ISSN 1024-588X. Вiсник Львiвського унiверситету. Серiя фiзична. 2021. Випуск 58. С. 30-38
Visnyk of the Lviv University. Series Physics. 2021. Issue 58. P. 30-38

УДК 530.145
PACS 03.65.-w, 02.40.Gh

https://doi.org/10.30970/vph.58.2021.30

Parameters of the deformed algebra with minimal uncertainties in position and momentum and the weak equivalence principle.

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Deformed algebra $[X, P] = i\hbar(1 + \beta P^2 + \alpha X^2)$ leading to the minimal uncertainties in position and momentum is considered. It is shown that the trajectory of motion of a free particle in the quantized space depends on its mass. Also the momentum of the particle is not proportional to mass. We conclude that the free motion does not depend on mass and the momentum is proportional to mass if the parameter of the deformed algebra β is inversely proportional to the squared mass and α does not depend on the mass. It is also shown that the same conditions lead to preservation of the weak equivalence principle in the quantized space.

Key words: minimal uncertainty in position, minimal uncertainty in momentum, free particle, weak equivalence principle.

1. Introduction

It follows from the String Theory that the Heisenberg uncertainty relation is generalized [1–3]. Inequality

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta\Delta P\right), \tag{1}$$

is known as generalized uncertainty principle (GUP) and result in minimal position uncertainty determined by the parameter of deformation β

$$\Delta X_{min} = \hbar\sqrt{\beta}. \tag{2}$$

Values ΔX, ΔP in (1) are defined as

$$\Delta X = \sqrt{\langle(\Delta X)^2\rangle}, \tag{3}$$

$$\Delta P = \sqrt{\langle(\Delta P)^2\rangle}. \tag{4}$$

Generalized uncertainty principle (1) follows from the deformed commutation relations for coordinate and momentum

$$[X, P] = i\hbar(1 + \beta P^2), \tag{5}$$

(see [4–7]). Therefore, deformation (5) leads to existence of the minimal length and can describe features of the space structure at the Planck scale, space quantization.

In a more general case deformed algebra (5) reads

$$[X, P] = i\hbar(1 + \beta P^2 + \alpha X^2), \tag{6}$$

where α, β are parameters of deformation $\alpha \geq 0$, $\beta \geq 0$, $\alpha\beta < 1/\hbar^2$ [8, 9]. The algebra results in to the minimal uncertainties in position and momentum that are defined as

$$\Delta X_{min} = \hbar\sqrt{\frac{\beta}{1 - \hbar^2\alpha\beta}}, \tag{7}$$

$$\Delta P_{min} = \hbar\sqrt{\frac{\alpha}{1 - \hbar^2\alpha\beta}}. \tag{8}$$

Algebra (6) with $\beta = 0$ can also be used to describe a particle with position dependent mass [10–12]. In the classical limit from deformed commutation relations (6) we obtain deformed Poisson brackets for coordinates and momenta as follows

$$\{X, P\} = 1 + \beta P^2 + \alpha X^2. \tag{9}$$

Influence of deformation of the commutation relation (6) on the spectrum of harmonic oscillator was studied in [12–14]. The authors of the papers obtained an exact expression for the energy levels of harmonic oscillator in the quantized space (6).

In the present paper we study motion of a free particle in a space with minimal uncertainties in position and in momentum (9). We show that the trajectory of a free particle in the space (9) depends on its mass. We conclude that the problem is solved if the parameter of deformation β depends on mass and parameter α does not depend on mass. We also study the weak equivalence principle in the space (9). We show that the principle is recovered if the parameter β is determined by mass and parameter α is the same for different particles.

The paper is organized as follows. In Section 2 the motion of a system of free particles is examined in the frame of the algebra (9). Section 3 is devoted to the studies of the weak equivalence principle in a space with minimal uncertainties in position and in momentum. Conclusions are presented in Section 4.

2. Free particle motion in a space with minimal uncertainties in position and momentum

Let us examine the motion of a free particle with mass m in a space with deformation of Poisson brackets (9). Considering Hamiltonian of the particle the same as in the ordinary space

$$H = \frac{P^2}{2m}, \tag{10}$$

and taking into account deformation of the Poisson brackets (9) one obtains the following equations of motion.

$$\dot{X} = \frac{P}{m}(1 + \beta P^2 + \alpha X^2), \tag{11}$$

$$\dot{P} = 0. \tag{12}$$

The solution of the equations with the initial conditions $X(0) = X_0$, $\dot{X}(0) = \upsilon_0$ reads

$$X(t) = \frac{\sqrt{1+\beta P^2}}{\sqrt{\alpha}} \tan\left(\sqrt{\alpha(1+\beta P^2)}\frac{P}{m}t + \arctan\left(\frac{X_0\sqrt{\alpha}}{\sqrt{1+\beta P^2}}\right)\right), \tag{13}$$

and the momentum P can be found from the equation

$$\upsilon_0 = \frac{P}{m}(1 + \beta P^2 + \alpha X_0^2). \tag{14}$$

The real solution of (14) reads

$$P = \frac{12^{\frac{1}{3}}}{6}\left(\left((81\upsilon_0^2 m^2\beta + 12(1+\alpha X_0^2)^3)^{\frac{1}{2}} + 9\beta^{\frac{1}{2}}\upsilon_0 m\right)^{\frac{2}{3}} - 12^{\frac{1}{3}}(1+\alpha X_0^2)^3\right) \times$$
$$\times\left((81\upsilon_0^2 m^2\beta^4 + 12(1+\alpha X_0^2)^3\beta^3)^{\frac{1}{2}} + 9\beta^2\upsilon_0 m\right)^{-\frac{1}{3}}. \tag{15}$$

Note that in the limit $\alpha \to 0$, $\beta \to 0$ from (13), (15) we obtain well known results $X(t) = \upsilon_0 t + X_0$, $P = m\upsilon_0$. It is important to mentioning that the trajectory of a free particle in deformed space depend on its mass (13). Besides, the momentum (15) is not proportional to mass.

Let us assume that the parameter of deformation β is determined by the mass of a particle as

$$\sqrt{\beta}m = \gamma, \tag{16}$$

where γ is a constant which is the same for particles of different masses. In addition we assume that parameter α does not depend on mass. Condition (16) was proposed in [15, 16] to recover the equivalence principle and to preserve the properties of the kinetic energy in deformed space (5).

Considering (16) and assuming that α does not depend on mass we obtain that the momentum of free particle is proportional to mass

$$P = \frac{12^{\frac{1}{3}}}{6}m\left(\left((81\upsilon_0^2\gamma^2 + 12(1+\alpha X_0^2)^3)^{\frac{1}{2}} + 9\gamma\upsilon_0\right)^{\frac{2}{3}} - 12^{\frac{1}{3}}(1+\alpha X_0^2)^3\right) \times$$
$$\times\left((81\upsilon_0^2\gamma^8 + 12(1+\alpha X_0^2)^3\gamma^6)^{\frac{1}{2}} + 9\gamma^4\upsilon_0\right)^{-\frac{1}{3}}, \tag{17}$$

as it is in the ordinary space. Also, due to condition (16) the trajectory of motion of a free particle does not depend on its mass. We have

$$X(t) = \frac{\sqrt{1+\gamma^2\tilde{P}^2}}{\sqrt{\alpha}} \tan\left(\sqrt{\alpha\left(1+\gamma^2\tilde{P}^2\right)}\tilde{P}t + \arctan\left(\frac{x_0\sqrt{\alpha}}{\sqrt{1+\gamma^2\tilde{P}^2}}\right)\right), \tag{18}$$

where

$$\tilde{P} = \frac{P}{m} = \frac{12^{\frac{1}{3}}}{6}\left(\left((81\upsilon_0^2\gamma^2 + 12(1+\alpha X_0^2)^3)^{\frac{1}{2}} + 9\gamma\upsilon_0\right)^{\frac{2}{3}} - \right.$$
$$\left. - 12^{\frac{1}{3}}(1+\alpha X_0^2)^3\right)\left((81\upsilon_0^2\gamma^8 + 12(1+\alpha X_0^2)^3\gamma^6)^{\frac{1}{2}} + 9\gamma^4\upsilon_0\right)^{-\frac{1}{3}}. \quad (19)$$

In the next section we show that the proposed conditions on the parameters of deformation α, β give a possibility to recover the weak equivalence principle in the space (6).

3. Recovering of the weak equivalence principle

Let us study the weak equivalence principle in the space with minimal uncertainties in position and momentum (6). For this purpose we examine a motion of a particle of mass m in gravitational field $V(X)$. The corresponding Hamiltonian reads

$$H = \frac{P^2}{2m} + mV(X). \quad (20)$$

Taking into account the deformation of the Poisson brackets (9), we find the following equations of motion

$$\dot{X} = \frac{P}{m}(1 + \beta P^2 + \alpha X^2), \quad (21)$$
$$\dot{P} = -m\frac{dV}{dX}(1 + \beta P^2 + \alpha X^2). \quad (22)$$

From (21), (22) follows that the velocity and the trajectory of a particle in gravitational field in the space with minimal uncertainties in position and momentum depend on its mass. Therefore, the weak equivalence principle stating that the trajectory and the velocity of a particle in gravitational field do not depend on its mass and composition is violated.

It is worth stressing that equations (21), (22) do not depend on mass if the parameter of deformation β satisfies condition (16) and the parameter α does not depend on mass. In this case we can write

$$\dot{X} = \tilde{P}(1 + \gamma^2\tilde{P}^2 + \alpha X^2), \quad (23)$$
$$\dot{\tilde{P}} = -\frac{dV}{dX}(1 + \gamma^2\tilde{P}^2 + \alpha X^2), \quad (24)$$

where for convenience we use notation

$$\tilde{P} = \frac{P}{m}. \quad (25)$$

Note that equations (23), (24) do not depend on mass. Therefore $X(t)$, $\tilde{P}(t)$ do not depend on mass either. Hence, the motion of a particle in gravitational field in the space (6) does not depend on its mass and the weak equivalence principle is preserved if the parameter of the deformed algebra α does not depend on mass and parameter β is determined by mass as (16).

At the end of this section it is worth mentioning that in [17] it was shown that relation of parameter of deformation with mass (16) gives a possibility to recover the weak equivalence principle in a space with nonrelativistic Snyder algebra [18–22]. Determining the parameters of noncommutativity by mass is also important for solving the soccer-ball problem and recovering the weak equivalence principle in a noncommutative space (see [23]). The question of the dynamics of a particle system in the frame of deformed algebra (9) worth to be studied separately.

4. Conclusions

Quantized space described by deformed commutation relation for coordinate and momenta (6) has been considered. The space is characterized by minimal uncertainties in position and momentum. The motion of a free particle has been examined in this space. We have found that the trajectory of free motion depends on the mass of the particle. Besides, it has been shown that the momentum is not proportional to mass. We have obtained that in the case when the parameter of the deformed algebra α does not depend on mass and parameter β is inversely proportional to the squared mass (16) the motion of free particle in quantized space (6) does not depend on its mass and the momentum is proportional to the mass as it is in the ordinary space. In addition, we have concluded that the same conditions on the parameters of deformed algebra lead to recovering of the weak equivalence principle in quantized space with minimal uncertainties in position and momentum (6).

Acknowledgments

This work was partly supported by the Projects ФФ – 11Нр (No. 0121U100058), ФФ – 83Ф (No. 0119U002203) from the Ministry of Education and Science of Ukraine.

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Статтю отримано: 19.10.2021
Прийнято до друку: 08.11.2021

Параметри деформованої алгебри з мінімальними невизначеностями координати та імпульсу і слабкий принцип еквівалентності.

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Розглянуто одновимірну деформовану алгебру з деформацією квадратичною за координатами та імпульсами $[X,P] = i\hbar(1+\beta P^2+\alpha X^2)$. Така алгебра приводить до існування мінімальних невизначеностей координати та імпульсу, які визначаються параметрами деформації α, β. У випадку, коли $\beta = 0$ алгебра описує частинки з масою залежною від координат, при $\alpha = 0$ алгебра описує простір з мінімальною довжиною, квантований простір. У класичній границі деформація комутаційного співвідношення для операторів координати та імпульсу приводить до деформованих дужок Пуассона. Вивчено рух вільної частинки у просторі з деформованими дужками Пуассона. Знайдено рівняння руху та отримано вираз для траєкторії. Встановлено, що вона залежить від маси частинки. Також показано, що імпульс частинки у деформованому просторі не є пропорційним до її маси. Ми дійшли до висновку, що у разі, коли параметр деформованої алгебри β є обернено пропорційним до квадрата маси частинки, а параметр α не залежить від маси частинки, траєкторія вільного руху в деформованому просторі залежить тільки від початкової координати та початкової швидкості та не залежить від маси. Також імпульс частинки пропорційний до маси, як це є у звичному просторі (просторі зі звичними комутаційними співвідношеннями для координати та імпульсу). Також досліджено рух частинки у гравітаційному полі в просторі з мінімальними невизначеностями координати та імпульсу. Ми зробили висновок, що швидкість та траєкторія руху частинки у гравітаційному полі залежить від маси, а тому слабкий принцип еквівалентності не виконується. Встановлено, що, якщо припустити, що параметр β є обернено пропорційним до квадрата маси, а параметр α є однаковим для різних частинок, принцип еквівалентності зберігається.

Ключові слова: мінімальна невизначеність координати, мінімальна невизначеність імпульсу, вільна частинка, слабкий принцип еквівалентності.