Classical dS and AdS cosmologies in the general case of deformed space with minimal length

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The effects of the minimal length uncertainty relation on classical de Sitter and Anti-de Sitter cosmological models is studied in the general case of deformed space. We obtain exact solutions for these models in case of some special choices of deformed spaces with minimal length and minimal or maximal momentum. It is shown that minimal length might affect and even change the inflationary nature of the de Sitter cosmology. Anti-de Sitter model with deformation has oscillatory behaviour, but depending on the choice of deformation function the period of oscillations can be larger or smaller in comparison to the undeformed model.

Key words: minimal length, generalized uncertainty principle, classical cosmology, cosmological constant, Hubble parameter

1 Introduction

String theory and quantum gravity independently suggest the existence of minimal length as a finite lower bound to the possible resolution of length [1–3]. Kempf et al. showed that minimal length can be introduced by modifying a canonical commutation relation [4–7]. The deformed commutation relation according to Kempf reads

\[ [\hat{X}, \hat{P}] = i\hbar(1 + \beta\hat{P}^2). \]  

Deformed algebra (1) can be generalized for a wider class of the deformed commutation relation

\[ [\hat{X}, \hat{P}] = i\hbar F(\hat{X}, \hat{P}). \]  

where \( F \) is a positive function of the position and momentum. Such a modification of the canonical commutation relation can lead to the existence of nonzero minimal uncertainties in position, or in momentum, or both. Function

\[ F(\hat{X}, \hat{P}) = 1 + \alpha \hat{X}^2 + \beta \hat{P}^2 \]  

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is an example of the deformation leading to the minimal length and minimal momentum. In this case \( \Delta X_{\text{min}} = \hbar \sqrt{\beta/(1 - \hbar^2 \alpha \beta)} \), \( \Delta P_{\text{min}} = \hbar \sqrt{\alpha/(1 - \hbar^2 \alpha \beta)} \).

In case when deformation function depends only on momentum, i.e. \( F(\hat{X}, \hat{P}) = f(\hat{P}) \), minimal length is [8]

\[
\Delta X_{\text{min}} = \frac{\pi}{2} \left( \int_{0}^{b} \frac{dP}{f(P)} \right)^{-1},
\]

where \( b \) denotes limits of \( P \in [-b, b] \). Here function of deformation \( f(P) \) is assumed to be strictly positive (\( f > 0 \)), even function. This means that minimal length is nonzero if

\[
\int_{0}^{b} \frac{dP}{f(P)} < \infty.
\]
2 A brief review of classical dS and AdS cosmologies

Let us assume that the universe is homogeneous and isotropic. It can be described by the flat Robertson-Walker metric

\[ ds^2 = -c^2 dt^2 + a(t) (dx^2 + dy^2 + dz^2). \] (6)

Here \( a(t) \) is the scale factor of the universe. Corresponding nonzero Christoffel symbols and Ricci tensor components are

\[ \Gamma^0_{ij} = a \dot{a} \delta_{ij}, \Gamma^i_0j = \frac{\dot{a}}{a} \delta^i_j, \] (7)

\[ R_{00} = \frac{3 \ddot{a}}{a}, R_{ij} = (2 \dot{a}^2 + a \ddot{a}), \] (8)

where a dot means differentiation with respect to time \( t \). The scalar curvature of Robertson-Walker metric is

\[ R = g^{\mu\nu} R_{\mu\nu} = 6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right). \] (9)

To construct the canonical formalism of the theory, let us start with the Einstein-Hilbert action

\[ S = \frac{1}{2\kappa} \int (R - 2\Lambda) \sqrt{-g} d^4x, \] (10)

Here \( \kappa = \frac{8\pi G}{c^4} \), \( g \) is the determinant of the spacetime metric and \( \Lambda \) is the cosmological constant representing the vacuum energy. Substituting (9) into (10) we obtain

\[ S = 3V c \kappa \int \left( \dot{a}^2 + \frac{\Lambda c^2 a^3}{3} \right) dt, \] (11)

which can be rewritten as

\[ S = -3V c \kappa \int \left( \dot{a}^2 a + \frac{\Lambda c^2 a^3}{3} \right) dt. \] (12)

Here integration over the spatial dimensions gives volume \( V \) and term with the total derivative over time is cancelled. Lagrangian of the system is the following

\[ L = \dot{a}^2 + \frac{\Lambda c^2 a^3}{3}, \] (13)

which yields the Hamiltonian

\[ H = p_a \dot{a} - L = \frac{p_a^2}{4a} - \frac{\Lambda c^2 a^3}{3}. \] (14)
Note that in (13) minus is omitted and $\frac{\partial V}{\partial \kappa} = 1$. We introduce canonical momentum $p_a = \frac{\partial L}{\partial a\dot{a}}$ satisfying

$$\{a, p_a\} = 1. \quad (15)$$

Making canonical transformation

$$u = a^{\frac{3}{2}}, \quad p_u = \frac{2p_a}{3\sqrt{a}}, \quad (16)$$

$$\{u, p_u\} = 1, \quad (17)$$

we rewrite the Hamiltonian into the following form

$$\mathcal{H} = \frac{9p_u^2}{16} - \frac{\Lambda c^2 u^2}{3}. \quad (18)$$

In case of $\Lambda > 0$ the Hamiltonian describes the simplest classical inflationary (dS) model and in case of $\Lambda < 0$ the oscillatory (AdS) model. For the dS model the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_u^2}{2m} - \frac{m\Omega^2 u^2}{2}, \quad (19)$$

with $m = \frac{8}{9}$ and $\Omega^2 = \frac{3}{4}\Lambda c^2$. Equations of motion are

$$\dot{u} = \{u, \mathcal{H}\} = \frac{p_u}{m}, \quad (20)$$

$$\dot{p}_u = \{p_u, \mathcal{H}\} = m\Omega^2 u, \quad (21)$$

$$\ddot{u} = \Omega^2 u, \quad (22)$$

which yields

$$u(t) = e^{\Omega(t-t_0)}, \quad (23)$$

where we have taken the initial condition $u(t_0) = 1$. The Hubble constant for considered model is

$$H_0 = \frac{\dot{a}}{a} = \frac{2}{3} \frac{\dot{u}}{u} = \frac{2}{3}\Omega \quad (24)$$

For AdS cosmological model the hamiltonian

$$\mathcal{H} = \frac{p_u^2}{2m} + \frac{m\omega^2 u^2}{2}, \quad (25)$$

with $\omega^2 = -\frac{3}{4}\Lambda c^2$.

$$u(t) = \sin(t), \quad (26)$$

where we have taken the initial condition $u(0) = 0$. 

3 Classical $dS$ cosmology with deformed Poisson algebra

Let us study the effects of the classical version of commutation relation (2) on the de Sitter cosmological model. In the classical limit quantum mechanical commutators should be replaced by the classical Poisson brackets as $[\hat{X}, \hat{P}] \rightarrow i\hbar \{X, P\}$. In the case of canonical variables $u$ and $p_u$ deformed Poisson bracket has the form

$$\{u, p_u\} = F(u, p_u).$$

(27)

Note that in case of $F(u, p_u) = f(p_u)$ the existence of minimal length imposes the constraint on the function of deformation $f(p_u)$

$$\int_0^a \frac{dp_u}{f(p_u)} < \infty.$$  

(28)

The equations of motion are

$$\dot{u} = \{u, H\} = \frac{p_u}{m} F(u, p_u),$$

(29)

$$\dot{p}_u = \{p_u, H\} = m\Omega^2 u F(u, p_u).$$

(30)

From equations (29) and (30) we obtain

$$\frac{\dot{u}}{\dot{p}_u} = \frac{1}{m^2 \Omega^2 u}$$

(31)

The last equation yields

$$p_u = \pm m\Omega u,$$

(32)

with the constant of integration which equals to zero. Finally, substituting (32) into (29) we obtain

$$\dot{u} = \pm \Omega u F(u, \pm m\Omega u).$$

(33)

Integration of the latter equation gives the law of inflation for $dS$ model which includes the effects caused by minimal length

$$\int_1^u \frac{du}{u F(u, \pm m\Omega u)} = \pm \Omega (t - t_0)$$

(34)

Hubble parameter then can be presented as

$$H = \frac{\dot{a}}{a} = \frac{2}{3} \frac{\dot{u}}{u} = \frac{2}{3} \Omega F(u, \pm m\Omega u) = H_0 F(u, \pm m\Omega u).$$

(35)

In the linear approximation on parameter of deformation formula (35) yields

$$H = H_0 F(e^{\Omega (t - t_0)}, \pm m\Omega e^{\Omega (t - t_0)}).$$

(36)
Now let us consider a few special examples of deformation function.

**Example 1.** Let us consider the deformation function proposed by Kempf $F(u, p_u) = 1 + \alpha u^2 + \beta p_u^2$. We obtain that the expansion law is

$$u(t) = \frac{e^{\Omega(t-t_0)}}{\sqrt{1 + \gamma (1 - e^{2\Omega(t-t_0)})}},$$

with

$$\gamma = \alpha + \beta m^2 \Omega^2. \quad (38)$$

In the linear approximation on the parameter of deformation $\gamma$ the expansion law writes

$$u(t) = e^{\Omega(t-t_0)} + \frac{\gamma}{2} \left( e^{3\Omega(t-t_0)} - e^{\Omega(t-t_0)} \right). \quad (39)$$

The last result was obtained in paper [39] in case of $\alpha = 0$. However, from the exact result (37) we obtain that inflation of the universe will last for finite time $t_r$, thus we have Big Rip scenario

$$t_r = t_0 + \frac{1}{2\Omega} \ln \left( \frac{1}{1 + \frac{1}{\gamma}} \right), \quad (40)$$

For this example of deformation function the Hubble parameter also can be found exactly

$$H = H_0 \left( \frac{1 + \gamma}{1 + \gamma (1 - e^{2\Omega(t-t_0)})} \right). \quad (41)$$

**Example 2.** The second example of deformation function that possesses the exact solution is the following

$$F(u, p_u) = f(p_u) = (1 + \beta p_u^2)^{\frac{3}{2}}. \quad (42)$$

The solution can be presented as

$$\frac{1}{\sqrt{1 + \gamma_\beta u^2}} = \frac{1}{\sqrt{1 + \gamma_\beta}} + \frac{1}{2} \ln \frac{(1 + \gamma_\beta u^2 - 1)(1 + \gamma_\beta + 1)}{(1 + \gamma_\beta u^2 + 1)(1 + \gamma_\beta - 1)} = \Omega(t-t_0), \quad (43)$$

with

$$\gamma_\beta = \beta m^2 \Omega^2. \quad (44)$$

In the linear approximation on parameter of deformation the expansion law writes

$$u(t) = e^{\Omega(t-t_0)} + \frac{3\gamma_\beta}{4} \left( e^{3\Omega(t-t_0)} - e^{\Omega(t-t_0)} \right). \quad (45)$$

Comparing (45) with (39) we see that for this example of deformation the expansion rate is higher than the one for the previous example. Plots of exact expansion laws (43) and (37) are present in Fig.1.
Similarly to the previous case, the end of the universe will occur in some moment $t_r$ in the future

$$t_r = t_0 - \frac{1}{\Omega \sqrt{1 + \gamma \beta}} + \frac{1}{2\Omega} \ln \frac{\sqrt{1 + \gamma \beta} + 1}{\sqrt{1 + \gamma \beta} - 1}.$$  \hspace{1cm} (46)$$

The dependencies of $t_r - t_0$ on deformation parameter $\gamma = \gamma \beta$ are given in Fig.2.

We again arrive at the Big Rip scenario. In fact, this ultimate fate of the universe can be predicted in the general case of deformed function $F(u, p_u) = f(p_u)$ with minimal length and without maximal momentum. Really, using constraint (28) in case of $u_{max} = \infty$ we can write

$$t_r - t_0 = \frac{1}{\Omega} \int_1^\infty \frac{du}{uf(\pm m\Omega u)} < \frac{1}{\Omega} \int_1^\infty \frac{du}{f(\pm m\Omega u)} < \infty.$$  \hspace{1cm} (47)$$

**Example 3.** The last example of deformation function with maximal momentum is $f(p_u) = \sqrt{1 - \beta p_u^2}$, $|p_u| < \frac{1}{\sqrt{\beta}}$. In such a case the expansion phase of the evolution of the universe will proceed until $u_{max} = \frac{1}{\sqrt{\gamma \beta}}$ by the law

$$u(t) = \frac{1}{\cosh \Omega(t - t_0) - \sqrt{1 - \gamma \beta} \sinh \Omega(t - t_0)}.$$  \hspace{1cm} (48)$$
In the linear approximation on parameter of deformation the expansion law writes
\[ u(t) = e^{\Omega(t-t_0)} - \frac{\gamma_k}{4} \left( e^{3\Omega(t-t_0)} - e^{\Omega(t-t_0)} \right). \] (49)

However, at the moment in time
\[ t_c = t_0 + \frac{1}{2\Omega} \ln \left( \frac{1 + \sqrt{1 - \gamma_k}}{1 - \sqrt{1 - \gamma_k}} \right), \] (50)

the expansion of the universe reverses and the universe recollapses by the law
\[ u(t) = \frac{1}{\sqrt{3}m\Omega \cosh [\Omega(t - t_c)]}. \] (51)

Thus, the ultimate fate of the universe in this model is the Big Crunch (see Fig.3).

In the general case of deformation function \( F(u, p_u) = f(p_u) \) with maximal momentum it can also be proven similarly as in (47) that expansion phase of the evolution of the universe lasts a finite period.
4 Classical AdS cosmology with deformed Poisson algebra

Now let us consider classical AdS cosmology with minimal length scenario. The Hamiltonian corresponding to this problem is

\[ \mathcal{H} = \frac{p_u^2}{2m} + \frac{m\omega^2 u^2}{2}, \]

(52)

with \( \omega^2 = -\frac{3}{4}\Lambda c^2 \).

The equations of motion are

\[ \dot{u} = \{u, H\} = \frac{p_u}{m} F(u, p_u), \]

(53)

\[ \dot{p}_u = \{p_u, H\} = -m\omega^2 u F(u, p_u). \]

(54)

The law of evolution of the universe for AdS model can be written as

\[ \int_0^u \frac{du}{\sqrt{u_0^3 - u^2 F(u, m\omega\sqrt{u_0^3} - u^2)}} = \omega t, \]

(55)

with \( u_0 = \sqrt{\frac{2E}{m\omega^2}} \) being the amplitude of the scale factor and \( E = \frac{p_u^2}{2m} + \frac{m\omega^2 u^2}{2} \) being the energy of the system.
We obtain exact analytical results for two special examples of deformation function.

**Example 1.** Let us consider the deformation function proposed by Kempf \( F(u, p_u) = 1 + \alpha u^2 + \beta p_u^2 \). The evolution of the universe is governed by the oscillatory law

\[
\begin{align*}
    u(t) &= \sqrt{\frac{2E(1 + \varepsilon_\beta)}{m\omega^2}} \frac{\sin\left[\sqrt{(1 + \varepsilon_\alpha)(1 + \varepsilon_\beta)} \omega t\right]}{\sqrt{1 + \varepsilon_\beta \sin^2[\sqrt{(1 + \varepsilon_\alpha)(1 + \varepsilon_\beta)} \omega t] + \varepsilon_\alpha \cos^2[\sqrt{(1 + \varepsilon_\alpha)(1 + \varepsilon_\beta)} \omega t]}} \tag{56}
\end{align*}
\]

with \( \varepsilon_\alpha = \frac{2\alpha E}{m^2}, \varepsilon_\beta = 2m\beta E \). The period of oscillations can be calculated exactly by the following formula

\[
    T = \frac{2\pi}{\omega \sqrt{(1 + \varepsilon_\alpha)(1 + \varepsilon_\beta)}} \tag{57}
\]

As it turned out this period is smaller than in ordinary AdS model.

**Example 2.** The last example of deformation function with maximal momentum is \( f(p_u) = \sqrt{1 - \beta p_u^2}, |p_u| < \frac{1}{\sqrt{\beta}} \). The exact solution can be presented by incomplete elliptic integral of the first kind as

\[
    \omega t = \frac{1}{\sqrt{1 - \varepsilon}} \text{EllipticF} \left( \frac{u}{u_0}, \sqrt{\frac{\varepsilon}{\varepsilon - 1}} \right) \tag{58}
\]
In the linear approximation on parameter of deformation (58) writes

\[ u(t) = u_0 \left(1 - \frac{\varepsilon}{4} \cos^2(\omega t)\right) \sin[(1 - \frac{\varepsilon}{4}) \omega t]. \]  

(59)

Comparison of results (56) and (58) with the undeformed one is presented on Fig. 4. From Fig. 4 we can conclude that depending on the choice of deformation function the period of oscillations can be larger or smaller in comparison to the ordinary model.

5 Conclusion

In this paper, we have studied the effects of minimal length uncertainty relation on classical dS and AdS cosmologies. We have shown in the general case that for the deformation with minimal length and without maximal momentum dS cosmology gives an inflationary universe with Big Rip ultimate fate.

We also obtain an exact solution for the dS model in case of deformed commutation relation with both minimal length and minimal momentum. In this case, Big Rip scenario is also realized.

For the deformation with minimal length and maximal momentum in dS model, we obtain that at some moment of time the expansion of the universe reverses and the universe recollapses. Thus, we arrive at Big-Crunch scenario.

In the classical AdS model, there are some differences between ordinary and deformed models. The model with deformation has oscillatory behaviour but the period of oscillations can be larger or smaller in comparison to the ordinary model. This means that depending on the choice of deformation function in AdS model with minimal length the corresponding Big-Crunch occurs later or earlier than in an ordinary model.

Acknowledgements

This work was supported by the project FF-63Hp (No. 0117U007190) from the Ministry of Education and Science of Ukraine.

Класичні dS та AdS космології в загальному випадку деформованого простору з мінімальною довжиною

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Вплив співвідношення невизначеності з мінімальною довжиною на класичні космологічні моделі де Сіттера та анти-де Сіттера виявляється в загальному випадку деформованого простору. Ми отримали точні розв'язки для цих моделей для деяких спеціальних випадків деформованих просторів з мінімальною довжиною та мінімальним чи максимальним імпульсом. Показано, що мінімальна довжина може впливати і навіть змінювати інфляційний характер космології де Сіттера. Модель анти-де Сіттера з деформацією має коливальну поведінку, але залежно від вибору функції деформації період коливань може бути більшим чи меншим в порівнянні зі не деформованою моделлю.

Ключові слова: мінімальна довжина, узагальнений принцип невизначеності, класична космологія, космологічна стала, параметр Хаббла