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DYNAMICS OF NEUTRAL AND CHARGED PARTICLE IN THE GRAVITATIONAL FIELD OF REISSNER–NORDSTRÖM BLACK HOLE IN THE PRESENCE OF EXTERNAL MAGNETIC FIELD AND DARK ENERGY

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I have studied the dynamics of a neutral and a charged particle around the Reissner–Nordström black hole surrounded by quintessence in the presence of an external magnetic field. The collision of a neutral particle moving around the black hole with another neutral particle has been discussed first. Then the collision of a charged particle moving around the black hole with another charged particle has been discussed. By numerical solution of the equations of motion it has been shown that after collision, if the particle gets sufficient energy, then it can escape to infinity. Dependence of orbits on the quintessence and the magnetic field for the charged particle has been studied in detail using the effective potential of the charged particle. Using the Lyapunov exponent effect of magnetic field and dark energy on the stability of the particle has been discussed. I have derived the expression for the center of mass energies of colliding charged particles moving around the black hole. Effective force on the charged particle due to dark energy and external magnetic field has been also discussed.

Key words: Euler–Lagrange equation, cyclic coordinate, escape velocity, center of mass energy and effective force.

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I. INTRODUCTION

In the history of the black hole astrophysics, it is very important to study the problem of the dynamics of a particle (massive or massless, charged or neutral) moving around a black hole. This type of problems helps us to understand the geometric structure of the space-time around a black hole [1,2]. High energy phenomena like formation of jets occurring near a black hole could be understood very nicely by the study of the dynamics of a particle moving around the black hole.

Different types of astrophysical observations, including the study of Supernova type Ia [3], cosmic microwave background (CMB) [4] and large scale structure (LSS) [5,6], indicate that the universe is flat and it has suffered two acceleration phases. The first one is the early acceleration phase (inflation) which occurred prior to the radiation dominated era and then again another accelerated expansion phase has started recently. These results are very nicely accommodated by introducing exotic matter which has large negative pressure, contrary to the ordinary baryonic matter. This exotic matter is called dark energy. In literature, the cosmological constant Λ is considered the simplest DE probe, which leads to the Λ CDM cosmology [7–9]. But it has not been very popular because of its extreme fine tuning and coincidence problem [10]. As a result, different types of dynamic dark energy having a (negative) variable equation of state are used widely to show recent accelerated expansion. Among them most popular candidates are quintessence, essence, tachyons, modifications of gravity, Chaplygin gas and others [11–16]. This dark energy constitutes about 78 percent of the energy density of the universe.

There exists a different type of external sources, such as a magnetic satellite (pulsar); plasma accreting onto the black hole can produce a magnetic field in the surrounding of the black hole [17–19]. The effect of a magnetic field near the event horizon is strong but it is not sufficient to disturb the geometry of the black hole. The presence of this magnetic field can highly influence the motion of the charged particle around the black hole [20]. This magnetic field can transfer the energy to the charged particle and due to this extra energy, there is a possibility for the charge particle to escape to spatial infinity. So in the presence of a magnetic field, collision among charged particles near a black hole can produce higher energy than in the absence of a magnetic field. Black holes with such type of magnetic field are known as weakly magnetized.

In [21–23] the influence of a magnetic field on a charged particle moving around a black hole has been investigated. Photon trajectory has been investigated in a modified gravity black hole in the presence of an axially symmetric magnetic field [24]. In literature different aspects of particle motion around the RN-black hole have been studied. In [25] critical escape velocity for a charged particle has been investigated around a weakly magnetized Schwarzschild black hole. In [26] the motion of a charged particle has been studied around weakly magnetized Reissner–Nordström black hole.

It is believed that black hole surrounded by dark energy plays important role in cosmology. Quintessence is one type of dark energy which is defined as scalar field coupled to gravity with the potential which decreases as field increases. Kiselev derived the solution for a spherically symmetric black hole surrounded by quintessence matter [27]. In [23] the dynamics of the charged particles has been studied around a Schwarschild — like black hole in the presence of Quintessence and magnetic field. In [28] the null geodesics has been investigated of the Reissner–Nordström black hole surrounded by quintessence.

In my paper, dynamics of a charged particle moving around a weakly magnetized Reissner–Nordström black hole surrounded by quintessence has been studied. The outline of the paper is as follows. In Section II, my model is explained and the dynamics of a neutral particle is studied. In this section, collision of a neutral particle moving around the black hole with another particle is also discussed. In Section III, I discuss the dynamics of a charged particle. The dimensionless form of equations is derived in Section IV. In this section, collision of a charged particle moving around the black hole with another charged particle is also discussed. Using the effective potential trajectories of the charged particle is discussed in Section V. After the collision, the trajectories of the charged particles are shown in this section. In Section VI, CME expressions for colliding particles are derived. The Lyapunov exponent is derived in Section VII. In Section VIII, effective force on a charged particle is calculated. In Section IX, concluding remarks are given.

II. DYNAMICS OF A NEUTRAL PARTICLE

Kiselev derived a static spherically symmetric exact solution of Einstein equations for a black hole surrounded by quintessence[27]. Kiselev expressed the metric of a charged black hole surrounded by quintessence as

$$ds^2 = -g(r)dt^2 + g(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad (1)$$

where

$$g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{c}{r^{3\omega_q+1}}, \quad (2)$$

where M and Q are the mass and charge of the black hole, ω_q is the quintessential state parameter of range $-1 < \omega_q < -\frac{1}{3}$, c is the positive normalization factor which is dependent on $\rho_q = -\frac{3c\omega_q}{2r^{3(1+\omega_q)}}$, where ρ_q is the density of quintessence which is positive.

Fernando [29] derived the geodesic structure of a massless particle of the Schwarzschild black hole surrounded by quintessence matter for $\omega_q = -\frac{2}{3}$. Mubasher Jamil [23] discussed the dynamics of a neutral and a charged particle around the Schwarzschild black hole in presence of quintessence and a magnetic field. Bushra Majeed [26] discussed the dynamics of a neutral and a charged particle around a weakly magnetized Reissner–Nordström black hole. B. Malakolkalami [28] studied the dynamics of a massless particle around the Reissner–Nordström black hole surrounded by quintessence for $\omega_q = -\frac{2}{3}$ and $Q^2 = \left(\frac{2}{27} \times \frac{\left(-1+9Mc+\sqrt{-(6Mc-1)^3}\right)}{c^2}\right)$. For this particular choice of the black hole charge parameter, it has been written in terms of the black hole mass parameter and the quintessence parameter.

In this paper, I am going to discuss the dynamics of a neutral and a charged particle around the Reissner–Nordström black hole in the presence of quintessence and magnetic field for $\omega_q = -\frac{2}{3}$ and $Q^2 = \left(\frac{2}{27} \times \frac{\left(-1+9Mc+\sqrt{-(6Mc-1)^3}\right)}{c^2}\right)$. Solving the equation $g(r) = 0$ for $\omega_q = -\frac{2}{3}$ and $Q^2 = \left(\frac{2}{27} \times \frac{\left(-1+9Mc+\sqrt{-(6Mc-1)^3}\right)}{c^2}\right)$ I get 3 roots as

$$r_1 = \left\{\frac{1}{3c} + \frac{\left(-(-1+6cM)^3\right)^{1/6}}{3c} - \frac{-729c^4 + 4374c^5M}{2187c^5\left(-(-1+6cM)^3\right)^{1/6}}\right\}, \quad (3)$$

$$r_2 = \left\{\frac{1}{3c} - \frac{(1+i\sqrt{3})\left(-(-1+6cM)^3\right)^{1/6}}{6c} + \frac{(1-i\sqrt{3})\left(-729c^4 + 4374c^5M\right)}{4374c^5\left(-(-1+6cM)^3\right)^{1/6}}\right\}, \quad (4)$$

$$r_3 = \left\{\frac{1}{3c} - \frac{(1-i\sqrt{3})\left(-(-1+6cM)^3\right)^{1/6}}{6c} + \frac{(1+i\sqrt{3})\left(-729c^4 + 4374c^5M\right)}{4374c^5\left(-(-1+6cM)^3\right)^{1/6}}\right\}. \quad (5)$$

Under the condition $0 < 6Mc < 1$, there exist two real roots. The reduced form of these two real roots are

$$r_{\rm in} = \frac{1}{3c} - \frac{1}{3}\sqrt{\frac{1-6cM}{c^2}} \quad \text{and} \quad r_{\rm out} = \frac{1}{3c} + \frac{2}{3}\sqrt{-\frac{-1+6cM}{c^2}},$$

where $r_{\rm out}$ is the event horizon. From the above two equations for $M = 1$ and $c = 0.1$ we get numerical values of $r_{\rm in}$ and $r_{\rm out}$ as $r_{\rm in} = 1.22515$ and $r_{\rm out} = 7.5497$. For $M = 1$ and $c = 0.1$, $g(r)$ verses r has been plotted in Fig. 1. For $M = 1$ and $c = 0.1$ solving $g(r) = 0$ we get 3 roots as $r_1 = 1.22515$, $r_2 = 1.22515$ and $r_3 = 7.5497$. So we can consider $r_1 = r_2 = r_{\rm in}$ and $r_3 = r_{\rm out}$.

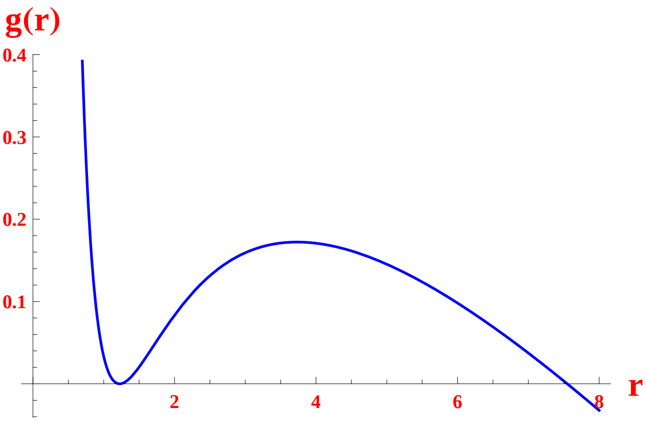


Fig. 1. $g(r)$ versus r for $M = 1$ and $c = 0.1$

First, I discuss the dynamics of a neutral particle in the absence of a magnetic field. In terms of Lagrangian mechanics, for the metric defined in (1) t and ϕ coordinates are cyclic, which leads to two conserved quantities, namely energy and angular momentum with the corresponding Noether symmetry generators $\xi_{(t)} = \xi^{\mu}_{(t)}\partial_\mu = \frac{\partial}{\partial t}$ and $\xi_{(\phi)} = \xi^{\mu}_{(\phi)}\partial_\mu = \frac{\partial}{\partial \phi}$ where $\xi^{\mu}_{(t)} = (1, 0, 0, 0)$ and $\xi^{\mu}_{\phi} = (0, 0, 0, 1)$. This shows that the black hole metric is invariant under time translation and rotation around symmetry axis. The corresponding conserved quantities are the energy E per unit mass and azimuthal angular momentum L_z per unit mass. Therefore, the equation of motion for the coordinates t and r can be written as

$$\dot{t} = \frac{E}{1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr} \tag{6}$$

and

$$\dot{\phi} = \frac{L_z}{r^2 \sin^2\theta}. \tag{7}$$

Since the metric is spherically symmetric, we can consider the motion of the particle in the equatorial plane $\theta = \frac{\pi}{2}$ and for the planar motion $\dot{\theta} = 0$. Therefore,

$$\dot{t} = \frac{E}{1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr} \tag{8}$$

and

$$\dot{\phi} = \frac{L_z}{r^2}. \tag{9}$$

Throughout in this paper, the over dot represents differentiation with respect to proper time τ. Using the normalization condition $u^\mu u_\mu = -1$, we get the equation of motion as

$$\begin{aligned} \dot{r}^2 &= E^2 - \left(1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr\right) r^2\dot{\theta}^2 \\ &- \left(1 + \frac{L_z^2}{r^2\sin^2\theta}\right)\left(1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr\right). \end{aligned} \tag{10}$$

From the above equation, effective potential can be defined as

$$U_{\text{eff}} \equiv \left(1 + \frac{L_z^2}{r^2\sin^2\theta}\right)\left(1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr\right).$$

For the circular motion $\dot{r} = 0$, hence equation (10) gives

$$E^2 = \left(1 + \frac{L_z^2}{r^2}\right)\left(1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr\right) \equiv U_{\text{eff}}. \tag{11}$$

Now, consider the particle in the inner most stable circular orbit r_0, where r_0 is the local minimum (which is also the convolution point) of the effective potential . The corresponding energy and azimuthal angular momentum are given by

$$L_{z0} = \left(\frac{{r_0}^2\left(2Mr_0 - 2Q^2 - c{r_0}^3\right)}{\left(2{r_0}^2 - 6Mr_0 + 4Q^2 - c{r_0}^3\right)}\right)^{\frac{1}{2}} \tag{12}$$

and

$$E_0 = \frac{\sqrt{2}\left({r_0}^2 - 2Mr_0 - Q^2 - c{r_0}^3\right)}{r_0\left(2{r_0}^2 - 6Mr_0 + 4Q^2 - c{r_0}^3\right)^{\frac{1}{2}}}. \tag{13}$$

Now, let us consider that the particle moving in an innermost stable circular orbit collides with another particle, so that after the collision the particle will move within a new plane tilted with respect to the original equatorial plane. In general, after the collision the particle will have a new integral of motion E, L_z and L. To simplify the problem and reduce the space of initial data to one parameter set, two restrictions are imposed. After the collision, the azimuthal angular momentum will not be changed and the initial radial velocity of the particle after the collision will remain the same, $\dot{r_0} = 0$. Under this restriction, only the new value of energy of the particle can determine the motion of the particle. As a result, after the collision the particle will get a velocity $v_\perp$ in the orthogonal direction to the equatorial plane. Therefore, after the collision the total angular momentum of the particle will be

$$L^2 = {r_0}^2{v_\perp}^2 + {L_{z0}}^2, \tag{14}$$

where L_{z0} is the angular momentum before collision, which is given in equation (12). The total energy of the particle after the collision, which we can determine from the equation (10), will be

$$E^2 = {E_0}^2 + {v_\perp}^2\left(1 + \frac{Q^2}{{r_0}^2} - \frac{2M}{r_0} - cr_0\right), \tag{15}$$

where E_0 is the total energy of the particle before the collision, which is given in equation (13), and $v_\perp$ is the velocity of the particle after the collision in the orthogonal direction of the equatorial plane. This velocity can be written as

$$
{v_\perp}^2 = \frac{E^2 - E_0^2}{\left(1 + \frac{Q^2}{{r_0}^2} - \frac{2M}{r_0} - cr_0\right)}. \tag{16}
$$

To study the final fate of the particle after the collision, this velocity is taken as the initial velocity and after the collision, the dynamics of the particle will be governed by the following dynamical equations:

$$
\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \frac{\cos\theta L_z^2}{r^4 \sin^3\theta}, \tag{17}
$$

$$
\begin{aligned}
\ddot{r} &= \dot{\theta}^2\left(r - 3M + \frac{2Q^2}{r} - \frac{cr^2}{2}\right) \\
&+ \frac{L_z^2}{r^2\sin^2\theta}\left(\frac{1}{r} - \frac{3M}{r^2} + \frac{2Q^2}{r^3} - \frac{c}{2}\right) \\
&- \frac{1}{2}\left(\frac{2M}{r^2} - \frac{2Q^2}{r^3} - c\right),
\end{aligned} \tag{18}
$$

$$
\dot{t} = \frac{E}{1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr} \tag{19}
$$

and

$$
\dot{\phi} = \frac{L_z}{r^2\sin^2\theta}. \tag{20}
$$

Here equations (17) and (18) have been derived from the following geodesic equation

$$
\ddot{x} + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0 \tag{21}
$$

and equation (19) and (20) are the first integrals of t and θ components of the equation (21). After the collision, there are three possibilities for the motion of the particle. The particle may moved on a bounded orbit or it may escape to infinity or it may be captured by the black hole. These three possibilities depend on the transferred energy and momentum. For small values of transferred energy and momentum, the orbits of the particle may be slightly perturbed but for larger values of $E - E_0$ the particle can go away from the initial plane and finally be captured by the black hole or escape to infinity.

III. DYNAMICS OF A CHARGED PARTICLE

Now, I investigate the effect of both the magnetic field in the black hole exterior and the gravitational field on the motion of the charged particle. The general killing vector equation is given by [30]

$$
\xi^\mu_{;\mu} = 0, \tag{22}
$$

where ξ^μ is a killing vector. The above equation coincides with the Maxwell equation for 4-potential A^μ in the Lorentz gauge $A^\mu_{;\mu} = 0$. I choose [31]

$$
A^\mu = \frac{\beta}{2}\xi^\mu_{(\phi)} \tag{23}
$$

as the test magnetic field, where β is the magnetic field strength. The 4-potential is invariant under the symmetries. Therefore,

$$
L_\xi A_\mu = A_{\mu,\nu}\xi^\nu + A_\nu \xi^\nu_\mu = 0. \tag{24}
$$

A magnetic field vector is defined as [32]

$$
\beta^\mu = -\frac{1}{2}\exp^{\mu\nu\lambda\sigma} F_{\lambda\sigma} u_\nu, \tag{25}
$$

where

$$
\exp^{\mu\nu\lambda\sigma} = \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}, \quad \epsilon_{0123} = 1, \quad g = \det(g_{\mu\nu}), \tag{26}
$$

$\epsilon^{\mu\nu\lambda\sigma}$ is the Levi Civita symbol. The Maxwell tensor is defined as

$$
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu;\mu} - A_{\mu;\nu}. \tag{27}
$$

The components of 4 velocity for a local observer at rest at the turning point $\dot{r} = 0$ are

$$
\begin{aligned}
u_0^\mu &= \frac{1}{\sqrt{1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr}}\xi^\mu_{(t)}, \\
u_1^\mu &= 0, \quad u_2^\mu = 0, \\
u_3^\mu &= \frac{1}{\sqrt{r^2\sin^2\theta}}\xi^\mu_{(\phi)}.
\end{aligned} \tag{28}
$$

From equations (25)–(28), the magnetic field is obtained as

$$
\beta^\mu = \beta\frac{1}{\sqrt{1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr}}\left(\cos\theta\delta^\mu_r - \frac{\sin\theta\delta^\mu_\theta}{r}\right). \tag{29}
$$

The Lagrangian of the moving particle of mass m and electric charge q in curved space time in the presence of an external magnetic field is given by [33]

$$
\mathcal{L} = \frac{1}{2}g_{\mu\nu}u^\mu u^\nu + \frac{q}{m}A_\mu u^\mu. \tag{30}
$$

Therefore, for the above Lagrangian,

$$
\dot{t} = \frac{E}{1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr} \tag{31}
$$

and

$$\dot{\phi} = \left(\frac{L_z}{r^2 \sin^2 \theta} - B \right), \tag{32}$$

where $B \equiv \frac{q\beta}{2m}$. At the equatorial plane ($\theta = \frac{\pi}{2}$), the above equations become

$$\dot{t} = \frac{E}{1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr} \tag{33}$$

and

$$\dot{\phi} = \left(\frac{L_z}{r^2} - B \right). \tag{34}$$

By the normalization condition ($u^\mu u_\mu = -1$) we can determine

$$\begin{aligned} E^2 &= \dot{r}^2 + \left(1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr\right) r^2 \dot{\theta}^2 \\ &+ \left(1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr\right) \\ &\times \left(1 + r^2 \sin^2 \theta \left(\frac{L_z}{r^2 \sin^2 \theta} - B \right)^2 \right). \end{aligned} \tag{35}$$

From the above equation, we can define effective potential as

$$\begin{aligned} U_{\text{eff}} &= \left(1 + \frac{Q^2}{r^2} - \frac{2M}{r} - cr\right) \\ &\times \left(1 + r^2 \sin^2 \theta \left(\frac{L_z}{r^2 \sin^2 \theta} - B \right)^2 \right). \end{aligned} \tag{36}$$

The equation of motion of a charged particle in an external electromagnetic field $F_{\mu\nu}$ is given by [26]

$$\ddot{x} + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = \frac{q}{m} F^\mu_{\ \alpha} \dot{x}^\alpha. \tag{37}$$

From equation (37) for the metric defined in (1), we get the dynamical equations for θ and r as

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} - B^2 \sin\theta \cos\theta + \frac{\cos\theta L_z^2}{r^4 \sin^3\theta} \tag{38}$$

and

$$\begin{aligned} \ddot{r} &= \dot{\theta}^2 \left(r - 3M + \frac{2Q^2}{r} - \frac{cr^2}{2} \right) + \frac{L_z^2}{r^2 \sin^2 \theta} \left(\frac{1}{r} - \frac{3M}{r^2} + \frac{2Q^2}{r^3} - \frac{c}{2}. \right) \\ &+ \left(BL_z - \frac{1}{2} \right) \left(\frac{2M}{r^2} - \frac{2Q^2}{r^3} - c \right) + B^2 \sin^2 \theta \left(-r + m + \frac{3cr^2}{2} \right). \end{aligned} \tag{39}$$

IV. DIMENSIONLESS FORM OF THE DYNAMICAL EQUATION

I shall numerically integrate the dynamical equations. For this purpose I introduce the following dimensionless quantities [32]:

$$\sigma = \frac{\tau}{2M}, \quad \rho = \frac{r}{2M}, \quad l = \frac{L_z}{2M}, \quad b = 2MB, \quad \bar{q} = \frac{Q}{2M}, \quad \bar{c} = 2Mc. \tag{40}$$

Using the above equations, the equations (38) and (39) can be written as

$$\frac{d^2\theta}{d\sigma^2} = -\frac{2}{\rho} + \frac{\cos\theta \ l^2}{\rho^4 \sin^3\theta} - b^2 \sin\theta \cos\theta \tag{41}$$

and

$$\begin{aligned} \frac{d^2\rho}{d\sigma^2} &= \left(\frac{d\theta}{d\sigma}\right)^2 \left(\rho - \frac{3}{2} + \frac{2\bar{q}^2}{\rho} \frac{\bar{c}\rho^2}{2} \right) + \frac{1}{2} b^2 \sin^2\theta \left(1 - 2\rho + 3\bar{c}\rho^2\right) \\ &+ \frac{\left(-\frac{1}{2} + bl\right)\left(-2\bar{q}^2 + \rho - \bar{c}\rho^3\right)}{\rho^3} + \frac{l^2 \left(4\bar{q}^2 - 3\rho + 2\rho^2 - \bar{c}\rho^3\right)}{2\rho^5 \sin^2\theta}. \end{aligned} \tag{42}$$

In the dimensionless form, equations (32), (35) and (36) become

$$\frac{d\phi}{d\sigma}=\left(\frac{l}{\rho^2\sin^2\theta}-b\right), \tag{43}$$

$$E^2=\left(\frac{d\rho}{d\sigma}\right)^2+\rho^2\left(\frac{d\theta}{d\sigma}\right)^2\left(1-\frac{1}{\rho}-\rho\bar{c}+\frac{\bar{q}^2}{\rho^2}\right)+\left(1+\frac{\left(l-b\rho^2\sin^2\theta\right)^2}{\rho^2\sin^2\theta}\right)\left(1-\frac{1}{\rho}-\rho\bar{c}+\frac{\bar{q}^2}{\rho^2}\right) \tag{44}$$

and

$$U_{\text{eff}}=\left(1+\frac{\left(l-b\rho^2\sin^2\theta\right)^2}{\rho^2\sin^2\theta}\right)\left(1-\frac{1}{\rho}-\rho\bar{c}+\frac{\bar{q}^2}{\rho^2}\right). \tag{45}$$

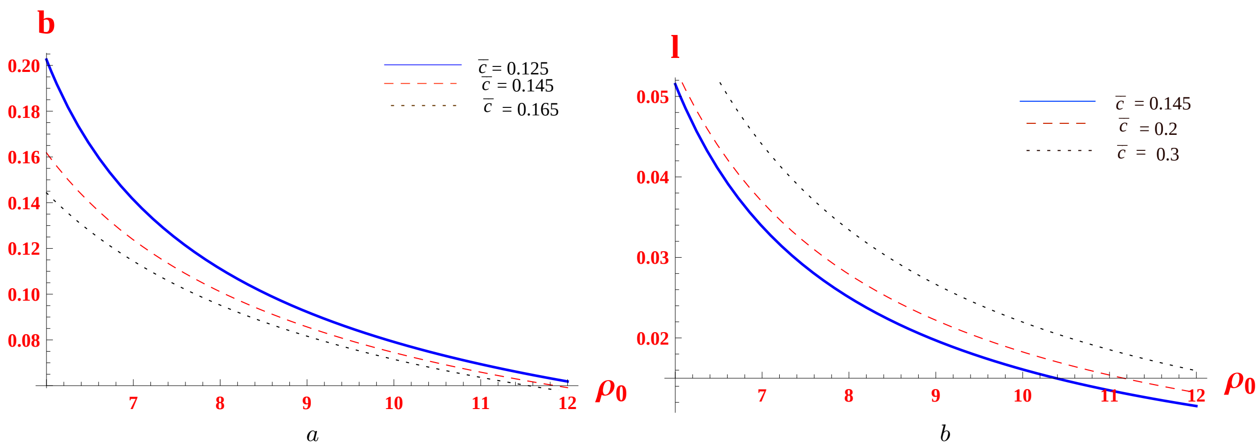


Fig. 2. a: Magnetic field b as a function of ρ_0 for different value of $\bar{c}$; b: Angular momentum l as a function of ρ_0 for different value of $\bar{c}$

Again, I consider the collision of a charged particle moving in an innermost stable circular orbit of radius ρ_0 with another charged particle. The collision mechanism is the same as in the case of a neutral particle. The energy of the charged particle moving in an innermost stable circular orbit of radius ρ_0 around the black hole at the equatorial plane $\theta=\frac{\pi}{2}$ is given by

$$E_0^2=\left(1+\frac{\left(l-b\rho_0^2\right)^2}{\rho_0^2}\right)\left(1-\frac{1}{\rho_0}-\rho_0\bar{c}+\frac{\bar{q}^2}{\rho_0^2}\right). \tag{46}$$

The first and second derivatives of the effective potential at the point $\rho=\rho_0$ can be written as

$$\left[\frac{dU_{\text{eff}}}{d\rho}\right]_{\rho=\rho_0}=\left(1+\frac{\bar{q}^2}{\rho_0^2}-\frac{1}{\rho_0}-\bar{c}\rho_0\right)\left(-\frac{4b\left(l-b\rho_0^2\right)}{\rho_0}-\frac{2\left(l-b\rho_0^2\right)^2}{\rho_0^3}\right)+\left(-\bar{c}-\frac{2\bar{q}^2}{\rho_0^3}+\frac{1}{\rho_0^2}\right)\left(1+\frac{\left(l-b\rho_0^2\right)^2}{\rho_0^2}\right) \tag{47}$$

and

$$\left[\frac{d^2U_{\text{eff}}}{d\rho^2}\right]_{\rho=\rho_0}=\left(1+\frac{\bar{q}^2}{\rho_0^2}-\frac{1}{\rho_0}-\bar{c}\rho_0\right)\left(8b^2+\frac{12b\left(l-b\rho_0^2\right)}{\rho_0^2}+\frac{6\left(l-b\rho_0^2\right)^2}{\rho_0^4}\right)$$

$$+2\left(-\bar{c}-\frac{2\bar{q}^2}{\rho_0^3}+\frac{1}{\rho_0^2}\right)\left(-\frac{4b\left(l-b\rho_0^2\right)}{\rho_0}-\frac{2\left(l-b\rho_0^2\right)^2}{\rho_0^3}\right)+\left(\frac{6\bar{q}^2}{\rho_0^4}-\frac{2}{\rho_0^3}\right)\left(1+\frac{\left(l-b\rho_0^2\right)^2}{\rho_0^2}\right). \tag{48}$$

Since the particle is moving in the innermost stable circular orbit of radius ρ_0, we can write the magnetic field strength b and the angular momentum l in terms of ρ_0 from the equations $\left[\frac{dU_{\text{eff}}}{d\rho}\right]_{\rho=\rho_0} = 0$ and $\left[\frac{d^2U_{\text{eff}}}{d\rho^2}\right]_{\rho=\rho_0} = 0$ as

$$b = \left(\frac{\alpha\left(\beta\lambda+\gamma+\delta\sqrt{\lambda}\right)}{(\beta\lambda+\gamma)^2-\delta^2}\right)^{\frac{1}{2}} \tag{49}$$

and

$$l = \sqrt{\lambda}\left(\frac{\alpha\left(\beta\lambda+\gamma+\delta\sqrt{\lambda}\right)}{(\beta\lambda+\gamma)^2-\delta^2}\right)^{\frac{1}{2}}, \tag{50}$$

where

$$\alpha = -2\bar{q}^2\rho_0^2+\rho_0^3-\bar{c}\rho_0^5,$$

$$\beta = -4\bar{q}^2+3\rho_0-2\rho_0^2+\bar{c}\rho_0^3,$$

$$\gamma = -\rho_0^5+2\rho_0^6-3\bar{c}\rho_0^7$$

$$\delta = -4\bar{q}^2\rho_0^2+2\rho_0^3-2\bar{c}\rho_0^5,$$

$$\lambda = \frac{-\rho_0+3\rho_0^2-6\bar{c}\rho_0^3-\bar{c}\rho_0^4+3\bar{c}^2\rho_0^5+\bar{q}^2\left(3-8\rho_0+15\bar{c}\rho_0^2\right)}{3\rho_0^2-\rho_0^3+\bar{q}^2\left(8\bar{q}^2-9\rho_0+11\bar{c}\rho_0^3\right)-\bar{c}\left(6\rho_0^4+3\rho_0^5-\rho_0^6\right)},$$

$$\bar{q} = \left(\frac{\frac{2}{27}\left(-1+\frac{9}{2}\bar{c}+\sqrt{-\left(3\bar{c}-1\right)^3}\right)}{\bar{c}^2}\right)^{\frac{1}{2}}.$$

We can determine the total energy of the charged particle after the collision from the equation (44) as

$$E^2 = {E_0}^2+{v_\perp}^2\left(1-\frac{1}{\rho_0}-\rho_0\bar{c}+\frac{\bar{q}^2}{\rho_0^2}\right), \tag{51}$$

where E_0 is the total energy before the collision, which is given in equation (46), and $v_\perp$ is the velocity of the charged particle just after the collision in the orthogonal direction of the equatorial plane. $v_\perp$ can be written as

$${v_\perp}^2 = \frac{E^2-E_0^2}{\left(1-\frac{1}{\rho_0}-\rho_0\bar{c}+\frac{\bar{q}^2}{\rho_0^2}\right)}. \tag{52}$$

To study the final fate of the charged particle, this velocity is taken as initial velocity and the motion of the charged particle after the collision will be governed by the equations (41), (42) and (43). If we know the radius ρ_0 for the innermost stable circular orbit and the total energy E of the charged particle after the collision, we can determine the final fate of the charged particle after the collision. After the collision, there are three possibilities for the motion of the charged particle. The particle may move on a bounded orbit or it may escape to infinity or it may be captured by the black hole. These three possibilities depend on the value of E and ρ_0.

In Fig. 2,a, I have plotted the magnetic field strength b as a function of ρ_0 for different value of $\bar{c}$. It can be concluded from the figure that a large value of dark energy decreases the magnetic field strength and the magnetic field strength decreases with the distance from the black hole.

In Fig. 2,b, angular momentum l versus ρ_0 is plotted. From the figure, it can be concluded that a large value of dark energy increases the angular momentum l and angular momentum l decreases with the distance from the black hole.

V. TRAJECTORIES FOR EFFECTIVE POTENTIAL AND ESCAPE VELOCITY

The behavior of effective potential is demonstrated by plotting it with respect to ρ in Fig. $3a$. Since the motion of the particles depends on the energy levels, then from Fig. $3a$ we can consider four cases according to different values of $E(E_1, E_2, E_3, E_4)$ for the motion as follows:

First, I consider the value of the energy $E = E_1$. In this case, if the particle starts the motion at $\rho > \rho_1$, it will be closer ρ until it reaches to $\rho = \rho_1$ and will rebound from $\rho = \rho_1$ to the infinity. Therefore, for all values of ρ the photons will never fall into the black hole.

When the energy of the particle is E_2 and $\dot{\rho} = 0$, then the orbit is circular and unstable at $\rho = \rho_c$. If the particle starts the motion at $\rho > \rho_c$, it moves in an unstable circular orbit at $\rho = \rho_c$, and if it starts the motion at $\rho_2 < \rho < \rho_c$, then it approaches $\rho = \rho_2$ and flies back, moving in an unstable circular orbit at $\rho = \rho_c$.

In the case of $E = E_3$, if the particle starts the motion at $\rho > \rho_5$, it will go to a minimum radius and then escape to the infinity. If the particle starts the motion at $\rho_3 < \rho < \rho_4$, then it will move to $\rho = \rho_3$ and fly back to $\rho = \rho_4$ and oscillate between ρ_3 and ρ_4.

Next, in the case of $E = E_4$, if the particle starts the motion at $\rho > \rho_{10}$, it will fall to a minimum radius and then escape to the infinity. If the particle starts the motion at $\rho_8 < \rho < \rho_9$ then it will move to $\rho = \rho_8$ and then fly back to $\rho = \rho_9$ and oscillate between ρ_8 and ρ_9. If the particle starts the motion at $\rho_c < \rho < \rho_7$ then it will move to $\rho = \rho_7$ and then fly back to $\rho = \rho_6$ and oscillate between ρ_6 and ρ_7. In Fig. 3,a, $U_{\min 1}$ corresponds to the innermost stable circular orbit and $U_{\min 2}$ corresponds to a stable circular orbit. $U_{\max 1}$ and $U_{\max 2}$ correspond to an unstable circular orbit.

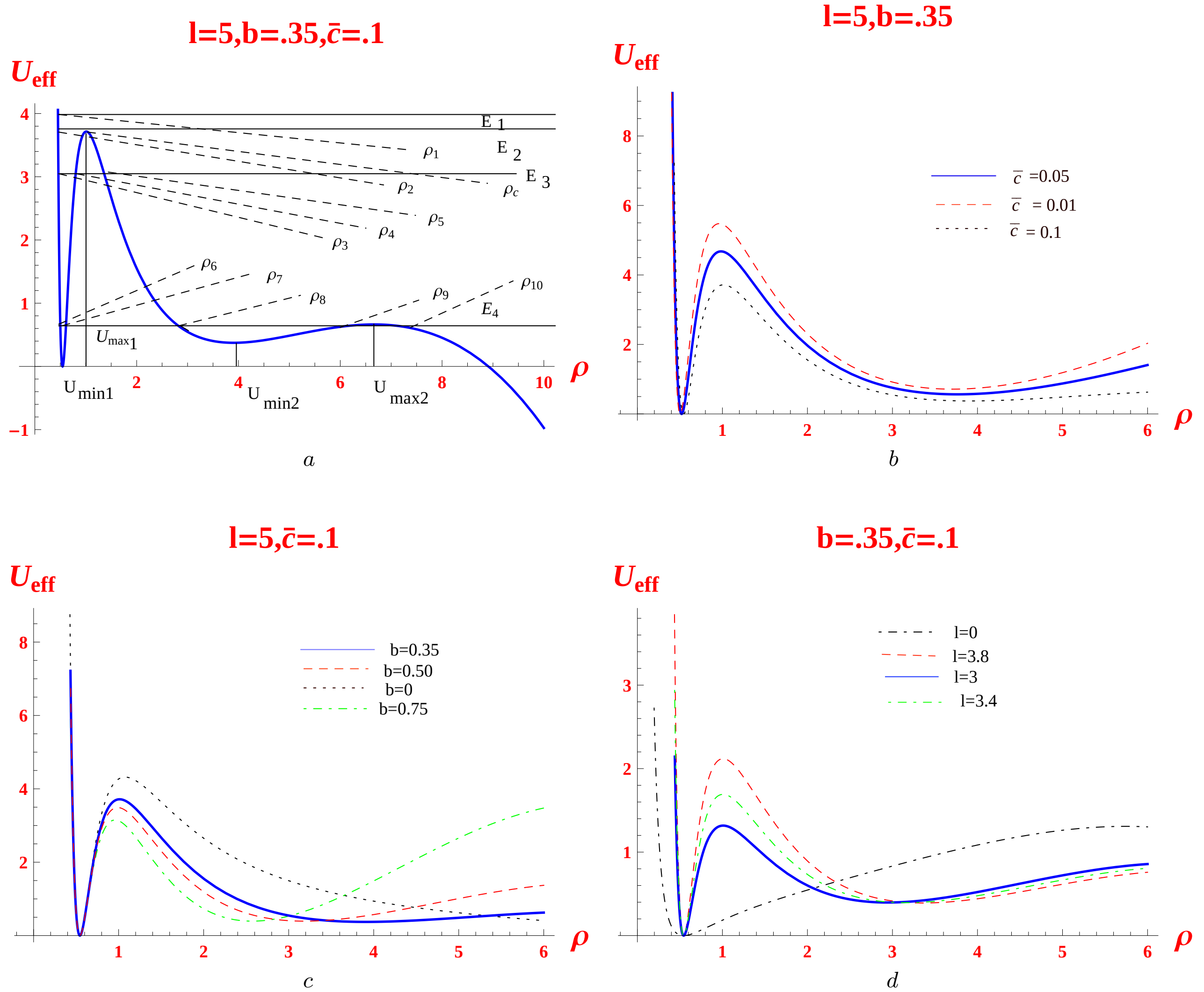
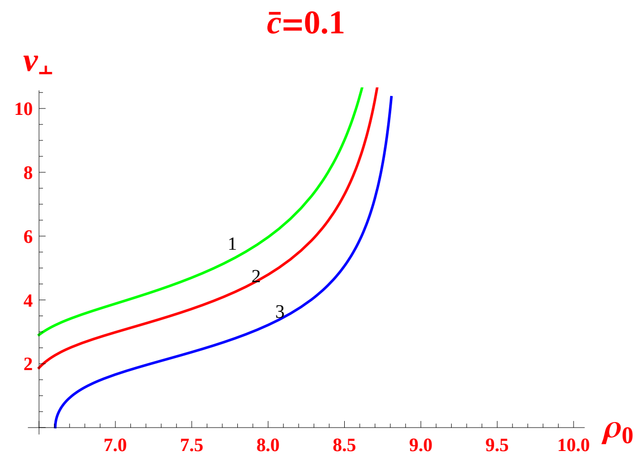


Fig. 3. a: Effective potential $U_{\rm eff}$ as a function of ρ;
b: Effective potential $U_{\rm eff}$ as a function of ρ for different values of $\bar{c}$;
c: Effective potential $U_{\rm eff}$ as a function of ρ for different value of magnetic field strength b;
d: Effective potential $U_{\rm eff}$ as a function of ρ for different value of angular momentum l

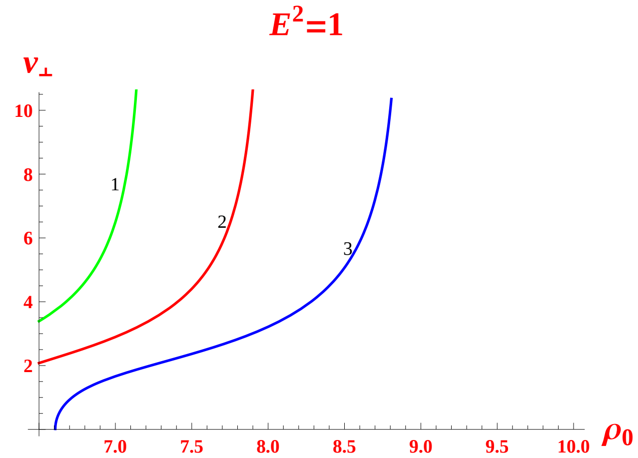
In Fig. 3,b, effective potential versus ρ has been plotted for different values of $\bar{c}$. For $\bar{c} = 0.01$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.503807$, 0.962903 and 3.70573, respectively. For $\bar{c} = 0.05$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.520304$, 0.983307 and 3.75975, respectively. For $\bar{c} = 0.1$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.544467$, 1.01143 and 3.90575, respectively. The minimum and maximum values of effective potential correspond to the stable circular orbit and unstable circular orbit, respectively. For an increasing value of the quintessence parameter, both the stable and the unstable circular orbits move away from the horizon. Also from the figure we can see that for an increasing value of the quintessence parameter effective potential decreases.

In Fig. 3,c, effective potential versus ρ has been plotted for different values of magnetic field b. For $b = 0.35$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.544467$, 1.01143 and 3.90575, respectively. For $b = 0.50$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.544467$, $0.989431$and 3.18512, respectively. For $b = 0.75$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.544467$, 0.957784 and 2.57065, respectively. We can observe from Fig. 3,c that orbits are more stable in the presence of magnetic field compared to the case when a magnetic field is absent, that is $b = 0$. It can also be seen that as the value of b increases, the first minimum of effective potential that is the innermost stable circular orbit is unchanged but the second minimum of the effective potential, which corresponds to a stable circular orbit, is shifting towards the horizon.

In Fig. 3,d, I have plotted effective potential versus ρ for different values of angular momentum l. For $l = 3$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.544467$, 1.00903 and 2.94107, respectively. For $l = 3.4$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.544467$, 1.00767 and 3.16082, respectively. For $l = 3.8$, the first minimum, the first maximum and the second minimum of effective potential are at $\rho = 0.544467$, 1.00783 and 3.36408, respectively. For an increasing value of the angular momentum, the position of the first minimum of effective potential is unchanged but the second minimum moves away from the horizon. From this figure, we can conclude that the particle having a larger value of angular momentum l has more possibility to escape compared to the particle with a lower value of angular momentum l.



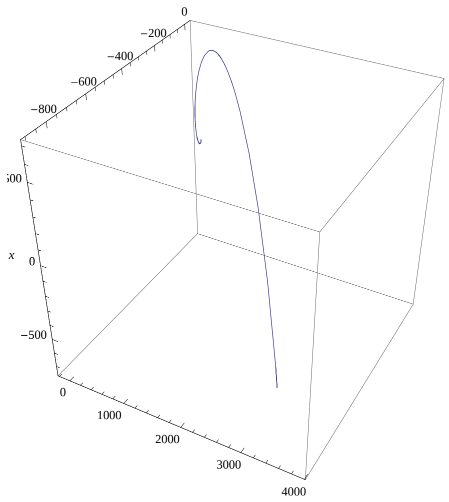
a



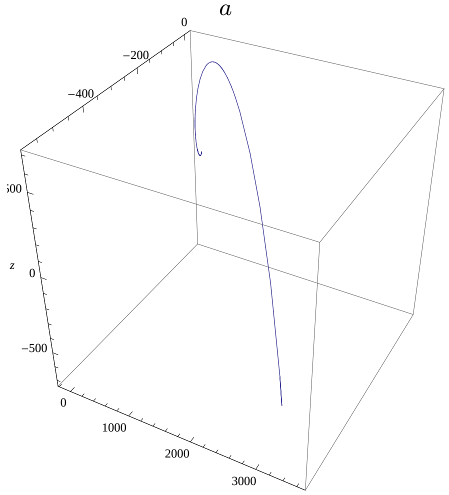
b

Fig. 4. *a*: Velocity of the charged particle after collision as a function of ρ_0 for different values of energy. Plots 1, 2, 3 correspond to the energy $E^2 = 3, 2, 1$;
b: Velocity of the charged particle after collision as a function of ρ_0 for different values of $\bar{c}$. Plots 1, 2, 3 correspond to $\bar{c} = 0.12, 0.11, 0.1$

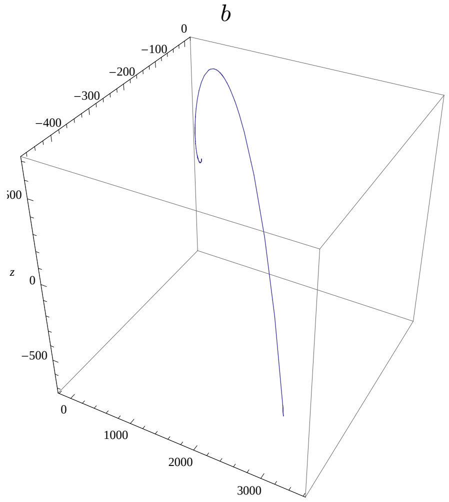
In Fig. 4,a, I have plotted the velocity of the charged particle after collision for different values of energy E. From this figure, we can see that the particle having greater energy has more possibility to escape to infinity from the vicinity of the black holes compared to the particle with a lesser value of energy. Also for a larger value of ρ_o, which corresponds to the innermost stable circular orbit, the possibility to escape from the vicinity of a black hole also increases.



a



b



c

Fig. 5. *a*: Trajectory of the charged particle after collision for $E^2 = 1$, $\bar{c} = 0.1$, $\rho_0 = 8.9$.
b: Trajectory of the charged particle after collision for $E^2 = 2$, $\bar{c} = 0.1$, $\rho_0 = 8.9$.
c: Trajectory of the charged particle after collision for $E^2 = 3$, $\bar{c} = 0.1$, $\rho_0 = 8.9$.

The velocity of the charged particle after collision for different values of $\bar{c}$ has been plotted in Fig. 4,b. It shows that for a greater value of $\bar{c}$, the velocity of the charged particle after collision will be greater; that is the possibility to escape to infinity increases.

By solving numerically the equations (41), (42) and (43), I have plotted the trajectories of the charged particle after collision in Fig. 5,a, Fig. 5,b and Fig. 5,c for different values of energy E. The initial velocity of the charged particle after collision has been determined from the equation (52). In Fig. 5,a, the trajectory of the charged particle after collision has been given for the initial velocity $v_\perp = 3.59409$ and the range of σ has been taken [0,2]. In Fig. 5,b, the trajectory of the charged particle after collision has been given for the initial velocity $v_\perp = 5.0853$ and the range of σ has been taken [0,1.5]. In Fig. 5,c, the trajectory of the charged particle after collision has been given for the initial velocity $v_\perp = 6.22921$ and the range of σ has been taken [0, 1.2]. From these trajectories, we can observe that at the time of collision, if the charged particle gets sufficient energy, then the charged particle after collision can escape to infinity from the vicinity of the black hole.

VI. CENTER OF MASS ENERGY OF A COLLIDING PARTICLE

First, I consider the collision of two neutral particles of masses m_1 and m_2 near a black hole coming from infinity in the absence of a magnetic field. The collision energy of the particles of masses $m_1 = m_2 = m_0$ in the center of mass frame is defined as [34]

$$E_{\rm cm} = m_0\sqrt{2}\sqrt{1 - g_{\mu\nu}u_1^\mu u_2\nu}, \tag{53}$$

where

$$u_i^\mu \equiv \frac{dx^\mu}{d\tau}, \qquad i = 1, 2 \tag{54}$$

is the four velocity of the particles. The CME for neutral particles, falling freely from the rest at infinity is given by

$$\begin{aligned} E_{\rm cm}^2 &= 2\left(1 + \frac{r^2E_1E_2}{Q^2 - cr - 2Mr + r^2} - \frac{L_1L_2}{r^2}\right)m_0^2 \\ &- 2\sqrt{\left(-1 + \frac{r^2E_1^2}{Q^2 - cr - 2Mr + r^2} - \frac{L_1^2}{r^2}\right)\left(-1 + \frac{r^2E_2^2}{Q^2 - cr - 2Mr + r^2} - \frac{L_2^2}{r^2}\right)}m_0^2, \end{aligned} \tag{55}$$

where

$$Q = \left(\frac{2}{27} \times \frac{\left(-1 + 9Mc + \sqrt{-(6Mc - 1)^3}\right)}{c^2}\right)^{\frac{1}{2}}.$$

In the presence of a magnetic field the expression for CME of charged particles coming from infinity colliding near the black hole is given by

$$\begin{aligned} E_{\rm cm}^2 &= 2\left(1 - B^2r^2 + \frac{r^2E_1E_2}{Q^2 - cr - 2Mr + r^2} - \frac{L_1L_2}{r^2} + B\left(L_1 + L_2\right)\right)m_0^2 \\ &- 2\sqrt{\left(-1 - B^2r^2 + \frac{r^2E_1^2}{Q^2 - cr - 2Mr + r^2} + BL_1 - \frac{L_1^2}{r^2}\right)\left(-1 - B^2r^2 + \frac{r^2E_2^2}{Q^2 - cr - 2Mr + r^2} + BL_2 - \frac{L_2^2}{r^2}\right)}m_0^2, \end{aligned} \tag{56}$$

where

$$Q = \left(\frac{2}{27} \times \frac{\left(-1 + 9Mc + \sqrt{-(6Mc - 1)^3}\right)}{c^2}\right)^{\frac{1}{2}}.$$

Here, E_1 and E_2 are the energies of the first and the second particle respectively. L_1 and L_2 are the angular momenta of the first and the second particle respectively.

In Fig. 6, I have plotted center of mass energy $E_{\rm cm}$ as a function of r for different values of the magnetic field strength B. From Fig. 6 it can be seen that the presence of a magnetic field increases the center of mass energy $E_{\rm cm}$.

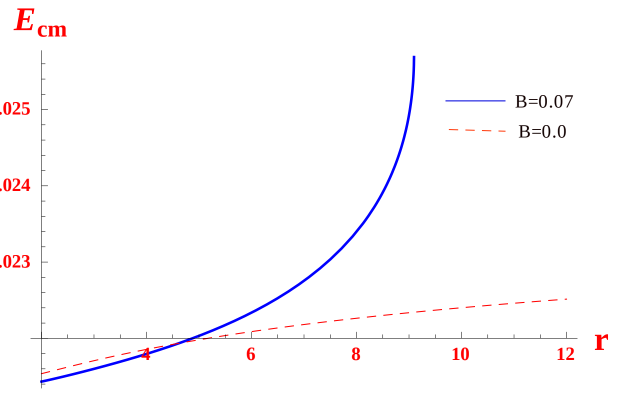


Fig. 6. Center of mass energy $E_{\rm cm}$ as a function of r for different values of B. Here $L_1 = 1$, $L_2 = 1.5$, $E_1 = 1$, $E_2 = 2$, $m_0 = 0.01$, $M = 1$, $c = 0.10$

VII. LYAPUNOV EXPONENT FOR THE INSTABILITY OF AN ORBIT

The Lyapunov exponent measures the average rate at which nearby trajectories in a phase space converge or diverge. A positive Lyapunov exponent indicates a divergence between nearby trajectories. The equation of motion for a Geodesic stability analysis in terms of the Lyapunov exponent can be written as [35]

$$\frac{dX_i}{dt} = H_i\left(X_j\right). \tag{57}$$

Its linearized form about a certain orbit is given by

$$\frac{d\delta X_i(t)}{dt} = K_{ij}(t)\delta X_j(t), \tag{58}$$

where

$$K_{ij} = \frac{\partial H_i}{\partial X_j}$$

is the linear stability matrix. The solution of the above equation can be written as

$$\delta X_i(t) = Z_{ij}(t)\delta X_j(0),$$

where $L_{ij}(t)$ is the evolution matrix which obey

$$\dot{L_{ij}}(t) = K_{im}L_{mj}(t) \text{ and } L_{ij}(0) = \delta_{ij}.$$

Determination of the eigenvalues of L_{ij} leads to the principal Lyapunov exponent λ as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{L_{ij}(t)}{L_{ij}(0)}\right). \tag{59}$$

In the case of a two dimensional phase space of the form $X_i(t) = (p_r, r)$ which includes circular orbits in a stationary spherically symmetric space-time, linearizing the equations of motion about orbits of constant r solution is given by

$$K_{11} = 0,\ K_{12} = \frac{d}{dr}\left(\dot{t}^{-1}\frac{\delta L}{\delta r}\right),\ K_{21} = -\left(\dot{t}g_{rr}\right)^{-1},$$

where L is the Lagrangian for geodesic motion. For circular orbits

$$\lambda = \sqrt{K_{12}K_{21}}.$$

Now from the equation of motion

$$\frac{d}{\tau}\frac{\delta L}{\delta \dot{r}} = \frac{\delta L}{\delta r}$$

using the definition of effective potential

$$\dot{r}^2 = E^2 - U_{\rm eff}$$

and conditions of circular geodesic

$$U_{\rm eff} = 0, U'_{\rm eff} = 0$$

Lyapunov exponent λ reduces to

$$\lambda = \sqrt{\frac{-U''_{\rm eff}(r)}{2\dot{t}(r)^2}}. \tag{60}$$

We can check the instability of a circular orbit by Lyapunov exponent λ. For a charged particle the Lyapunov exponent can be written as

$$\lambda^2 = \left(-\frac{\left(Q^2 - 2Mr + r^2 - cr^3\right)}{2r^6\left(L^2 + r^2 - 2BLr^2 + B^2r^4\right)}\right),$$

$$\Big(20L^2Q^2 - 24L^2Mr + 6L^2r^2 + 6Q^2r^2 - 12BLQ^2r^2$$
$$-2cL^2r^3 - 4Mr^3 + 8BLMr^3 + 2B^2r^6 - 6B^2cr^7\Big), \tag{61}$$

where

$$Q = \left(\frac{2}{27} \times \frac{\left(-1 + 9Mc + \sqrt{-(6Mc-1)^3}\right)}{c^2}\right)^{\frac{1}{2}}.$$

In Fig. 7, Lyapunav exponent (λ) versus r has been plotted. It is observed that a greater value of the magnetic field strength increases the instability of the circular orbit.

VIII. EFFECTIVE FORCE ON A PARTICLE

The effective force on a particle is given by [29]

$$F = -\frac{1}{2}\frac{dU_{\rm eff}}{dr}. \tag{62}$$

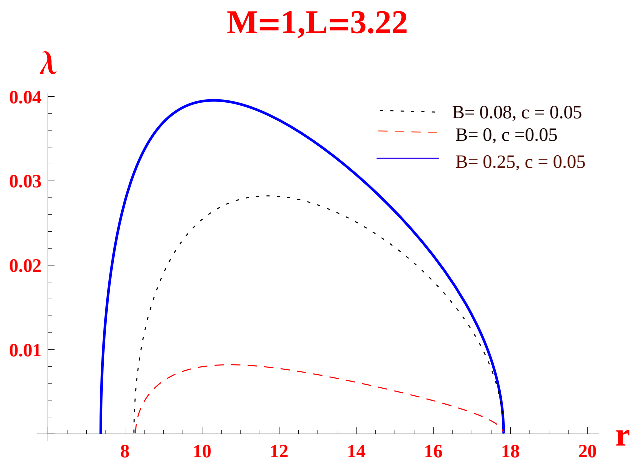


Fig. 7. Lyapunov exponent λ for different values of B as a function of radial coordinate r

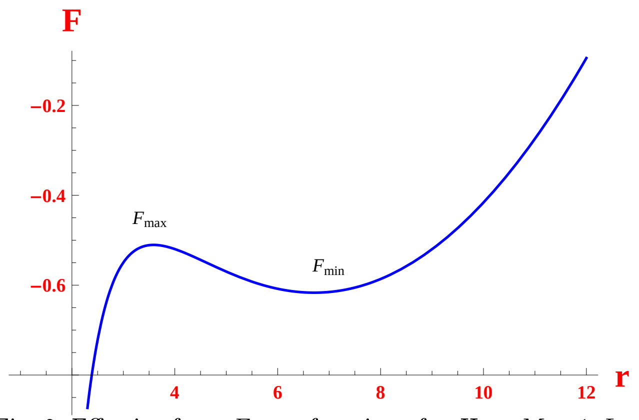


Fig. 8. Effective force F as a function of r. Here $M = 1, L = 3.22, B = 0.5, c = 0.05$

Therefore effective force for the charged particle can be written as

$$
\begin{aligned}
F &= c\left(\frac{1}{2} - BL\right) + B^2 M + \frac{2L^2Q^2}{r^5} - \frac{3L^2M}{r^4} \\
&+ \frac{(1-2BL)Q^2}{r^3} + \frac{-\frac{cL^2}{2} + (-1+2BL)M}{r^2} \\
&- B^2 r + \frac{3}{2}B^2 c r^2,
\end{aligned} \tag{63}
$$

where

$$
Q = \left(\frac{2}{27} \times \frac{\left(-1 + 9Mc + \sqrt{-(6Mc-1)^3}\right)}{c^2}\right)^{\frac{1}{2}}.
$$

In Fig. 8, I have plotted the effective force versus r for a charged particle. In the figure, $F_{\max}$ corresponds to the unstable circular orbit and $F_{\min}$ corresponds to the stable circular orbit.

IX. CONCLUSION

I have investigated the dynamics of neutral and charged particles moving around the Reissner–Nordström black hole in the presence of quintessence and a magnetic field. Depending on the energy, initial velocity and position of the charged particle, the charged particle may either fall into the black hole, or escape to infinity, or move in a stable circular orbit, or oscillate between two orbits. The presence of dark energy affects the motion of the charged particle. For anincreasing value of the quintesence parameter, both the stable and the unstable circular orbit move away from the horizon. For an increasing value of the quintesence parameter, effective potential decreases. The orbits are more stable in the presence of a magnetic field as compared to the case when a magnetic field is absent, that is $b = 0$. It can also be seen that as the value of b increases, the first minimum of effective potential, that is the first stable circular orbit, is unchanged but the second minimum of the effective potential which corresponds to a stable circular orbit is shifting towards the horizon. For an increasing value of angular momentum, the position of the first minimum of effective potential is unchanged but the second minimum moves away from the horizon. The particle having a larger value of angular momentum l has more possibility to escape compared to the particle with a lower value of angular momentum l.

When we consider the collision of a charged particle moving in the innermost stable circular orbit with another charged particle, then the particle getting greater energy after the collision has more possibility to escape to infinity from the vicinity of the black hole compared to the particle getting lesser energy. Also for the larger value of ρ_o which corresponds to the innermost stable circular orbit, the possibility to escape to infinity after the collision from the vicinity of the black hole also increases. For a greater value of $\bar{c}$, the velocity of the charged particle after the collision will be greater; that is the possibility to escape to infinity increases. It can be concluded that at the time of the collision if the charged particle gets sufficient energy, then the charged particle can escape to infinity after the collision from the vicinity of the black hole.

The presence of a magnetic field increases the center of mass energy E_{cm}. It is observed that a greater value of the magnetic field strength increases the instability of the circular orbit.

In this paper, the dynamics of neutral and charged particles has been investigated around the Reissner–Nordström black hole in the presence of quintessence and the magnetic field which was derived from the Kiselev black hole space-time. In [36] Matt Visser showed that for the single component Kiselev's black hole space time

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r} - \frac{k}{r^{1+3\omega}}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} - \frac{k}{r^{1+3\omega}}\right)} \\
&+ r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)
\end{aligned}
$$

energy density and pressure can be written as

$$\rho = -p_r = -\frac{3k\omega}{8\pi r^{3(1+\omega)}}, \qquad p_t = -\frac{3k\omega(1+3\omega)}{16\pi r^{3(1+\omega)}}.$$

These are not isotropic. He also showed anisotropic nature of energy density and pressure is also true for the generalized N-component Kiselev black hole. So stress energy tensor in Kiselev's space time cannot be the perfect fluid. Within the cosmology community, quintessence refers to a scalar field with a time-like gradient. The stress energy tensor associated with quintessence is that of a zero-vorticity perfect fluid. The Kiselev spacetime does not represent quintessence in the sense that this term used within the cosmology community. Despite these terminological the issues, the Kiselev black hole has some interesting physical and mathematical properties and does merit investigation.

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ДИНАМІКА НЕЙТРАЛЬНОЇ ТА ЗАРЯДЖЕНОЇ ЧАСТИНКИ В ҐРАВІТАЦІЙНОМУ ПОЛІ ЧОРНОЇ ДІРИ РАЙССНЕРА–НОРДСТРЬОМА ЗА НАЯВНОСТІ ЗОВНІШНЬОГО МАГНІТНОГО ПОЛЯ ТА ТЕМНОЇ ЕНЕРҐІЇ

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У статті досліджено динаміку нейтральної та зарядженої частинок навколо чорної діри Райсснера–Нордстрьома, оточеної квінтесенцією, за наявності зовнішнього магнітного поля. Спершу розглянуто зіткнення нейтральної частинки, що рухається навколо чорної діри, з іншою нейтральною частинкою. Потім обговорено зіткнення зарядженої частинки, що рухається навколо чорної діри, з іншою зарядженою частинкою. За допомогою чисельних розв'язків рівнянь руху показано, що після зіткнення, якщо частинка отримує достатню енерґію, вона може втекти в безмежність. Залежність орбіт від квінтесенції та магнітного поля для зарядженої частинки детально вивчено за допомогою ефективного потенціалу зарядженої частинки. Використовуючи показник Ляпунова, обговорено вплив магнітного поля та темної енерґії на стабільність частинки. Виведено вираз для енерґій центра мас заряджених частинок, що стикаються, рухаючись навколо чорної діри. За наявності темної енерґії та магнітного поля обговорено вплив ефективної сили на заряджену частинку.

Ключові слова: рівняння Ейлера–Лаґранжа, циклічна координата, швидкість утечі, енерґія центра мас та ефективна сила.