right| \leq 10^{-13}. \tag{134}$$

From the inequalities, using (130), (131), we find [44]

$$|\alpha_{\rm E} - \alpha_{\rm M}| \leq 10^{-20}\ {\rm s}^{-1}, \quad |\gamma_{\rm E} - \gamma_{\rm M}| \leq 10^{-7}\ {\rm s}. \tag{135}$$

It is important to stress that considering conditions on the parameters of noncommutativity proposed in the previous section, namely, assuming that $\alpha_{\rm E} = \alpha_{\rm M}$, $\gamma_{\rm E} = \gamma_{\rm M}$, we obtain that the Eötvös parameter for Earth and the Moon (129) is equal to zero. So, the weak equivalence principle is preserved in a noncommutative phase space of a canonical type.

IV. QUANTIZED SPACE WITH PRESERVED ROTATIONAL AND TIME-REVERSAL SYMMETRIES AND WEAK EQUIVALENCE PRINCIPLE

A. Rotationally-invariant noncommutative algebra of a canonical type

In a six-dimensional noncommutative phase space of a canonical type (a three dimensional configuration space and a three dimensional momentum space) (18)–(20), the rotational and time reversal symmetries are not preserved [55].

Algebra which is rotational invariant and equivalent to a noncommutative algebra of a canonical type and does not cause the time reversal symmetry breaking, was proposed in [55]. It reads

$$[X_i, X_j] = i\hbar\theta_{ij} = ic_\theta \sum_k \varepsilon_{ijk} p^a_k, \quad (136)$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \gamma_{ij}) = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar^2}(\mathbf{p}^a \cdot \mathbf{p}^b)\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar^2} p^a_j p^b_i\right), \quad (137)$$

$$[P_i, P_j] = i\hbar\eta_{ij} = ic_\eta \sum_k \varepsilon_{ijk} p^b_k. \quad (138)$$

The algebra is constructed, considering tensors of noncommutativity defined as

$$\theta_{ij} = \frac{c_\theta}{\hbar} \sum_k \varepsilon_{ijk} p^a_k, \quad (139)$$

$$\eta_{ij} = \frac{c_\eta}{\hbar} \sum_k \varepsilon_{ijk} p^b_k, \quad (140)$$

here p^a_i, p^b_i are additional momenta, c_θ, c_η are constants, $\lim_{\hbar\to 0} c_\theta/\hbar = \text{const}$, $\lim_{\hbar\to 0} c_\eta/\hbar = \text{const}$ [55]. From the symmetric representation of noncommutative coordinates and noncommutative momenta (see, for instance, [25, 56, 57]) follows that parameters σ_{ij} are defined as $\sigma_{ij} = \sum_k \theta_{ik}\eta_{jk}/4$. So, using (139), (140), we obtain

$$\sigma_{ij} = \frac{c_\theta c_\eta}{4\hbar^2}(\mathbf{p}^a \cdot \mathbf{p}^b)\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar^2} p^a_j p^b_i. \quad (141)$$

The symmetric representation for noncommutative coordinates and noncommutative momenta reads

$$X_i = x_i + \frac{1}{2}[\boldsymbol{\theta} \times \mathbf{p}]_i, \quad P_i = p_i - \frac{1}{2}[\boldsymbol{\eta} \times \mathbf{x}]_i. \quad (142)$$

Coordinates and momenta x_i, p_i satisfy the ordinary commutation relations

$$[x_i, x_j] = [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}. \quad (143)$$

In (142) we use notations $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$

$$\theta_i = \sum_{jk} \frac{\varepsilon_{ijk}\theta_{jk}}{2} = \frac{c_\theta p^a_i}{\hbar}, \qquad \eta_i = \sum_{jk} \frac{\varepsilon_{ijk}\eta_{jk}}{2} = \frac{c_\eta p^b_i}{\hbar}. \quad (144)$$

Additional momenta p^a_i, p^b_i and additional coordinates a_i, b_i satisfy the ordinary commutation relations $[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0$, $[p^a_i, p^a_j] = [p^b_i, p^b_j] = [p^a_i, p^b_j] = 0$, $[a_i, p^a_j] = [b_i, p^b_j] = i\hbar\delta_{ij}$, $[a_i, p^b_j] = [b_i, p^a_j] = 0$, $[a_i, X_j] = [a_i, P_j] = [p^b_i, X_j] = [p^b_i, P_j] = 0$. So, the tensors of noncommutativity commute with coordinates and momenta

$$[\theta_{ij}, X_k] = [\theta_{ij}, P_k] = [\eta_{ij}, X_k] = [\eta_{ij}, P_k] = 0, \quad (145)$$

$$[\sigma_{ij}, X_k] = [\sigma_{ij}, P_k] = 0. \quad (146)$$

The same relations (145), (146) are satisfied within the frame of the noncommutative algebra of a canonical type (18)–(20). In this sense, algebra (136)-(138) is equivalent to (18)-(20) [55].

To preserve the rotational symmetry, additional coordinates and momenta a_i, b_i, p^a_i, p^b_i have to be governed by rotationally-symmetric systems. For simplicity, in [55] these systems were considered to be harmonic oscillators

$$H^a_{\text{osc}} = \frac{(\mathbf{p}^a)^2}{2m_{\text{osc}}} + \frac{m_{\text{osc}}\,\omega^2_{\text{osc}}\mathbf{a}^2}{2}, \qquad H^b_{\text{osc}} = \frac{(\mathbf{p}^b)^2}{2m_{\text{osc}}} + \frac{m_{\text{osc}}\,\omega^2_{\text{osc}}\mathbf{b}^2}{2}, \quad (147)$$

with $\sqrt{\hbar}/\sqrt{m_{\text{osc}}\omega_{\text{osc}}} = l_P$ and very large frequency ω_{osc} (oscillators put into the ground states remain in the states) [55].

B. Particle in a gravitational field in a noncommutative phase space with preserved rotational and time reversal symmetries

Let us study the motion of a particle in a uniform field within the frame of the algebra (136)–(138) and examine the weak equivalence principle. We consider the following Hamiltonian

$$H_{\text{P}} = \frac{\mathbf{P}^2}{2m} + mgX_1, \quad (148)$$

here m is the mass of the particle, g is the free fall acceleration. The X_1 axis is chosen to correspond to the field direction. Coordinates and momenta of the particle satisfy relations of noncommutative algebra (136)–(138) which contain additional momenta. So, to study the motion of the particle in a gravitational field, we have to take into account additional terms corresponding to harmonic oscillators. Therefore, we consider the total Hamiltonian as follows

$$H = H_{\text{P}} + H^a_{\text{osc}} + H^b_{\text{osc}}. \quad (149)$$

It is convenient to use representation (142) and rewrite the Hamiltonian in the following form

$$H = \frac{\mathbf{p}^2}{2m} + mgx_1 - \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m} + \frac{mg}{2}[\boldsymbol{\theta}\times\mathbf{p}]_1 + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} + H^a_{\text{osc}} + H^b_{\text{osc}}, \quad (150)$$

here $\mathbf{L} = [\mathbf{x} \times \mathbf{p}]$. The Hamiltonian also can be represented as

$$H = H_0 + \Delta H, \tag{151}$$

$$H_0 = \langle H_{\rm P}\rangle_{ab} + H^a_{\rm osc} + H^b_{\rm osc}, \tag{152}$$

$$\Delta H = H - H_0 = H_{\rm P} - \langle H_{\rm P}\rangle_{ab}, \tag{153}$$

where $\langle ...\rangle_{ab} = \langle \psi^a_{0,0,0}\psi^b_{0,0,0}|...|\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle$, $\psi^a_{0,0,0}$, $\psi^b_{0,0,0}$ are the well known eigenfunctions of the harmonic oscillators $H^a_{\rm osc}$, $H^b_{\rm osc}$ in the ground states.

For a particle in a uniform field, we have

$$H_0 = \frac{\mathbf{p}^2}{2m} + mgx_1 + \frac{\langle\eta^2\rangle\mathbf{x}^2}{12m} + H^a_{\rm osc} + H^b_{\rm osc}, \tag{154}$$

$$\Delta H = -\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m} + \frac{mg}{2}[\boldsymbol{\theta}\times\mathbf{p}]_1 + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} - \frac{\langle\eta^2\rangle\mathbf{x}^2}{12m}. \tag{155}$$

To find these expressions, the following results are used

$$\langle\psi^a_{0,0,0}|\theta_i|\psi^a_{0,0,0}\rangle = 0, \quad \langle\psi^b_{0,0,0}|\eta_i|\psi^b_{0,0,0}\rangle = 0, \tag{156}$$

$$\langle\theta^2\rangle = \sum_i\langle\theta_i^2\rangle = \sum_i \frac{c_\theta^2}{\hbar^2}\langle\psi^a_{0,0,0}|(p^a_i)^2|\psi^a_{0,0,0}\rangle = \frac{3c_\theta^2}{2l_{\rm P}^2}, \tag{157}$$

$$\langle\eta^2\rangle = \sum_i\langle\eta_i^2\rangle = \sum_i \frac{c_\eta^2}{\hbar^2}\langle\psi^b_{0,0,0}|(p^b_i)^2|\psi^b_{0,0,0}\rangle = \frac{3c_\eta^2}{2l_{\rm P}^2}. \tag{158}$$

In [58] it was shown that the corrections to H_0 caused by term ΔH vanish up to the second order of the perturbation theory. So, up to the second order in ΔH (or up to the second order in the parameters of noncommutativity), we can study Hamiltonian (154) and write the following equations of motion

$$\dot{x}_i = \frac{p_i}{m}, \quad \dot{p}_i = -mg\delta_{i,1} - \frac{\langle\eta^2\rangle x_i}{6m}. \tag{159}$$

Ther solution of the equations with initial conditions $x_i(0) = x_{0i}$, $\dot{x}_i(t) = \upsilon_{0i}$ reads [59]

$$x_i(t) = \left(x_{0i} + 6g\frac{m^2}{\langle\eta^2\rangle}\delta_{1,i}\right)\cos\left(\sqrt{\frac{\langle\eta^2\rangle}{6m^2}}t\right) + \upsilon_{0i}\sqrt{\frac{6m^2}{\langle\eta^2\rangle}}\sin\left(\sqrt{\frac{\langle\eta^2\rangle}{6m^2}}t\right) - 6g\frac{m^2}{\langle\eta^2\rangle}\delta_{1,i}. \tag{160}$$

From this result, we can conclude that up to the second order in the parameters of noncommutativity the motion of a particle in a uniform gravitational field is not affected by noncommutativity of coordinates. Also, it is worth noting that in limit $\langle\eta^2\rangle \to 0$ from (160) we find the well known result $x_i(t) = \delta_{1,i}gt^2/2 + x_{0i}$, which corresponds to the motion of a particle in a gravitational field in the ordinary space.

It is important to mention, that the trajectory of a particle in a gravitational field (160) depends on its mass. So, the weak equivalence principle is violated in a noncommutative phase space of a canonical type with preserved rotational and time reversal symmetries.

Note that if we consider the tensor of momentum noncommutativity to be dependent on mass as

$$\eta_{ij} = \tilde{\alpha}m\hbar\sum_k \varepsilon_{ijk}p^b_k, \tag{161}$$

namely, if constant $c^{(n)}_\eta$ in (140) satisfies condition

$$\frac{c^{(n)}_\eta}{m_n} = \tilde{\alpha} = {\rm const}, \tag{162}$$

(here $\tilde{\alpha}$ is the same for different particles), the motion of a particle in a uniform field does not depend on mass and the weak equivalence principle is recovered [59, 60].

From (162) follows that

$$\frac{\langle\eta^2\rangle}{m^2} = \frac{3\tilde{\alpha}^2}{2l_{\rm P}^2} = B = {\rm const}, \tag{163}$$

and the trajectory of a particle reads

$$x_i(t) = \left(x_{0i} + \frac{6g}{B}\delta_{1,i}\right)\cos\left(\sqrt{\frac{B}{6}}t\right) + \upsilon_{0i}\sqrt{\frac{6}{B}}\sin\left(\sqrt{\frac{B}{6}}t\right) - \frac{6g}{B}\delta_{1,i}. \tag{164}$$

In the case of a non-uniform gravitational field for a particle with mass m, we consider the following Hamiltonian

$$H = H_{\rm P} + H^a_{\rm osc} + H^b_{\rm osc}, \quad H_{\rm P} = \frac{P^2}{2m} - \frac{G\tilde{M}m}{X}, \tag{165}$$

here $X = \sqrt{\sum_i X_i^2}$. The Hamiltonian $H_{\rm P}$ written in representation (142) up to the second order in the parameters of noncommutativity has the following form

$$H_{\rm P} = \frac{p^2}{2m} - \frac{G\tilde{M}m}{x} - \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m} + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} - \frac{G\tilde{M}m}{\sqrt{x^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{[\boldsymbol{\theta}\times\mathbf{p}]^2}{4}}} = \frac{p^2}{2m} - \frac{G\tilde{M}m}{x} - \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m} + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} - \frac{G\tilde{M}m}{2x^3}(\boldsymbol{\theta}\cdot\mathbf{L}) - \frac{3G\tilde{M}m}{8x^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + \frac{G\tilde{M}m}{16}\left(\frac{1}{x^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x} + \frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x^2} + \frac{\hbar^2}{x^7}[\boldsymbol{\theta}\times\mathbf{x}]^2\right), \tag{166}$$

where $x=|\mathbf{x}|$ (the details of calculations of the expansion can be found in [61]). So, for ΔH we have

$$\Delta H=-\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m}+\frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m}-\frac{\langle\eta^2\rangle x^2}{12m}-\frac{G\tilde{M}m}{2x^3}(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{G\tilde{M}mL^2\langle\theta^2\rangle}{8x^5} \tag{167}$$

$$+\frac{G\tilde{M}m}{16}\left(\frac{1}{x^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x}+\frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x^2}+\frac{\hbar^2}{x^7}[\boldsymbol{\theta}\times\mathbf{x}]^2\right)-\frac{3G\tilde{M}m}{8x^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2-\frac{G\tilde{M}m\langle\theta^2\rangle}{24}\left(\frac{1}{x^2}p^2\frac{1}{x}+\frac{1}{x}p^2\frac{1}{x^2}+\frac{\hbar^2}{x^5}\right).$$

Up to the second order in the parameters of noncommutativity to study the motion of a particle in a nonuniform gravitational field, we can consider the following Hamiltonian

$$H_0=\frac{p^2}{2m}-\frac{G\tilde{M}m}{x}+\frac{\langle\eta^2\rangle x^2}{12m}-\frac{G\tilde{M}mL^2\langle\theta^2\rangle}{8x^5}+\frac{G\tilde{M}m\langle\theta^2\rangle}{24}\left(\frac{2}{x^3}p^2+\frac{6i\hbar}{x^5}(\mathbf{x}\cdot\mathbf{p})-\frac{\hbar^2}{x^5}\right)+H^a_{\mathrm{osc}}+H^b_{\mathrm{osc}}, \tag{168}$$

and find the following equations of motion

$$\dot{\mathbf{x}}=\frac{\mathbf{p}}{m}-\frac{G\tilde{M}m\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\mathbf{p}-\frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\mathbf{p})\right), \tag{169}$$

$$\dot{\mathbf{p}}=-\frac{G\tilde{M}m\mathbf{x}}{x^3}-\frac{\langle\eta^2\rangle\mathbf{x}}{6m}-\frac{G\tilde{M}m\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\mathbf{p})\mathbf{p}-\frac{2\mathbf{x}}{x^5}p^2+\frac{5\mathbf{x}}{2x^7}L^2+\frac{5\hbar^2\mathbf{x}}{6x^7}-\frac{5i\hbar}{x^7}\mathbf{x}(\mathbf{x}\cdot\mathbf{p})\right). \tag{170}$$

These equations in the classical limit ($\hbar\to 0$) transform to

$$\dot{\mathbf{x}}=\boldsymbol{p}'-\frac{G\tilde{M}m^2\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\boldsymbol{p}'-\frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{p}')\right), \tag{171}$$

$$\dot{\boldsymbol{p}}'=-\frac{G\tilde{M}\mathbf{x}}{x^3}-\frac{\langle\eta^2\rangle\mathbf{x}}{6m^2}-\frac{G\tilde{M}m^2\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{p}')\boldsymbol{p}'-\frac{2\mathbf{x}}{x^5}(p')^2+\frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{p}']^2\right), \tag{172}$$

here $\boldsymbol{p}'=\mathbf{p}/m$ [59]. Let us consider the dependence of the tensor of coordinates noncommutativity on mass as follows

$$c_\theta^{(n)}m_n=\tilde{\gamma}=\mathrm{const}, \tag{173}$$

$$\theta_{ij}=\frac{\tilde{\gamma}}{m}\hbar\sum_k\varepsilon_{ijk}p^a_k, \tag{174}$$

$$\langle\theta^2\rangle m^2=\frac{3\tilde{\gamma}^2}{2l_{\mathrm{P}}^2}=A=\mathrm{const}, \tag{175}$$

where constants A, $\tilde{\gamma}$ are the same for different particles. So, in the case when the relations (162), (173) hold, from (169), (170) we obtain

$$\dot{\mathbf{x}}=\boldsymbol{p}'-\frac{G\tilde{M}B}{12}\left(\frac{1}{x^3}\boldsymbol{p}'-\frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{p}')\right), \tag{176}$$

$$\dot{\boldsymbol{p}}'=-\frac{G\tilde{M}\mathbf{x}}{x^3}-\frac{B\mathbf{x}}{6}-\frac{G\tilde{M}A}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{p}')\boldsymbol{p}'-\frac{2\mathbf{x}}{x^5}(p')^2+\frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{p}']^2+\frac{5\hbar^2\mathbf{x}}{6m^2x^7}-\frac{5i\hbar}{mx^7}\mathbf{x}(\mathbf{x}\cdot\boldsymbol{p}')\right). \tag{177}$$

In the classical limit, on the basis of (176), (177) we find

$$\dot{\mathbf{x}}=\boldsymbol{p}'-\frac{G\tilde{M}A}{12}\left(\frac{1}{x^3}\boldsymbol{p}'-\frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{p}')\right), \tag{178}$$

$$\dot{\boldsymbol{p}}'=-\frac{G\tilde{M}\mathbf{x}}{x^3}-\frac{B\mathbf{x}}{6}-\frac{G\tilde{M}A}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{p}')\boldsymbol{p}'-\frac{2\mathbf{x}}{x^5}(p')^2+\frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{p}']^2\right). \tag{179}$$

The equations of motion of a particle in a gravitational field in the quantum case (176), (177) depend on the ratio $\hbar/m$, as it has to be. This is caused by the dependence of the commutation relation on mass $[\mathbf{x}, \boldsymbol{p}'] = i\hbar\hat{I}/m$ [62]. Classical equations of motion (178), (179) do not depend on mass. So, the weak equivalence principle is satisfied in a noncommutative phase space with preserved rotational and time reversal symmetries if tensors of noncommutativity are related with mass (162), (173) [59].

It is worth noting that the conditions (162), (173) considered in this section are in agreement with those presented in section III (76), (75) to recover the weak equivalence principle in noncommutative phase space of a canonical type.

V. WEAK EQUIVALENCE PRINCIPLE WITHIN THE FRAME OF THE NONCOMMUTATIVE ALGEBRA OF THE LIE TYPE

A. The Lie algebra with space coordinates commuting to time and the weak equivalence principle

Let us study the motion of a particle in a gravitational field in a space with noncommutativity of the Lie type in the case when space coordinates commute to time

$$[X_i, X_j] = \frac{i\hbar t}{\kappa}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right), \tag{180}$$

$$[X_i, P_j] = i\hbar\delta_{ij}, \quad [P_i, P_j] = 0, \tag{181}$$

here $i, j = (1, 2, 3)$, indexes ρ, τ are fixed and different, κ is a parameter [35, 63]. The deformed Poisson brackets corresponding to (180)-(181) are as follows

$$\{X_i, X_j\} = \frac{t}{\kappa}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right), \tag{182}$$

$$\{X_i, P_j\} = \delta_{ij}, \quad \{P_i, P_j\} = 0, \tag{183}$$

(see [35]).

For a particle with mass m in a gravitational field $V = V(X_1, X_2, X_3)$, the Hamiltonian reads

$$H = \frac{\mathbf{P}^2}{2m} + mV(X_1, X_2, X_3). \tag{184}$$

Taking into account (182), (183), we can write equations of motion as follows

$$\dot{X}_i = \{X_i, H\} = \frac{P_i}{m} + \frac{tm}{\kappa}\frac{\partial V}{\partial X_k}\left(\delta_{i\rho}\delta_{k\tau} - \delta_{i\tau}\delta_{k\rho}\right), \tag{185}$$

$$\dot{P}_i = \{P_i, H\} = -m\frac{\partial V}{\partial X_i}, \tag{186}$$

(see [35, 64]). Note that in (185) because of noncommutativity of the Lie type, we have a term proportional to mass m. Therefore the weak equivalence principle is violated. Similarly as in a noncommutative space of a canonical type, let us consider the dependence of the parameter of the noncommutative algebra on mass and write the following condition

$$\frac{\kappa}{m} = \gamma_\kappa = \text{const}, \tag{187}$$

here γ_κ does not depend on mass and is the same for different particles. Taking into account relation (187), the equations of motion of a particle in a gravitational field can be rewritten as

$$\dot{X}_i = P'_i + \frac{t}{\gamma_\kappa}\frac{\partial V}{\partial X_k}\left(\delta_{i\rho}\delta_{k\tau} - \delta_{i\tau}\delta_{k\rho}\right), \ \dot{P}'_i = -\frac{\partial V}{\partial X_i}, \tag{188}$$

where $P'_i = P_i/m$. So, on the basis of the obtained result, we have that $X_i(t)$, $P'_i(t)$ do not depend on mass and the weak equivalence principle is recovered if condition (187) is satisfied [64].

Let us also study the case of the motion of a composite system in a gravitational field and examine the weak equivalence principle. For coordinates and momenta of different particles, the noncommutative algebra of the Lie type (182), (183) can be generalized as

$$\{X_i^{(a)}, X_j^{(b)}\} = \frac{t}{\kappa_a}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right)\delta_{ab}, \tag{189}$$

$$\{X_i^{(a)}, P_j^{(b)}\} = \delta_{ab}\delta_{ij}, \quad \{P_i^{(a)}, P_j^{(b)}\} = 0, \tag{190}$$

here $X_i^{(a)}$, $P_i^{(a)}$, κ_a are coordinates, momenta and parameters of the noncommutative algebra corresponding to the particle with index a [64]. The noncommutative algebra for coordinates and momenta of the center-of-mass, coordinates and momenta of the relative motion introduced in the traditional way ($\tilde{\mathbf{P}} = \sum_a \mathbf{P}^{(a)}$, $\tilde{\mathbf{X}} = \sum_a \mu_a \mathbf{X}^{(a)}$, $\Delta\mathbf{P}^a = \mathbf{P}^{(a)} - \mu_a\tilde{\mathbf{P}}$, $\Delta\mathbf{X}^{(a)} = \mathbf{X}^{(a)} - \tilde{\mathbf{X}}$, $\mu_a = m_a/M$, $M = \sum_a m_a$) is as follows

$$\{\tilde{X}_i, \tilde{X}_j\} = t\sum_a \frac{\mu_a^2}{\kappa_a}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right), \tag{191}$$

$$\{\tilde{X}_i, \tilde{P}_j\} = \delta_{ij}, \quad \{\tilde{P}_i, \tilde{P}_j\} = 0 \tag{192}$$

$$\{\Delta X_i^{(a)}, \Delta X_j^{(b)}\} = t\left(\frac{\delta^{ab}}{\kappa_a} - \frac{\mu_a}{\kappa_a} - \frac{\mu_b}{\kappa_b} + \sum_c \frac{\mu_c^2}{\kappa_c}\right) \times \left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right), \tag{193}$$

$$\{\Delta X_i^{(a)}, \Delta P_j^{(b)}\} = \delta_{ab} - \mu_b, \tag{194}$$

$$\{\Delta X_i^{(a)}, \tilde{X}_j\} = t\left(\frac{\mu_a}{\kappa_a} - \sum_c \frac{\mu_c^2}{\kappa_c}\right) \times \left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right), \tag{195}$$

$$\{\Delta P_i^{(a)}, \Delta P_i^{(b)}\} = \{\tilde{P}_i, \Delta P_j^{(b)}\} = 0. \tag{196}$$

The Poisson brackets for coordinates of the center-of-mass and coordinates of the relative motion vanish

$$\{\Delta X_i^{(a)}, \tilde{X}_j\} = 0, \tag{197}$$

if the parameters of the noncommutative algebra are determined by mass as (187) [64]. Namely, if relation $\kappa_a = m_a\gamma_\kappa$ is satisfied. Also, in this case the effective parameter of noncommutativity depends on the total mass of the system and is independent of its composition

$$\tilde{\theta}^0_{ij} = \sum_a \frac{\mu_a^2}{\kappa_a}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{j\tau}\delta_{i\rho}\right) = \frac{1}{\kappa_{eff}}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right) = \frac{1}{\gamma_\kappa M}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{j\tau}\delta_{i\rho}\right). \quad (198)$$

Let us study the motion of a composite system of mass M in a gravitational field in the space with the Lie algebraic noncommutativity (182), (183) on the basis of the obtained results. The Hamiltonian reads

$$H = \frac{\tilde{\mathbf{P}}^2}{2M} + MV(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) + H_{\text{rel}}. \quad (199)$$

The term H_{rel} corresponds to the relative motion, $\tilde{X}_i$, $\tilde{P}_i$ are coordinates and momenta of the center-of-mass of the composite system that are defined in the traditional way.

Considering the condition on the parameter of noncommutative algebra (187), we have (197) and

$$\left\{\frac{\tilde{\mathbf{P}}^2}{2M} + MV(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3), H_{\text{rel}}\right\} = 0, \quad (200)$$

So, for a composite system we can write the following equations of motion in the gravitational field

$$\dot{\tilde{X}}_i = \frac{\tilde{P}_i}{M} + tM\sum_a \frac{\mu_a^2}{\kappa_a}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right)\frac{\partial V}{\partial \tilde{X}_j} = \tilde{P}'_i + t\sum_a \frac{1}{\gamma_\kappa}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right)\frac{\partial V}{\partial \tilde{X}_j}, \quad (201)$$

$$\dot{\tilde{P}}_i = -M\frac{\partial V}{\partial \tilde{X}_i} = -\frac{\partial V}{\partial \tilde{X}_i}, \quad (202)$$

here $\tilde{P}'_i = \tilde{P}_i/M$. From equations (201), (202) follows that expressions for $\tilde{X}_i(t)$, $\tilde{P}'_i(t)$ do not depend on the mass of the composite system or its composition. So, the weak equivalence principle is recovered within the frame of the algebra (182), (183) due to condition (187) [64].

B. Preserving of the weak equivalence principle in the general case of the Lie algebraic noncommutativity

In a more general case of the noncommutative algebra of the Lie type, the Poisson brackets are as follows

$$\{X_i, X_j\} = \theta^0_{ij}t + \theta^k_{ij}X_k, \quad (203)$$

$$\{X_i, P_j\} = \delta_{ij} + \bar{\theta}^k_{ij}X_k + \tilde{\theta}^k_{ij}P_k, \quad \{P_i, P_j\} = 0, \quad (204)$$

here $i, j, k = (1, 2, 3)$, θ^0_{ij}, θ^k_{ij}, $\bar{\theta}^k_{ij}$, $\tilde{\theta}^k_{ij}$ are constants, $\theta^0_{ij} = -\theta^0_{ji}$, $\bar{\theta}^k_{ij} = -\bar{\theta}^k_{ji}$, $\tilde{\theta}^k_{ij} = -\tilde{\theta}^k_{ij}$ [38]. These constants have to be chosen to satisfy the Jacobi identity. This issue was studied in [38]. The author of the paper considered the following algebras of the Lie type

$$\{X_k, X_\gamma\} = -\frac{t}{\kappa} + \frac{X_l}{\tilde{\kappa}}, \quad \{X_l, X_\gamma\} = \frac{t}{\kappa} - \frac{X_k}{\tilde{\kappa}}, \quad (205)$$

$$\{X_k, X_l\} = \frac{t}{\kappa}, \quad \{P_k, X_\gamma\} = \frac{P_l}{\tilde{\kappa}}, \quad (206)$$

$$\{P_l, X_\gamma\} = -\frac{P_k}{\tilde{\kappa}}, \quad \{X_i, P_j\} = \delta_{ij}, \quad (207)$$

$$\{X_\gamma, P_\gamma\} = 1, \quad \{P_{\text{M}}, P_n\} = 0, \quad (208)$$

and the second ones

$$\{X_k, X_\gamma\} = -\frac{t}{\kappa} + \frac{X_l}{\tilde{\kappa}}, \quad \{X_l, X_\gamma\} = \frac{t}{\kappa} - \frac{X_k}{\tilde{\kappa}}, \quad (209)$$

$$\{X_k, X_l\} = 0, \quad \{P_k, X_\gamma\} = \frac{X_l}{\bar{\kappa}} + \frac{P_l}{\tilde{\kappa}}, \quad (210)$$

$$\{P_l, X_\gamma\} = \frac{X_k}{\bar{\kappa}} - \frac{P_k}{\tilde{\kappa}}, \quad \{X_i, P_j\} = \delta_{ij}, \quad (211)$$

$$\{X_\gamma, P_\gamma\} = 1, \quad \{P_{\text{M}}, P_n\} = 0, \quad (212)$$

The algebras correspond to the cases when parameters of noncommutativity satisfy the following relations

$$\theta^0_{kl} = -\theta^0_{k\gamma} = \frac{1}{\kappa}, \quad \theta^0_{l\gamma} = \frac{1}{\kappa}, \quad (213)$$

$$\theta^l_{k\gamma} = -\theta^k_{l\gamma} = \tilde{\theta}^l_{k\gamma} = -\tilde{\theta}^k_{l\gamma} = \frac{1}{\tilde{\kappa}}, \quad (214)$$

and

$$\theta^0_{l\gamma} = -\theta^0_{k\gamma} = \frac{1}{\kappa}, \quad \theta^l_{k\gamma} = -\theta^k_{l\gamma} = \frac{1}{\tilde{\kappa}}, \quad (215)$$

$$\tilde{\theta}^l_{k\gamma} = -\tilde{\theta}^k_{l\gamma} = \frac{1}{\tilde{\kappa}}, \quad (216)$$

$$\bar{\theta}^l_{k\gamma} = -\bar{\theta}^k_{l\gamma} = \frac{1}{\bar{\kappa}}, \quad (217)$$

respectively.

For a particle in a gravitational field (184) taking into account (203), (204), we obtain that the equations of motion depend on mass

$$\dot{X}_i = \frac{P_i}{m} + \bar{\theta}^k_{ij}\frac{P_jX_k}{m} + \tilde{\theta}^k_{ij}\frac{P_jP_k}{m} + m(\theta^0_{ij}t + \theta^k_{ij}X_k)\frac{\partial V}{\partial X_j}, \quad (218)$$

$$\dot{P}_i = -m\frac{\partial V}{\partial X_i} - m(\bar{\theta}^k_{ij}X_k + \tilde{\theta}^k_{ij}P_k)\frac{\partial V}{\partial X_j}. \quad (219)$$

Due to the dependence of the parameters of noncommutativity on mass proposed in [64]

$$\theta^{0(a)}_{ij}m_a = \gamma^0_{ij} = \text{const}, \quad \theta^{k(a)}_{ij}m_a = \gamma^k_{ij} = \text{const}, \quad (220)$$

$$\tilde{\theta}^{k(a)}_{ij}m_a = \tilde{\gamma}^k_{ij} = \text{const}, \quad \bar{\theta}^{k(a)}_{ij} = \bar{\theta}^k_{ij}. \quad (221)$$

we obtain

$$\dot{X}_i = P'_i + \bar{\theta}^k_{ij} P'_j X_k + \tilde{\gamma}^k_{ij} P'_j P'_k + (\gamma^0_{ij} t + \gamma^k_{ij} X_k) \frac{\partial V}{\partial X_j}, \quad (222)$$

$$\dot{P}'_i = -\frac{\partial V}{\partial X_i} - (\bar{\theta}^k_{ij} X_k + \tilde{\gamma}^k_{ij} P'_k) \frac{\partial V}{\partial X_j}. \quad (223)$$

Here constants γ^0_{ij}, γ^k_{ij}, $\tilde{\gamma}^k_{ij}$ do not depend on mass $\gamma^0_{ij} = -\gamma^0_{ji}$, $\gamma^k_{ij} = -\gamma^k_{ji}$, $\tilde{\gamma}^k_{ij} = -\tilde{\gamma}^k_{ji}$, $P'_i = P_i/m$. So, if conditions (220), (221) hold, the weak equivalence principle is preserved in a noncommutative space of the Lie type (203), (204).

Let us also study the motion of a composite system in a gravitational field in the space (203), (204) and examine the weak equivalence principle. The noncommutative algebra (203), (204) can be generalized for coordinates and momenta of different particles $X^{(a)}_i$, $P^{(a)}_i$ (index a label a particle) as

$$\{X^{(a)}_i, X^{(b)}_j\} = \delta_{ab}\theta^{0(a)}_{ij} t + \delta_{ab}\theta^{k(a)}_{ij} X^{(a)}_k, \quad (224)$$

$$\{X^{(a)}_i, P^{(b)}_j\} = \delta_{ab}\delta_{ij} + \delta_{ab}\bar{\theta}^{k(a)}_{ij} X^{(a)}_k + \delta_{ab}\tilde{\theta}^{k(a)}_{ij} P^a_k, \quad (225)$$

$$\{P^{(a)}_i, P^{(b)}_j\} = 0, \quad (226)$$

$\theta^{0(a)}_{ij}$, $\theta^{k(a)}_{ij}$, $\bar{\theta}^{k(a)}_{ij}$, $\tilde{\theta}^{k(a)}_{ij}$ are parameters of the noncommutative algebra corresponding to a particle with index a [64]. The relations of the noncommutative algebra for coordinates and momenta of the center-of-mass read

$$\{\tilde{X}_i, \tilde{X}_j\} = \sum_a \mu^2_a \theta^{0(a)}_{ij} t + \sum_a \mu^2_a \theta^{k(a)}_{ij} X^{(a)}_k, \quad (227)$$

$$\{\tilde{X}_i, \tilde{P}_j\} = \delta_{ij} + \sum_a \mu_a \bar{\theta}^{k(a)}_{ij} X^{(a)}_k + \sum_a \mu_a \tilde{\theta}^{k(a)}_{ij} P^a_k, \quad (228)$$

$$\{\tilde{P}_i, \tilde{P}_j\} = 0. \quad (229)$$

Note that the relations (227), (229) do not reproduce relations of the Lie algebra (203)–(204). In the right-hand side of (227), (228), we do not have coordinates and momenta of the center-of-mass. It is important to mention that the problem is solved due to conditions (220), (221) [64]. For coordinates and momenta of the center-of-mass, one obtains relations of the noncommutative algebra of the Lie type

$$\{\tilde{X}_i, \tilde{X}_j\} = \theta^{0(\text{eff})}_{ij} t + \theta^{k(\text{eff})}_{ij} \tilde{X}_k, \quad (230)$$

$$\{\tilde{X}_i, \tilde{P}_j\} = \delta_{ij} + \bar{\theta}^k_{ij} \tilde{X}_k + \tilde{\theta}^{k(\text{eff})}_{ij} \tilde{P}_k, \quad (231)$$

with parameters

$$\theta^{0(\text{eff})}_{ij} = \frac{\gamma^0_{ij}}{M}, \quad \theta^{k(\text{eff})}_{ij} = \frac{\gamma^k_{ij}}{M}, \quad \tilde{\theta}^{k(\text{eff})}_{ij} = \frac{\tilde{\gamma}^k_{ij}}{M}, \quad (232)$$

here $M = \sum_a m_a$ is the total mass of the system [64].

So, on the basis of these results, one can write the equations of motion of a composite system in a gravitational field in a quantized space with algebra (203), (204). Introducing notation $\tilde{P}'_i = \tilde{P}_i/M$ for a composite system in a gravitational field, we find

$$\dot{\tilde{X}}_i = \tilde{P}'_i + \left(\bar{\theta}^k_{ij} \tilde{X}_k + \tilde{\gamma}^k_{ij} \tilde{P}'_k\right) \tilde{P}'_j + \left(\gamma^0_{ij} t + \gamma^k_{ij} \tilde{X}_k\right) \frac{\partial V}{\partial \tilde{X}_j}, \quad (233)$$

$$\dot{\tilde{P}}'_i = -\frac{\partial V}{\partial \tilde{X}_i} - \left(\bar{\theta}^k_{ij} \tilde{X}_k + \tilde{\gamma}^k_{ij} \tilde{P}'_k\right) \frac{\partial V}{\partial \tilde{X}_j}. \quad (234)$$

Writing (233), (234), we assume that the influence of the relative motion on the motion of the center-of-mass of the system can be neglected. Equations of motion of a composite system in a gravitational field (233), (234) do not depend on its total mass, masses of particles forming it, its composition. So, the weak equivalence principle is preserved in a general case of the noncommutative algebra of the Lie type (203), (204) due to relations (220), (221) [64].

VI. CONCLUSIONS

We have examined quantum spaces with different deformed Heisenberg algebras (noncommutative algebra of a canonical type, noncommutative algebra of the Lie type, the Snyder algebra, the Kempf algebra and their generalizations). The motion of a particle in a gravitational field has been studied within frame of the deformed algebras and the implementation of the weak equivalence principle has been analyzed.

We have concluded that different types of deformation of the commutation relations for coordinates and momenta (canonical, Lie and nonlinear deformations) lead to the dependence of the motion of a particle (composite system) in a gravitational filed on mass and its composition. Therefore, the weak equivalence principle is violated. The principle is violated even in the case when the gravitational mass is equal to the inertial mass. It is worth stressing that the deformation of the algebra leads to a great violation of the principle, which can be easily seen in an experiment. But from the observations we know that the weak equivalence principle is preserved with hight accuracy. The problem is solved if one considers parameters of deformed algebras to be dependent on mass. In this case, the motion of a particle (composite system) in a gravitational field does not depend on its mass and composition or the weak equivalence principle is recovered.

It is important to add that the idea to relate parameters of deformed algebra to mass is also important for preserving the properties of the kinetic energy (additivity property, independence of composition) in a quantum space and, therefore for recovering the law of conservation of energy. Also, in the case when parameters of the deformed algebra depend on mass, the problem of description of the motion of a composite system in a space with minimal length is solved. The problem is well known in the literature as the soccer-ball problem.

So, the idea of the dependence of parameters of deformed algebras on mass leads to solving fundamental problems in a space with minimal length, among them violation of the weak equivalence principle, nonadditivity of the kinetic energy and its dependence on composition, the soccer-ball problem.

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ДЕФОРМОВАНІ АЛҐЕБРИ ГАЙЗЕНБЕРҐА РІЗНИХ ТИПІВ ЗІ ЗБЕРЕЖЕНИМ ПРИНЦИПОМ ЕКВІВАЛЕНТНОСТІ

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Розглянуто ідею опису квантованості простору (існування кванта довжини) за допомогою модифікації комутаційних співвідношень для операторів координат та операторів імпульсів. Вивчено різні типи деформації алґебри Гайзенберґа, а саме: канонічна (комутатори координат та імпульсів дорівнюють константам), типу Лі (комутатори координат та імпульсів пропорційні до координат та імпульсів) та нелінійна деформація (комутатори координат та імпульсів дорівнюють нелінійній функції цих координат та імпульсів). Досліджено некомутативну алґебру з некомутативністю координат та некомутативністю імпульсів канонічного типу, некомутативна алґебра типу Лі, алґебра Снайдера, алґебра Кемпфа та їх узагальнення в разі, коли комутатор координат та імпульсів дорівнює довільній функції, що залежить від імпульсів. У межах різних деформованих алґебр вивчено рух частинки (макроскопічного тіла) у ґравітаційному полі та проаналізовано виконання слабкого принципу еквівалентності. Показано, що у квантованому просторі рух у ґравітаційному полі залежить від маси та композиції. Параметр Етвеша не дорівнює нулеві, навіть якщо інерційна маса дорівнює ґравітаційній. Слабкий принцип еквівалентності порушується у квантованому просторі, причому деформація комутаційних співвідношень для операторів координат та операторів імпульсів зумовлює значні поправки до параметра Етвеша, які легко можна спостерігати в експерименті. З іншого боку, відповідно до експериментальних даних слабкий принцип еквівалентності виконується з великою точністю. Цю проблему можна розв'язати, припустивши, що параметри деформованих алґебр залежать від маси. Така ідея дає змогу відновити слабкий принцип еквівалентності, а також зберегти властивості кінетичної енерґії, розв'язати проблему опису руху макроскопічного тіла (ця проблема добре відома в літературі під назвою проблема футбольного м'яча) у квантованому просторі. Отже, залежність параметрів деформації від маси є важливою для побудови теорії квантованого простору зі збереженими фундаментальними законами та принципами.

Ключові слова: квантовий простір, мінімальна довжина, деформована алґебра Гайзенберґа, слабкий принцип еквівалентності, макроскопічне тіло, проблема футбольного м'яча, кінетична енерґія.