Kh. P. Gnatenko, V. M. Tkachuk

Rotational and time-reversal symmetries in noncommutative space

Monograph

Monograph is devoted to studies of symmetrical properties in quantum space realized with noncommutative algebra of canonical type. In three-dimensional case of noncommutative algebra the rotational symmetry is violated. Also, in noncommutative phase space of canonical type we face a problem of time reversal symmetry breaking. Algebra which is equivalent to noncommutative algebra of canonical type and does not cause violation of the rotational and time reversal symmetries is constructed. Properties of physical systems in the frame of the algebra are studied. Among them are spectra of hydrogen and exotic atoms, systems of harmonic oscillators, particles with harmonic oscillator interaction. Also, the problem of violation of the weak equivalence principle in the noncommutative phase space of canonical type with preserved rotational and time reversal symmetries is solved.

PACS numbers: 02.40.Gh, 03.65.-w, 01.55.+b

Contents

Preface

In the monograph, the idea of description of features of space structure at the Planck scale is considered. Algebra with noncommutativity of coordinates and noncommutativity of momenta of canonical type is examined. Rotational and time-reversal symmetries are studied in the quantum space. We construct algebra which is rotationally-invariant equivalent to noncommutative algebra of canonical type and does not lead to violation of the time-reversal symmetry. In the frame of the algebra, various one and many-particle systems are exained. Based on the obtained results upper bounds for the parameters of coordinate and momentum noncommutativity are estimated.

In Chapter 2 effect of noncommutativity of coordinates and noncommutativity of momenta on the spectrum of hydrogen atom is studied. We find corrections to the energy levels of the atom up to the second order in the parameter of noncommutativity. Based on the obtained results with the experimental data for $1S - 2S$ transition frequency the upper bound for the minimal length is obtained. Also, two-particle system with Coulomb interaction is examined and hydrogen-like exotic atoms are studied in rotationally-invariant noncommutative phase space.

In Chapter 3 a system of interacting harmonic oscillators in a uniform field is considered. We find energy levels of the system up to the second order in the parameters of noncommutativity. It is obtained that the noncommutativity of coordinates and noncommutativity of momenta affect the mass and frequencies of the system. Also, particular cases of a system of particles with harmonic oscillator interaction and a system of free particles are examined.

In Chapter 4 time-reversal symmetry is studied in noncommutative phase space of canonical type. We construct noncommutative algebra which does not lead to violation of the rotational and time-reversal symmetries besides it is equivalent to noncommutative algebra of canonical type. In the frame of the algebra composite systems are studied. The motion of a system of free particles is examined. Also, the motion in a gravitational field is studied and the weak equivalence principle is analyzed. We propose expressions for tensors of noncommutativity which preserve the weak equivalence in rotationally-invariant and time-reversal invariant noncommutative phase space of canonical type.

Conclusions are presented in Chapter 5.

Chapter 1

Noncommutative space of canonical type with preserved rotational symmetry

1.1 Introduction

One of the important predictions of the String Theory and the Quantum Gravity is the existence of the minimal length which is of the order of the Planck length (see, for instance, [1–7]). This feature of space structure can be described with modifications of the ordinary commutation relations for operators of coordinates and operators of momenta.

The first article in which the idea that coordinates may not commute was published by Snyder [8]. Before Snyder the idea was suggested by Heisenberg. The scientist proposed such a modification to solve the problem of ultraviolet divergences in quantum field theory.

Many different modifications of the commutation relations were proposed to describe features of space structure on the Planck scale. The most simple and well-known is algebra with noncommutativity of coordinates of canonical type. The algebra is characterized by the modification of commutation relation for operators of coordinates. It reads

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1.1}$$
$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{1.2}$$
$$[P_i, P_j] = 0, \tag{1.3}$$

where θ_{ij} are parameters of coordinates noncommutativity which are elements of the constant antisymmetric matrix. The algebra describes a space with minimal length. Note, that the noncommutativity of coordinates can be used to describe motion of a particle in a strong magnetic field (see, for instance, [9–13]). Various physical problems have been examined in the frame of noncommutative algebra of canonical type. Among the first papers on the subject it is worth mention [14–18].

It is important to note that in $2D$ case the noncommutative algebra of canonical type is rotationally invariant

$$[X_1, X_2] = -[X_2, X_1] = i\hbar\theta, \tag{1.4}$$
$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{1.5}$$
$$[P_i, P_j] = 0, \tag{1.6}$$

where $i, j = (1, 2)$, θ is a parameter of noncommutativity. But in 3D case of noncommutative algebra (1.1)-(1.3) one faces a problem of rotational symmetry breaking [19, 20].

It is evident that the same problem appears in more general case when the noncommutativity of momenta is also considered. The noncommutative phase space of canonical type is characterized by the following commutation relations for coordinates and momenta

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1.7}$$
$$[X_i, P_j] = i\hbar(\delta_{ij} + \gamma_{ij}), \tag{1.8}$$
$$[P_i, P_j] = i\hbar\eta_{ij}. \tag{1.9}$$

Here θ_{ij}, η_{ij}, γ_{ij} are parameters of the algebra which in the case of noncommutative algebra of canonical type are considered to be elements of constant matrixes, θ_{ij} are parameters of coordinate noncommutativity, η_{ij} are parameters of momentum noncommutativity.

There are different ways of representation of the coordinates X_i and momenta P_i which do not commute (1.7), (1.9) Symmetrical representation is well known. It reads

$$X_i = x_i - \frac{1}{2}\sum_j \theta_{ij} p_j, \tag{1.10}$$

$$P_i = p_i + \frac{1}{2}\sum_j \eta_{ij} x_j, \tag{1.11}$$

here x_i, p_i are coordinates and momenta that satisfy the ordinary algebra. We have

$$[x_i, x_j] = 0, \tag{1.12}$$

$$[x_i, p_j] = i\hbar\delta_{ij}, \tag{1.13}$$

$$[p_i, p_j] = 0. \tag{1.14}$$

On the basis of expressions (1.10), (1.11), one find [21]

$$[X_i, P_j] = i\hbar\delta_{ij} + i\hbar\sum_k \frac{\theta_{ik}\eta_{jk}}{4}. \tag{1.15}$$

So, from the symmetrical representation follows that parameters γ_{ij} read

$$\gamma_{ij} = \sum_k \frac{\theta_{ik}\eta_{jk}}{4}. \tag{1.16}$$

New classes of noncommutative algebras were developed to preserve the rotational symmetry. In paper, [22] the idea of foliating the space with concentric fuzzy spheres was proposed to preserve the rotational symmetry. Rotationally-invariant noncommutative space was constructed as a sequence of fuzzy spheres in [23]. Author of paper [24] introduced the curved noncommutative space. In [25] promotion of the parameter of noncommutativity to an operator in Hilbert space was implemented to construct rotationally-invariant noncommutative algebra. Rotation invariance in N dimensional case was studied in [26].

In the present chapter, we present noncommutative algebra which is rotationally-invariant and besides it is equivalent to noncommutative algebra of canonical type. The algebra is constructed with the help of generalization of the parameters of noncommutativity to tensors. The tensors are defined by introducing additional coordinates and additional momenta governed by a system with rotational symmetry. The basis problems are studied in the frame of the algebra. They are free particle, and harmonic oscillator.

The Chapter is organized as follows. In section 1.2 rotationally-invariant algebra with noncommutativity of coordinates and noncommutativity of momenta which is equivalent to algebra of canonical type is introduced. The spectrum of free particle in rotationally-invariant noncommutative phase space is examined in section 1.3. The harmonic oscillator in noncommutative phase space with preserved rotational symmetry is studied in section 1.4. Section 1.5 is devoted to studies of minimal length in coordinate and momentum space based on expressions for eigenvalues of squared length operator. Conclusions are presented in section 1.6.

1.2 Noncommutative phase space of canonical type with preserved rotational symmetry

To construct an algebra which is rotationally-invariant and describes a noncommutative phase space we propose to generalize parameters of noncommutativity θ_{ij}, η_{ij} to tensors. The tensors are considered to be constructed with the help of additional coordinates and additional momenta. Tensors of coordinate noncommutativity read

$$\theta_{ij} = \frac{l_0}{\hbar}\varepsilon_{ijk}a_k. \tag{1.17}$$

Hare l_0 is a constant with the dimension of length and a_i are additional coordinates. For tensors of momentum noncommutativity we have the following expression

$$\eta_{ij} = \frac{p_0}{\hbar}\varepsilon_{ijk}p^b_k, \tag{1.18}$$

here p_0 is a constant, p^b_k are additional momenta.

We consider the additional coordinates a_i, b_i and momenta p^a_i, p^b_i

to satisfy the ordinary commutation relations. Namely, we have

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0, \tag{1.19}$$

$$[a_i, p^a_j] = [b_i, p^b_j] = i\hbar\delta_{ij}, \tag{1.20}$$

$$[p^a_i, p^a_j] = [p^b_i, p^b_j] = [p^a_i, p^b_j] = 0, \tag{1.21}$$

$$[a_i, p^b_j] = [b_i, p^a_j] = 0. \tag{1.22}$$

To preserve the rotational symmetry the additional coordinates and additional momenta are assumed to be governed by spherically-symmetric systems. For simplicity they are considered to be harmonic oscillators

$$H^a_{osc} = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}, \tag{1.23}$$

$$H^b_{osc} = \frac{(p^b)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 b^2}{2}. \tag{1.24}$$

Parameters of the oscillators are assumed to be as follows

$$\sqrt{\frac{\hbar}{m_{osc}\omega}} = l_P, \tag{1.25}$$

where l_P is the Planck's length. We also consider the frequency ω to be very large. In this case because of large distance between energy levels $\hbar\omega$ the oscillators are in the ground state.

Taking into account (1.16), (1.17), (1.18), we can write

$$\gamma_{ij} = \frac{l_0 p_0}{4\hbar^2}\left((\mathbf{a}\cdot\mathbf{p^b})\delta_{ij} - a_j p^b_i\right). \tag{1.26}$$

As a result, the noncommutative algebra is characterized by the following relations

$$[X_i, X_j] = i\varepsilon_{ijk} l_0 a_k, \tag{1.27}$$

$$[X_i, P_j] = i\hbar\left(\delta_{ij} + \frac{l_0 p_0}{4\hbar^2}(\mathbf{a}\cdot\mathbf{p^b})\delta_{ij} - \frac{l_0 p_0}{4\hbar^2} a_j p^b_i\right), \tag{1.28}$$

$$[P_i, P_j] = \varepsilon_{ijk} p_0 p^b_k. \tag{1.29}$$

Additional coordinates a_i, b_i can be treated as some internal coordinates of a particle. Quantum fluctuations of a_i, b_i lead effectively to a non-point-like particle. The size of the particle is of the order of the Planck scale.

It is important to note that γ_{ij}, θ_{ij}, η_{ij} commute with each other

$$[\theta_{ij}, \eta_{ij}] = [\theta_{ij}, \gamma_{ij}] = [\gamma_{ij}, \eta_{ij}] = 0. \tag{1.30}$$

Also, we have that the following commutation relations are satisfied

$$\begin{aligned}[t] [\theta_{ij}, X_k] = [\theta_{ij}, P_k] = [\eta_{ij}, X_k] = \\ = [\eta_{ij}, P_k] = [\gamma_{ij}, X_k] = [\gamma_{ij}, P_k] = 0. \end{aligned} \tag{1.31}$$

So, similarly as in the case when θ_{ij}, η_{ij}, and γ_{ij} are constants, tensors θ_{ij}, η_{ij} and γ_{ij} commute with coordinates and momenta. In this sense we have that the constructed algebra (1.27)-(1.29) is equivalent to noncommutative algebra of canonical type (1.7)-(1.9).

Commutation relations of algebra (1.27)-(1.29) remain the same after rotation

$$[X_i', X_j'] = i\varepsilon_{ijk} l_0 a_k', \tag{1.32}$$

$$[X_i', P_j'] = i\hbar \left(\delta_{ij} + \frac{l_0 p_0}{4\hbar^2} (\mathbf{a}' \cdot \mathbf{p}^{\mathbf{b}\prime}) \delta_{ij} - \frac{l_0 p_0}{4\hbar^2} a_j' p_i^{b\prime} \right), \tag{1.33}$$

$$[P_i', P_j'] = \varepsilon_{ijk} p_0 p_k^{b\prime}. \tag{1.34}$$

Here we use the following notations

$$X_i' = U(\varphi) X_i U^+(\varphi), \tag{1.35}$$

$$P_i' = U(\varphi) P_i U^+(\varphi), \tag{1.36}$$

$$a_i' = U(\varphi) a_i U^+(\varphi), \tag{1.37}$$

$$p_i^{b\prime} = U(\varphi) p_i^b U^+(\varphi). \tag{1.38}$$

The rotation operator reads

$$U(\varphi) = e^{\frac{i}{\hbar}\varphi(\mathbf{n}\cdot\tilde{\mathbf{L}})}, \tag{1.39}$$

where $\tilde{\mathbf{L}}$ is the total angular momentum defined as

$$\tilde{\mathbf{L}} = [\mathbf{r} \times \mathbf{p}] + [\mathbf{a} \times \mathbf{p}^a] + [\mathbf{b} \times \mathbf{p}^b], \tag{1.40}$$

$\mathbf{r} = (x_1, x_2, x_3)$. It is easy to show that $\tilde{\mathbf{L}}$ satisfies the following relations

$$[\tilde{L}_i, (\mathbf{a}\cdot\mathbf{p})] = [\tilde{L}_i, (\mathbf{b}\cdot\mathbf{p})] = [\tilde{L}_i, (\mathbf{a}\cdot\mathbf{b})] = 0, \quad (1.41)$$

$$[\tilde{L}_i, (\mathbf{r}\cdot\mathbf{a})] = [\tilde{L}_i, (\mathbf{r}\cdot\mathbf{b})] = 0, \quad (1.42)$$

$$[\tilde{L}_i, (\mathbf{a}\cdot\mathbf{L})] = [\tilde{L}_i, (\mathbf{b}\cdot\mathbf{L})] = [\tilde{L}_i, (\mathbf{p^a}\cdot\mathbf{L})] = [\tilde{L}_i, (\mathbf{p^b}\cdot\mathbf{L})] = 0, \quad (1.43)$$

$$[\tilde{L}_i, r^2] = [\tilde{L}_i, p^2] = [\tilde{L}_i, a^2] = [\tilde{L}_i, b^2] = 0, \quad (1.44)$$

$$[\tilde{L}_i, (p^a)^2] = [\tilde{L}_i, (p^b)^2] = 0. \quad (1.45)$$

Here for convenience we introduce notation $L = [\mathbf{r}\times\mathbf{p}]$. So, taking these relations into account we have that

$$[\tilde{L}_i, R] = 0, \quad (1.46)$$

where R is the operator of distance. This operator on the basis of (1.10), (1.11), (1.17), (1.18) can be rewritten as

$$R = \sqrt{\sum_i X_i^2} =$$

$$= \sqrt{r^2 + \frac{l_0^2}{4\hbar^2}a^2p^2 - \frac{l_0}{4\hbar^2}(\mathbf{a}\cdot\mathbf{p})^2 - \frac{l_0}{\hbar}(\mathbf{a}\cdot\mathbf{L})}. \quad (1.47)$$

So, after rotation, we obtain the same distance

$$R' = U(\varphi)RU^{+}(\varphi) = R. \quad (1.48)$$

Also, the operator of the total angular momentum commutes with momentum $P = \sqrt{\sum_i P_i^2}$. We have

$$[\tilde{L}_i, P] = 0, \quad (1.49)$$

$$P = \sqrt{p^2 + \frac{p_0^2}{4\hbar^2}r^2(p^b)^2 - \frac{p_0^2}{4\hbar^2}(\mathbf{r}\cdot\mathbf{p^b})^2 + \frac{p_0}{\hbar}(\mathbf{p^b}\cdot\mathbf{L})}. \quad (1.50)$$

So, the absolute value of momentum does not change after rotation

$$P' = U(\varphi)PU^{+}(\varphi) = P. \quad (1.51)$$

Commutators for coordinates and total angular momentum are the same as in the ordinary space (space with ordinary commutation relations for operators of coordinates and momenta)

$$[X_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}X_k, \tag{1.52}$$

$$[P_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}P_k, \tag{1.53}$$

$$[a_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}a_k, \tag{1.54}$$

$$[p^a_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}p^a_k, \tag{1.55}$$

$$[b_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}b_k, \tag{1.56}$$

$$[p^b_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}p^b_k. \tag{1.57}$$

Using (1.10), (1.11), (1.17), (1.18), for noncommutative coordinates and noncommutative momenta we have the following representation

$$X_i = x_i + \frac{l_0}{2\hbar}[\mathbf{a} \times \mathbf{p}]_i, \tag{1.58}$$

$$P_i = p_i - \frac{p_0}{2\hbar}[\mathbf{r} \times \mathbf{p^b}]_i. \tag{1.59}$$

The existence of such a representation guarantees that the Jacobi identity is satisfied for all possible triplets of operators. Also, it is important to note that from (1.58), (1.59) follows the following relations

$$[X_i, p^a_j] = i\varepsilon_{ijk}\frac{l_0}{2}p_k, \tag{1.60}$$

$$[P_i, b_j] = i\varepsilon_{ijk}\frac{l_0}{2}x_k, \tag{1.61}$$

$$[X_i, a_j] = [X_i, b_j] = [X_i, p^b_j] = 0, \tag{1.62}$$

$$[P_i, a_j] = [P_i, p^a_j] = [P_i, p^b_j] = 0. \tag{1.63}$$

1.3 Free particle in rotationally-invariant quantum phase space

Let us consider a free particle of mass m

$$H_p = \sum_i \frac{P_i^2}{2m}, \tag{1.64}$$

and study its energy levels in the frame of rotationally-invariant noncommutative algebra (1.27)-(1.29). So, momenta in the Hamiltonian do not commute, we have (1.29).

To construct algebra (1.27)-(1.29) we involve additional coordinates and additional momenta $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{p}_i^a$, $\tilde{p}_i^b$, So, to find energy levels of free particle in noncommutative phase space we have to consider the total Hamiltonian as follows

$$H = \sum_i \frac{P_i^2}{2m} + H^a_{osc} + H^b_{osc}. \tag{1.65}$$

Here H^a_{osc}, H^b_{osc} are Hamiltonians of harmonic oscillators, that are given by (1.23), (1.24). For convenience, we introduce the following operator

$$\Delta H = H_p - \langle H_p \rangle_{ab}. \tag{1.66}$$

Here $\langle ... \rangle_{ab}$ denotes averaging over the eigenstates of oscillators (1.23), (1.24) in the ground states $\psi^a_{0,0,0}$, $\psi^b_{0,0,0}$.

$$\langle ... \rangle_{ab} = \langle \psi^a_{0,0,0} \psi^b_{0,0,0} | ... | \psi^a_{0,0,0} \psi^b_{0,0,0} \rangle. \tag{1.67}$$

So, we can rewrite Hamiltonian (1.64) as follows

$$H = H_0 + \Delta H, \tag{1.68}$$

$$H_0 = \langle H_p \rangle_{ab} + H^a_{osc} + H^b_{osc}. \tag{1.69}$$

Up to the second order in ΔH in the rotationally-invariant noncommutative phase space we can study (1.69). To show this we find corrections caused by the term ΔH to the energy levels of the total Hamiltonian

$$H = H_s + H^a_{osc} + H^b_{osc}, \tag{1.70}$$

here H_s is a Hamiltonian of a system. It is important that

$$[\langle H_s \rangle_{ab}, H^a_{osc} + H^b_{osc}] = 0. \tag{1.71}$$

So, the eigenfunctions and the eigenvalues of Hamiltonian H_0 read

$$\psi^{(0)}_{\{n_s\},\{0\},\{0\}} = \psi^s_{\{n_s\}} \psi^a_{0,0,0} \psi^b_{0,0,0}, \tag{1.72}$$

$$E^{(0)}_{\{n_s\}} = E^s_{\{n_s\}} + 3\hbar\omega_{osc}, \tag{1.73}$$

Here for convenience we introduce the following notations $\psi^s_{\{n_s\}}$ are eigenfunctions and $E^s_{\{n_s\}}$ and eigenvalues of $\langle H_s\rangle_{ab}$, $\{n_s\}$ are quantum numbers. In the first order of the perturbation theory, the correction reads

$$\Delta E^{(1)} = \langle \psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0}|\Delta H|\psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle = \\ = \langle \psi^s_{\{n_s\}}|\langle H_s\rangle_{ab} - \langle H_s\rangle_{ab}|\psi^s_{\{n_s\}}\rangle = 0. \qquad (1.74)$$

Now, let us find corrections of the second order. We can write

$$\Delta E^{(2)} = \\ = \sum_{\{n'_s\},\{n^a\},\{n^b\}} \frac{\left|\left\langle \psi^{(0)}_{\{n'_s\},\{n^a\},\{n^b\}}\left|\Delta H\right|\psi^{(0)}_{\{n_s\},\{0\},\{0\}}\right\rangle\right|^2}{E^s_{\{n'_s\}} - E^s_{\{n_s\}} - \hbar\omega_{osc}(n^a_1+n^a_2+n^a_3+n^b_1+n^b_2+n^b_3)}. \qquad (1.75)$$

It is important to mention that the set $\{n'_s\}$, $\{n^a\}$, $\{n^b\}$ does not coincide with $\{n_s\}$,$\{0\}$, $\{0\}$. So, in the denominator of all terms in the sum we have oscillator frequency ω_{osc}. Mean values

$$\left\langle \psi^{(0)}_{\{n'_s\},\{n^a\},\{n^b\}}\left|\Delta H\right|\psi^{(0)}_{\{n_s\},\{0\},\{0\}}\right\rangle, \qquad (1.76)$$

do not depend on ω_{osc} This follows from the relation (1.25). In the limit $\omega_{osc}\to\infty$ the second order corrections are equal to zero

$$\lim_{\omega_{osc}\to\infty}\Delta E^{(2)} = 0. \qquad (1.77)$$

This result will be used in our studies of energy levels of different physical systems in the monograph.

So, let us apply the result for finding energy levels of free particle in rotationally-invariant noncommutative phase space.

To find $\langle H_p\rangle_{ab}$ we use representation of noncommutative coordinates and noncommutative momenta by x_i, p_i satisfying the ordinary commutation relations

$$X_i = x_i - \sum_j \frac{1}{2}\theta_{ij}p_j = x_i + \frac{1}{2}[\boldsymbol{\theta} \times \mathbf{p}]_i, \tag{1.78}$$

$$P_i = p_i + \sum_j \frac{1}{2}\eta_{ij}x_j = p_i - \frac{1}{2}[\boldsymbol{\eta} \times \mathbf{x}]_i, , \tag{1.79}$$

$$\theta_i = \sum_{jk} \varepsilon_{ijk}\frac{\theta_{jk}}{2} = \frac{c_\theta l_P^2}{\hbar}\tilde{a}_i, \tag{1.80}$$

$$\eta_i = \sum_{jk} \varepsilon_{ijk}\frac{\eta_{jk}}{2} = \frac{c_\eta \hbar}{l_P^2}\tilde{p}_i^b, \tag{1.81}$$

here $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{p} = (p_1, p_2, p_3)$. For convenience we introduce dimensionless constants c_θ, c_η and dimensionless coordinates and momenta

$$\tilde{a}_i = \frac{a_i}{l_P}, \quad \tilde{b}_i = \frac{b_i}{l_P}, \tag{1.82}$$

$$\tilde{p}_i^a = \frac{p_i^a l_P}{\hbar}, \quad \tilde{p}_i^b = \frac{p_i^a l_P}{\hbar}. \tag{1.83}$$

So, the Hamiltonian of a particle reads

$$H_p = \frac{p^2}{2m} - \frac{(\boldsymbol{\eta} \cdot [\mathbf{x} \times \mathbf{p}])}{2m} + \frac{[\boldsymbol{\eta} \times \mathbf{x}]^2}{8m}. \tag{1.84}$$

Note that H_p does not depend on the a_i, p_i^a. So, we have

$$\langle H_p \rangle_{ab} = \langle \psi_{0,0,0}^b | H_p | \psi_{0,0,0}^b \rangle. \tag{1.85}$$

It is easy to calculate

$$\langle \psi_{0,0,0}^b | \tilde{p}_i^b | \psi_{0,0,0}^b \rangle = 0, \tag{1.86}$$

$$\langle \psi_{0,0,0}^b | \tilde{p}_i^b \tilde{p}_j^b | \psi_{0,0,0}^b \rangle = \frac{1}{2}\delta_{ij}. \tag{1.87}$$

So, for $\langle \eta_i \rangle_{ab}$, and $\langle \eta^2 \rangle_{ab}$ we obtain

$$\langle \eta_i \rangle_{ab} = 0, \tag{1.88}$$

$$\langle \eta^2 \rangle = \langle \eta^2 \rangle_{ab} = \frac{3(\hbar c_\eta)^2}{2l_P^4}. \tag{1.89}$$

Therefore after averaging H_p over the eigenfunctions of the harmonic oscillators we obtain

$$\langle H_p \rangle_{ab} = \frac{p^2}{2m} + \frac{\langle \eta^2 \rangle x^2}{12m}. \tag{1.90}$$

On the basis of this result (1.90), we find

$$\Delta H = -\frac{(\boldsymbol{\eta} \cdot [\mathbf{x} \times \mathbf{p}])}{2m} + \frac{[\boldsymbol{\eta} \times \mathbf{x}]^2}{8m} - \frac{\langle \eta^2 \rangle x^2}{12m}. \tag{1.91}$$

Hamiltonian $\langle H_p \rangle_{ab}$ corresponds to the Hamiltonian of harmonic oscillator with mass m and frequency

$$\omega = \sqrt{\frac{\langle \eta^2 \rangle}{6m^2}}, \tag{1.92}$$

in the ordinary space (coordinates and momenta x_i, p_j satisfy the ordinary commutation relations).

Expression for ΔH contains terms of the first and second orders in the parameter of momentum noncommutativity. So, the energy levels of free particle in rotationally-invariant noncommutative phase space up to the second order in the parameter of momentum noncommutativity are as follows

$$E_{n_1,n_2,n_3} = \sqrt{\frac{\hbar^2 \langle \eta^2 \rangle}{6m^2}} \left(n_1 + n_2 + n_3 + \frac{3}{2} \right), \tag{1.93}$$

$n_1 = 0, 1, 2...$, $n_2 = 0, 1, 2...$, $n_3 = 0, 1, 2....$

So, we can conclude that because of the noncommutativity of momenta, the energy levels of free particles are quantized. They correspond to the energy levels of a harmonic oscillator with frequency determined by the parameter of momentum noncommutativity and given by (1.93)

1.4 Harmonic oscillator in rotationally-invariant space with noncommutativity of coordinates and noncommutativity of momenta

We consider three-dimensional harmonic oscillator with mass m and frequency ω in the frame of noncommutative algebra (1.27)-(1.29)

$$H_{osc} = \sum_i \frac{P_i^2}{2m} + \sum_i \frac{m\omega^2 X_i^2}{2}. \tag{1.94}$$

Similarly, as in the previous section let us write the total Hamiltonian

$$H = H_0 + \Delta H, \tag{1.95}$$
$$H_0 = \langle H_{osc}\rangle_{ab} + H^a_{osc} + H^b_{osc}, \tag{1.96}$$
$$\Delta H = H_{osc} - \langle H_{osc}\rangle_{ab}. \tag{1.97}$$

To find $\langle H_{osc}\rangle_{ab}$ we use representation (2.145)-(2.146) and rewrite the Hamiltonian as follows

$$H_{osc} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} - \frac{(\boldsymbol{\eta}\cdot[\mathbf{x}\times\mathbf{p}])}{2m} - \frac{m\omega^2(\boldsymbol{\theta}\cdot[\mathbf{x}\times\mathbf{p}])}{2} + \\ + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} + \frac{m\omega^2[\boldsymbol{\theta}\times\mathbf{p}]^2}{8}, \tag{1.98}$$

Using (4.27), (4.30), (1.86), (1.87) we obtain

$$\langle\psi^a_{0,0,0}|\tilde{a}_i|\psi^a_{0,0,0}\rangle = 0, \tag{1.99}$$
$$\langle\psi^a_{0,0,0}|\tilde{a}_i\tilde{a}_j|\psi^a_{0,0,0}\rangle = \frac{1}{2}\delta_{ij}. \tag{1.100}$$
$$\tag{1.101}$$

So, finally, we find

$$\langle H_{osc}\rangle_{ab} = \left(\frac{1}{2m} + \frac{m\omega^2\langle\theta^2\rangle}{12}\right)p^2 + \left(\frac{m\omega^2}{2} + \frac{\langle\eta^2\rangle}{12m}\right)x^2, \tag{1.102}$$

where we use the following notation

$$\langle\eta^2\rangle = \langle\eta^2\rangle_{ab} = \frac{3(\hbar c_\eta)^2}{2l_P^4}. \tag{1.103}$$

Note, that ΔH reads

$$\Delta H = -\frac{(\boldsymbol{\eta}\cdot[\mathbf{x}\times\mathbf{p}])}{2m} - \frac{m\omega^2(\boldsymbol{\theta}\cdot[\mathbf{x}\times\mathbf{p}])}{2} + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} + \\ + \frac{m\omega^2[\boldsymbol{\theta}\times\mathbf{x}]^2}{8}\frac{m\omega^2\langle\theta^2\rangle}{12}p^2 - \frac{\langle\eta^2\rangle}{12m}x^2, \quad (1.104)$$

and it contains terms of the first and second orders in the parameters of noncommutativity. So, up to the second order in the parameters of coordinates and momentum noncommutativity, we obtain the following energy levels of the harmonic oscillator

$$E_{n_1,n_2,n_3} = \\ = \hbar\sqrt{\left(m\omega^2+\frac{\langle\eta^2\rangle}{6m}\right)\left(\frac{1}{m}+\frac{m\omega^2\langle\theta^2\rangle}{6}\right)}\left(n_1+n_2+n_3+\frac{3}{2}\right) \quad (1.105)$$

n_1, n_2, n_3 are quantum numbers, $n_1 = 0, 1, 2...$, $n_2 = 0, 1, 2...$, $n_3 = 0, 1, 2...$.

Note that we have correspondence of the spectrum of harmonic oscillators in the quantum space written up to the second order in the parameters of noncommutativity and spectrum of harmonic oscillator in the ordinary space. Noncommutativity affects the mass and the frequency of the oscillator and does not affect the form of its spectrum. From (1.102), we have

$$m_{eff} = \frac{6m}{6+m^2\omega^2\langle\theta^2\rangle}, \quad (1.106)$$

$$\omega_{eff} = \sqrt{\left(m\omega^2+\frac{\langle\eta^2\rangle}{6m}\right)\left(\frac{1}{m}+\frac{m\omega^2\langle\theta^2\rangle}{6}\right)}. \quad (1.107)$$

Note, that in the limits $\langle\theta^2\rangle \to 0$, $\langle\eta^2\rangle \to 0$ we obtain $m_{eff} = m$, $\omega_{eff} = \omega$. So, the limits expression (1.105) reduces to the spectrum of the harmonic oscillator in the ordinary space.

Based on the results obtained in this section in the next section we will study the length in quantum space with preserved rotational symmetry.

1.5 Length in quantum space with preserved rotational symmetry

We study squared length operator defined as

$$\mathbf{Q}^2 = \alpha^2 \sum_i P_i^2 + \beta^2 \sum_i X_i^2, \tag{1.108}$$

with α and β being constants. Let us find the eigenvalues of the operator in noncommutative phase space with preserved rotational symmetry. So, we consider X_i, P_i satisfying relations of algebra (1.27)-(1.29).

Operator $\mathbf{Q}^2$ can be considered as Hamiltonian of a tree-dimensional harmonic oscillator with mass

$$m = \frac{1}{2\alpha^2}, \tag{1.109}$$

and frequency

$$\omega = 2\alpha\beta. \tag{1.110}$$

So, we can use results presented in the previous section and write eigenvalues of the operator $\mathbf{Q}^2$ up to the second order in the parameters of noncommutativity as follows

$$q^2_{n_1,n_2,n_3} = \\ = \hbar\sqrt{\left(2\beta^2 + \frac{\alpha^2\langle\eta^2\rangle}{3}\right)\left(2\alpha^2 + \frac{\beta^2\langle\theta^2\rangle}{3}\right)}\left(n_1 + n_2 + n_3 + \frac{3}{2}\right), \tag{1.111}$$

$n_1 = 0, 1, 2...$, $n_2 = 0, 1, 2...$, $n_3 = 0, 1, 2...$. Let us analyze the minimal length on the basis of result (1.111). We have

$$q^2_{min} = \sqrt{q^2_{0,0,0}} = \sqrt{\hbar}\sqrt[4]{2\beta^2 + \frac{\alpha^2\langle\eta^2\rangle}{3}}\sqrt[4]{2\alpha^2 + \frac{\beta^2\langle\theta^2\rangle}{3}} \tag{1.112}$$

So, the expression for the minimal length depends on the parameters of coordinate and momentum noncommutativity $\langle\theta^2\rangle$, $\langle\eta^2\rangle$.

Let us study particular cases. Namely, $\alpha = 0$, $\beta = 1$. In this case one has the squared length operator in the coordinate space

$$\mathbf{R}^2 = \sum_{i=1}^{3} X_i^2. \tag{1.113}$$

Using (1.111), the eigenvalues of the operator read

$$r^2_{n_1,n_2,n_3} = \sqrt{\frac{2\hbar^2\langle\theta^2\rangle}{3}}\left(n_1 + n_2 + n_3 + \frac{3}{2}\right), \tag{1.114}$$

here $n_1 = 0, 1, 2...$, $n_2 = 0, 1, 2...$, $n_3 = 0, 1, 2....$ From (1.114) expression follows that the squared length is quantized. This is caused by the noncommutativity of coordinates. The minimal length in the coordinate space reads

$$r_{min} = \sqrt{r^2_{0,0,0}} = \sqrt{\frac{3\hbar^2\langle\theta^2\rangle}{2}}. \tag{1.115}$$

It is determined by the parameter of coordinate noncommutativity $\langle\theta^2\rangle$.

Let us also study another particular case when $\alpha = 1$, $\beta = 0$. In this case we have squared length operator in momentum space. It reads

$$\mathbf{P}^2 = \sum_{i=1}^{3} P_i^2. \tag{1.116}$$

$$p^2_{n_1,n_2,n_3} = \sqrt{\frac{2\hbar^2\langle\eta^2\rangle}{3}}\left(n_1 + n_2 + n_3 + \frac{3}{2}\right), \tag{1.117}$$

$n_1 = 0, 1, 2...$, $n_2 = 0, 1, 2...$, $n_3 = 0, 1, 2....$ And the minimal length in the momentum space is determined by parameter of momentum noncommutativity. We have

$$p_{min} = \sqrt{p^2_{0,0,0}} = \sqrt[4]{\frac{3\hbar^2\langle\eta^2\rangle}{2}}. \tag{1.118}$$

1.6 Conclusions

A way to construct algebra with noncommutativity of coordinates and noncommutativity of momenta which is rotationally-invariant and equivalent to noncommutative algebra of canonical type has been proposed. The idea of generalization of the parameters of noncommutativity to tensors has been used to construct algebra (1.7)-(1.9). The tensors have been defined with the help of additional coordinates and conjugate momenta of them that are governed by harmonic oscillators. The length of the oscillators has been considered to be the Planck length. The frequency of the oscillators is assumed to be very large. Therefore harmonic oscillators that are in the ground state remains in them.

The spectrum of free particle has been studied in the frame of rotationally-invariant noncommutative algebra. Up to the second order in the parameters of noncommutativity it is shown that the energy levels of a free particle in noncommutative phase space correspond to the energy levels of harmonic oscillator (1.93) with frequency defined by the parameter of momentum noncommutativity (1.92).

Also, harmonic oscillator has been examined in rotationally-invariant noncommutative phase space. We have found energy levels of the oscillator up to the second order in the parameters of noncommutativity. We have concluded that noncommutativity of coordinates and noncommutativity of momenta affect on the mass and the frequency of the oscillator. The expression for the energy levels of the harmonic oscillator in noncommutative phase space corresponds to that in the ordinary space.

Based on the obtained results the minimal length has been studied in the frame of rotationally-invariant noncommutative algebra. Squared length operator has been considered in coordinate, momentum space, and phase space. The eigenvalues of the operators (1.111), (1.114), (1.117) have been obtained up to the second order in the parameters of coordinate and momentum noncommutativity. Based on the results the minimal lengths in coordinate space, momentum space, and phase space have been obtained (1.115), (1.118), (1.112).

Chapter 2

Hydrogen and exotic atoms in noncommutative space with preserved rotational symmetry

2.1 Introduction

Studies of a hydrogen atom in noncommutative space have received much attention (see [27–37]). In paper [27] energy levels of hydrogen atom were obtained up to the first order in the parameter of noncommutativity. In the paper, the Lamb shift in noncommutative space was studied. In paper [28] the case when particles of opposite charges feel opposite noncommutativity was examined. In the frame of such an algebra the hydrogen atom as a two-particle system was considered. In [30] the quadratic Stark effect was studied. In [31] shifts in the spectrum of hydrogen atom caused by space quantization were presented. In [32] the noncommutative Klein-Gordon equation was studied and the hydrogen atom energy levels were calculated. The influence of noncommutativity on the Dirac equation with a Coulomb field was studied in [33, 34].

Effect of noncommutativity of coordinates and noncommutativity of momenta on the energy levels of hydrogen atom was examined in [35–37]. Hydrogen atom problem in the frame of space-time noncommutativity was considered in [38–42].

In the present chapter, we examine hydrogen atom and hydrogen-like exotic atoms in the frame of rotationally-invariant noncommutative algebra of canonical type. The energy levels of the hydrogen atom are found up to the second order in the parameters of coordinate and momentum noncommutativity. Based on the obtained results the upper bounds for the parameters are estimated. Also, a two-particle system with Coulomb interaction is studied in the frame of rotationally-invariant noncommutative algebra. We examine the influence of space quantization on the energy levels of the system. Based on the obtained results the energy levels of muonic hydrogen and antiprotonic helium are examined.

The chapter is organized as follows. In section 2.2 the Hamiltonian of a hydrogen atom is examined in the frame of rotationally-invariant noncommutative algebra. In section 2.2 corrections to the energy levels of hydrogen atom are found up to the second order in the perturbation theory. Section 2.3 is devoted to studies of the corrections to the ns energy levels of hydrogen atom. The effect of noncommutaivity on energy levels of the hydrogen-like atoms is examined in section 2.5. Upper founds for the parameters of coordinate and momentum noncommutativity are obtained in section 2.6. Section 2.7 is devoted to conclusions. Results presented in this chapter are published in [43–46].

2.2 Hamiltonian of hydrogen atom in noncommutative phase space with preserved rotational symmetry

Let us consider hydrogen atom and find corrections to the energy levels of the atom in rotationally invariant noncommutative phase space (1.27)-(1.29). So, we consider the total Hamiltonian

$$H = H_h + H^a_{osc} + H^b_{osc}, \tag{2.1}$$

where

$$H_h = \frac{P^2}{2M} - \frac{e^2}{R}, \tag{2.2}$$

is the Hamiltonian of the hydrogen atom. Here $R = \sqrt{\sum_i X_i^2}$, coordinates X_i and momenta P_i satisfy relations of noncommutative algebra (1.27)-(1.29) Hamiltonians H^a_{osc}, H^b_{osc} correspond to harmonic oscillators and are given by (1.23), (1.24).

Using representation for coordinates and momenta that satisfy relations of noncommutative algebra by coordinates and momenta satisfying the ordinary relations we can write

$$H_h = \frac{p^2}{2M} + \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M} + \frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{8M} - \frac{e^2}{\sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}. \quad (2.3)$$

To find effect of noncommutativity of the energy levels of the hydrogen atom we expand Hamiltonian of the hydrogen atom in the series over $\boldsymbol{\theta}$. For $1/R$ we obtain

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}} = \\ = \frac{1}{r} + \frac{1}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{3}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 - \\ -\frac{1}{16}\left(\frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2\right). \quad (2.4)$$

To find expansion for $1/R$, firstly we solve problem of finding the expansion of R up to the second order in $\boldsymbol{\theta}$. Expression for the distance reads

$$R = \sqrt{(\mathbf{r} + \frac{1}{2}[\boldsymbol{\theta}\times\mathbf{p}])^2} = \sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}. \quad (2.5)$$

It is important to stress that the operators under the square root do not commute. Therefore we introduce unknown function $f(\mathbf{r})$ and find the expansion in the following form

$$R = r - \frac{1}{2r}(\boldsymbol{\theta}\cdot\mathbf{L}) - \frac{1}{8r^3}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + \\ +\frac{1}{16}\left(\frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2 + [\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \theta^2 f(\mathbf{r})\right). \quad (2.6)$$

Then to obtain $f(\mathbf{r})$ we square left- and right-hand sides of equation (2.6). Up to the second order in $\boldsymbol{\theta}$ we can write

$$r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2 = r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \\ +\frac{1}{16}\left(2[\boldsymbol{\theta}\times\mathbf{p}]^2 + r[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2 r + 2r\theta^2 f(\mathbf{r})\right). \quad (2.7)$$

From (2.7) we have

$$\frac{\hbar^2}{r^4}[\boldsymbol{\theta} \times \mathbf{r}]^2 - r\theta^2 f(\mathbf{r}) = 0. \tag{2.8}$$

And finally, function $f(\mathbf{r})$ reads

$$\theta^2 f(\mathbf{r}) = \frac{\hbar^2}{r^5}[\boldsymbol{\theta} \times \mathbf{r}]^2. \tag{2.9}$$

So, expansion for the distance is as follows

$$R = r - \frac{1}{2r}(\boldsymbol{\theta} \cdot \mathbf{L}) - \frac{1}{8r^3}(\boldsymbol{\theta} \cdot \mathbf{L})^2 + \\ + \frac{1}{16}\left(\frac{1}{r}[\boldsymbol{\theta} \times \mathbf{p}]^2 + [\boldsymbol{\theta} \times \mathbf{p}]^2 \frac{1}{r} + \frac{\hbar^2}{r^5}[\boldsymbol{\theta} \times \mathbf{r}]^2\right). \tag{2.10}$$

Then on the basis of this result we can easily write (2.4). As a result the total Hamiltonian reads

$$H = H_0 + V. \tag{2.11}$$

Here V is the perturbation operator

$$V = \frac{(\boldsymbol{\eta} \cdot \mathbf{L})}{2M} + \frac{[\boldsymbol{\eta} \times \mathbf{r}]^2}{8M} - \frac{e^2}{2r^3}(\boldsymbol{\theta} \cdot \mathbf{L}) - \frac{3e^2}{8r^5}(\boldsymbol{\theta} \cdot \mathbf{L})^2 + \\ + \frac{e^2}{16}\left(\frac{1}{r^2}[\boldsymbol{\theta} \times \mathbf{p}]^2 \frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta} \times \mathbf{p}]^2 \frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta} \times \mathbf{r}]^2\right), \tag{2.12}$$

and H_0 contains Hamiltonian of the hydrogen atom in the ordinary space and terms corresponding to harmonic oscillators

$$H_0 = \frac{p^2}{2M} - \frac{e^2}{r} + H^a_{osc} + H^b_{osc}. \tag{2.13}$$

2.3 Effect of noncommutativity of the energy levels of the hydrogen atom

Let us calculate corrections to the energy levels of hydrogen atom caused by noncommutativity of coordinates and noncommutativity of momenta. It is important that

$$\left[\frac{p^2}{2M} - \frac{e^2}{r}, H^a_{osc}\right] = \left[\frac{p^2}{2M} - \frac{e^2}{r}, H^b_{osc}\right] = \left[H^a_{osc}, H^b_{osc}\right] = 0. \tag{2.14}$$

So, the eigenvalues and eigenstates of Hamiltonian H_0 can be written as follows

$$E^{(0)}_{n,\{n^a\},\{n^b\}} = -\frac{e^2}{2a_B n^2} + \\ +\hbar\omega(n_1^a + n_2^a + n_3^a + n_1^b + n_2^b + n_3^b + 3), \tag{2.15}$$

$$\psi^{(0)}_{n,l,m,\{n^a\},\{n^b\}} = \psi_{n,l,m}\psi^a_{n_1^a,n_2^a,n_3^a}\psi^b_{n_1^b,n_2^b,n_3^b}. \tag{2.16}$$

Here a_B is the Bohr radius, $\psi_{n,l,m}$ are well known eigenfunctions of the hydrogen atom in the ordinary space ($\theta_{ij} = \eta_{ij} = 0$) and $\psi^a_{n_1^a,n_2^a,n_3^a}$, $\psi^b_{n_1^b,n_2^b,n_3^b}$ are eigenfunctions of three-dimensional harmonic oscillators H^a_{osc}, H^b_{osc}. Using perturbation theory and taking into account that the frequency of the oscillators is large and they are in the ground states we can write

$$\Delta E^{(1)}_{n,l} = \langle \psi^{(0)}_{n,l,m,\{0\},\{0\}}|V|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\rangle. \tag{2.17}$$

Note, that

$$\langle \psi^a_{0,0,0}|\theta_i|\psi^a_{0,0,0}\rangle = 0, \tag{2.18}$$

$$\langle \psi^b_{0,0,0}|\eta_i|\psi^b_{0,0,0}\rangle = 0. \tag{2.19}$$

So, we can write

$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M} \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle = 0, \tag{2.20}$$

$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| \frac{e^2}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L}) \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle = 0. \tag{2.21}$$

Effect of momentum noncommutativity is represented by the terms $[\boldsymbol{\eta}\times\mathbf{r}]^2/8M$. The corrections caused by the term reads

$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| \frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{8M} \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle = \\ = \left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| \frac{\eta^2 r^2}{8M} - \frac{(\boldsymbol{\eta}\cdot\mathbf{r})^2}{8M} \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle = \\ = \frac{a_B^2 n^2}{24M}(5n^2 + 1 - 3l(l+1))\langle\eta^2\rangle. \tag{2.22}$$

To write the expression we use

$$\langle\psi^b_{0,0,0}|\eta_i\eta_j|\psi^b_{0,0,0}\rangle = \frac{m_{osc}\omega p_o^2}{2\hbar}\delta_{ij} = \frac{1}{3}\langle\eta^2\rangle\delta_{ij}, \tag{2.23}$$

where $\langle\eta^2\rangle$ is given by

$$\langle\eta^2\rangle = \frac{p_o^2}{\hbar^2}\langle\psi^b_{0,0,0}|(p^b)^2|\psi^b_{0,0,0}\rangle = \frac{3m_{osc}\omega p_o^2}{2\hbar} = \frac{3p_o^2}{2l_P^2}. \tag{2.24}$$

We also take into account the following result for the mean value (see, for example, [47])

$$\left\langle\psi_{n,l,m}\left|r^2\right|\psi_{n,l,m}\right\rangle = a_B^2\frac{n^2}{2}(5n^2+1-3l(l+1)). \tag{2.25}$$

To find correction caused by term $3e^2(\boldsymbol{\theta}\cdot\mathbf{L})^2/8r^5$ we take into account the following result for the mean value (see for instance [47])

$$\left\langle\psi_{n,l,m}\left|\frac{1}{r^5}\right|\psi_{n,l,m}\right\rangle = \\ = \frac{4(5n^2-3l(l+1)+1)}{a_B^5 n^5 l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}. \tag{2.26}$$

We also calculate

$$\langle\psi^a_{0,0,0}\psi^b_{0,0,0}|\theta_i\theta_j|\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle = \frac{1}{2}\left(\frac{\alpha}{m\omega}\right)^2\delta_{ij} = \frac{1}{3}\langle\theta^2\rangle\delta_{ij}. \tag{2.27}$$

As a result, on the basis of (2.26), (2.27), we find

$$\left\langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{3e^2}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle = \\ = \frac{\hbar^2e^2(5n^2-3l(l+1)+1)\langle\theta^2\rangle}{2a_B^5n^5(l+2)(2l+1)(2l+3)(l-1)(2l-1)}. \tag{2.28}$$

Then, let us rewrite last terms in the perturbation as follows

$$\frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2 = \theta^2\frac{1}{r^2}p^2\frac{1}{r} + \\ +\theta^2\frac{1}{r}p^2\frac{1}{r^2} + \theta^2\frac{\hbar^2}{r^5} - \frac{1}{r^2}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r} - \frac{1}{r}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r^2} - \frac{\hbar^2}{r^7}(\boldsymbol{\theta}\cdot\mathbf{r})^2. \tag{2.29}$$

So, after averaging over the eigenfunctions of the harmonic oscillators we find

$$\left\langle \psi^a_{0,0,0}\psi^b_{0,0,0} \left| \frac{1}{r^2}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r} + \frac{1}{r}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}(\boldsymbol{\theta}\cdot\mathbf{r})^2 \right| \psi^a_{0,0,0}\psi^b_{0,0,0} \right\rangle = = \frac{1}{3}\left(\frac{1}{r^2}p^2\frac{1}{r} + \frac{1}{r}p^2\frac{1}{r^2} + \frac{\hbar^2}{r^5}\right)\langle\theta^2\rangle. \tag{2.30}$$

So, we can write

$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| \frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2} \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle + + \left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2 \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle = = \frac{2}{3}\left\langle \psi_{n,l,m} \left| \frac{1}{r^2}p^2\frac{1}{r} + \frac{1}{r}p^2\frac{1}{r^2} + \frac{\hbar^2}{r^5} \right| \psi_{n,l,m} \right\rangle \langle\theta^2\rangle. \tag{2.31}$$

We represent $\frac{1}{r^2}p^2\frac{1}{r} + \frac{1}{r}p^2\frac{1}{r^2} + \frac{\hbar^2}{r^5}$ as follows

$$\frac{1}{r^2}p^2\frac{1}{r} + \frac{1}{r}p^2\frac{1}{r^2} + \frac{\hbar^2}{r^5} = \frac{1}{r^3}p^2 + p^2\frac{1}{r^3} + \frac{5\hbar^2}{r^5}. \tag{2.32}$$

So, the correction reads

$$\left\langle \psi_{n,l,m} \left| \frac{1}{r^2}p^2\frac{1}{r} + \frac{1}{r}p^2\frac{1}{r^2} + \frac{\hbar^2}{r^5} \right| \psi_{n,l,m} \right\rangle = = -\frac{2\hbar^2}{a_B^2 n^2}\left\langle \psi_{n,l,m} \left| \frac{1}{r^3} \right| \psi_{n,l,m} \right\rangle + + \frac{4\hbar^2}{a_B}\left\langle \psi_{n,l,m} \left| \frac{1}{r^4} \right| \psi_{n,l,m} \right\rangle + 5\hbar^2\left\langle \psi_{n,l,m} \left| \frac{1}{r^5} \right| \psi_{n,l,m} \right\rangle. \tag{2.33}$$

Taking into account well-known results for the mean values (see for instance [47])

$$\left\langle \psi_{n,l,m} \left| \frac{1}{r^3} \right| \psi_{n,l,m} \right\rangle = \frac{2}{a_B^3 n^3 l(l+1)(2l+1)}, \tag{2.34}$$

$$\left\langle \psi_{n,l,m} \left| \frac{1}{r^4} \right| \psi_{n,l,m} \right\rangle = \frac{4(3n^2 - l(l+1))}{a_B^4 n^5 l(l+1)(2l+1)(2l+3)(2l-1)}, \tag{2.35}$$

and all the obtained results we can write explicit expression for the corrections to the energy levels caused by coordinates noncommutativity. It reads

$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| -\frac{3e^2}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + \frac{e^2}{16r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle +$$
$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} \left| \frac{e^2}{16}\left(+\frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2 \right) \right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle =$$
$$-\frac{\hbar^2 e^2\langle\theta^2\rangle}{a_B^5 n^5}\left(\frac{1}{6l(l+1)(2l+1)} - \frac{6n^2-2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} + \frac{5n^2-3l(l+1)+1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} - \frac{5}{6}\frac{5n^2-3l(l+1)+1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)} \right), \tag{2.36}$$

where $\langle\theta^2\rangle$ is given by

$$\langle\theta^2\rangle = \frac{l_0^2}{\hbar^2}\langle\psi^a_{0,0,0}|a^2|\psi^a_{0,0,0}\rangle = \frac{3l_0^2}{2\hbar}\left(\frac{1}{m_{osc}\omega}\right) = \frac{3l_0^2 l_P^2}{2\hbar^2}. \tag{2.37}$$

So, using (2.20), (2.21), (2.22) and (2.36) in the first order of perturbation theory effect of noncommutativity on the energy levels of hydrogen atom is as follows

$$\Delta E^{(1)}_{n,l} = \Delta E^{(\eta)}_{n,l} + \Delta E^{(\theta)}_{n,l}, \tag{2.38}$$

where

$$\Delta E^{(\eta)}_{n,l} = \frac{a_B^2 n^2\langle\eta^2\rangle}{24M}(5n^2+1-3l(l+1)), \tag{2.39}$$

are corrections to the spectrum caused by the noncommutativity of momenta and

$$\Delta E^{(\theta)}_{n,l} = -\frac{\hbar^2 e^2 \langle\theta^2\rangle}{a_B^5 n^5}\left(-\frac{6n^2 - 2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} + \right.$$
$$+\frac{1}{6l(l+1)(2l+1)} + \frac{5n^2 - 3l(l+1)+1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} -$$
$$\left. -\frac{5}{6}\frac{5n^2 - 3l(l+1)+1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right), \tag{2.40}$$

being corrections caused by coordinates noncommutativity.

Note that in the second order of the perturbation theory we have

$$\Delta E^{(2)}_{n,l,m,\{0\}} = \sum_{n',l',m',\{n^a\},\{n^b\}} \frac{\left|\left\langle \psi^{(0)}_{n',l',m',\{n^a\},\{n^b\}} \left|V\right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle\right|^2}{E^{(0)}_n - E^{(0)}_{n'} - \hbar\omega(n^a_1 + n^a_2 + n^a_3 + n^b_1 + n^b_2 + n^b_3)}, \tag{2.41}$$

$$E^{(0)}_n = -\frac{e^2}{2a_B n^2}. \tag{2.42}$$

In the limit $\omega \to \infty$ this correction vanish

$$\lim_{\omega\to\infty} \Delta E^{(2)}_{n,l,m,\{0\}} = 0. \tag{2.43}$$

So, up to the second order in the parameters of coordinates noncommutativity and parameters of momentum noncommutativity the corrections to the energy levels of the hydrogen atom are as follows

$$\Delta E_{n,l} = \Delta E^{(1)}_{n,l}. \tag{2.44}$$

It is important to stress that the obtained corrections to the energy levels of the hydrogen atom (2.44) are divergent for $l = 0$ and $l = 1$. From this follows that we can not use expansion of hamiltonian into the series over the parameter of coordinate noncommutativity. In the next section we find finite result for corrections to the ns energy levels of the hydrogen atom. We are interested in the corrections because on the basis of the results, stringent upper bound for the minimal length can be found.

2.4 Corrections to the ns energy levels of the hydrogen atom in noncommutative phase space

To calculate corrections to the ns energy levels we rewrite perturbation V caused by noncommutativity of coordinates and noncommutativity of momenta as follows

$$V=\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M}+\frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{2M}-\frac{e^2}{R}+\frac{e^2}{r}= \\ =-\frac{e^2}{\sqrt{r^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}+\frac{e^2}{r}. \tag{2.45}$$

So, the corrections read

$$\Delta E_{ns}= \\ =\left\langle\psi^{(0)}_{n,0,0,\{0\},\{0\}}\left|\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M}+\frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{8M}\right|\psi^{(0)}_{n,0,0,\{0\},\{0\}}\right\rangle+ \\ +\left\langle\psi^{(0)}_{n,0,0,\{0\},\{0\}}\left|-\frac{e^2}{r}-\frac{e^2}{\sqrt{r^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}\right|\psi^{(0)}_{n,0,0,\{0\},\{0\}}\right\rangle. \tag{2.46}$$

It is important to note that

$$[(\boldsymbol{\theta}\cdot\mathbf{L}),[\boldsymbol{\theta}\times\mathbf{p}]^2]=[(\boldsymbol{\theta}\cdot\mathbf{L}),r^2]=0. \tag{2.47}$$

Also, we have

$$(\boldsymbol{\theta}\cdot\mathbf{L})\psi^{(0)}_{n,0,0,\{0\},\{0\}}(\mathbf{r},\mathbf{a},\mathbf{b})=0. \tag{2.48}$$

So, we can write

$$\Delta E_{ns}=\frac{a_B^2n^2\langle\eta^2\rangle}{24M}(5n^2+1)+ \\ \left\langle\psi^{(0)}_{n,0,0,\{0\},\{0\}}(\mathbf{r},\mathbf{a},\mathbf{b})\left|\frac{e^2}{r}-\frac{e^2}{\sqrt{r^2+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}\right|\psi^{(0)}_{n,0,0,\{0\},\{0\}}(\mathbf{r},\mathbf{a},\mathbf{b})\right\rangle. \tag{2.49}$$

We introduce $\mathbf{a}' = \mathbf{a}/l_p$, $\mathbf{b}' = \mathbf{b}/l_p$,

$$\mathbf{r}' = \sqrt{\frac{2}{\alpha}}\frac{\mathbf{r}}{l_p}, \tag{2.50}$$

with l_p being the Planck length. So, we can rewrite $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta} = \frac{\alpha l_p^2}{\hbar}\boldsymbol{\theta}', \tag{2.51}$$

$$\boldsymbol{\theta}' = [\mathbf{a}' \times \mathbf{b}']. \tag{2.52}$$

So, the corrections caused by noncommutativity of coordinates $\Delta E_{ns}^{(\theta)}$ reads

$$\Delta E_{ns}^{(\theta)} = \frac{\chi^2 e^2}{a_B} I_{ns}(\chi), \tag{2.53}$$

where

$$I_{ns}(\chi) = \int d\mathbf{a}' \tilde{\psi}^a_{0,0,0}(\mathbf{a}') \int d\mathbf{b}' \tilde{\psi}^b_{0,0,0}(\mathbf{b}') \int d\mathbf{r}' \tilde{\psi}_{n,0,0}(\chi\mathbf{r}')$$
$$\left(\frac{1}{r'} - \frac{1}{\sqrt{(r')^2 + [\boldsymbol{\theta}' \times \mathbf{p}']^2}}\right) \tilde{\psi}_{n,0,0}(\chi\mathbf{r}') \tilde{\psi}^a_{0,0,0}(\mathbf{a}') \tilde{\psi}^b_{0,0,0}(\mathbf{b}'), \tag{2.54}$$

and

$$\chi = \sqrt{\frac{\alpha}{2}}\frac{l_p}{a_B}. \tag{2.55}$$

Eigenfunctions of harmonic oscillators and hydrogen atom read

$$\tilde{\psi}^a_{0,0,0}(\mathbf{a}') = \pi^{-\frac{3}{4}} e^{-\frac{(a')^2}{2}}, \tag{2.56}$$

$$\tilde{\psi}^b_{0,0,0}(\mathbf{b}') = \pi^{-\frac{3}{4}} e^{-\frac{(b')^2}{2}}, \tag{2.57}$$

$$\tilde{\psi}_{n,0,0}(\chi\mathbf{r}') = \sqrt{\frac{1}{\pi n^5}} e^{-\frac{\chi r'}{n}} L^1_{n-1}\left(\frac{2\chi r'}{n}\right), \tag{2.58}$$

$L^1_{n-1}\left(\frac{2\chi r'}{n}\right)$ are the generalized Laguerre polynomials.

Integral (2.54) is finite for $\chi = 0$. So, the asymptotic of $\Delta E_{ns}^{(\theta)}$ for $\chi \to 0$ reads

$$\Delta E_{ns}^{(\theta)} = \frac{\chi^2 e^2}{a_B} I_{ns}(0). \tag{2.59}$$

So, to obtain the asymptotic of $\Delta E_{ns}^{(\theta)}$ we have to calculate integral $I_{ns}(0)$. As the firs step we consider the integral over $\mathbf{r}'$. We have

$$I_{ns}(\chi, \boldsymbol{\theta}') = \\ = \int d\mathbf{r}' \tilde{\psi}_{n,0,0}(\chi \mathbf{r}') \left(\frac{1}{r'} - \frac{1}{\sqrt{(r')^2 + [\boldsymbol{\theta}' \times \mathbf{p}']^2}} \right) \tilde{\psi}_{n,0,0}(\chi \mathbf{r}'). \tag{2.60}$$

In the momentum representation the integral reads

$$I_{ns}(\chi, \boldsymbol{\theta}') = \\ = \frac{1}{\chi^6} \int d\mathbf{p}' \tilde{\psi}_{n,0,0} \left(\frac{\mathbf{p}'}{\chi} \right) \times \\ \times \left(\frac{1}{\sqrt{-\nabla_{p'}^2}} - \frac{1}{\sqrt{-\nabla_{p'}^2 + [\boldsymbol{\theta}' \times \mathbf{p}']^2}} \right) \tilde{\psi}_{n,0,0} \left(\frac{\mathbf{p}'}{\chi} \right), \tag{2.61}$$

here

$$\nabla_{p'}^2 = \sum_i \frac{\partial^2}{(\partial p_i')^2}. \tag{2.62}$$

Integral $I_{ns}(\chi, \boldsymbol{\theta}')$ does not depend on the direction of the vector $\boldsymbol{\theta}'$.

So, we can rewrite the integral as

$$I_{ns}(\chi,\boldsymbol{\theta}') = I_{ns}(\chi,\theta') = \frac{1}{4\pi\chi^6}\int d\Omega \int d\mathbf{p}'\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right)\times$$
$$\times\left(\frac{1}{\sqrt{-\nabla^2_{p'}}}-\frac{1}{\sqrt{-\nabla^2_{p'}+[\boldsymbol{\theta}'\times\mathbf{p}']^2}}\right)\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right)=$$
$$=\frac{1}{4\pi\chi^6}\int d\Omega \int d\mathbf{p}'\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right)\times$$
$$\times\left(\frac{1}{\sqrt{-\nabla^2_{p'}}}-\frac{1}{\sqrt{-\nabla^2_{p'}+(\theta')^2(p')^2\sin^2\Theta}}\right)\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right), \tag{2.63}$$

here Θ is an angle between vectors $\boldsymbol{\theta}'$ and $\mathbf{p}'$, $\theta' = |\boldsymbol{\theta}'|$, and $d\Omega = \sin\Theta d\Theta d\Phi$.

Let us use substitution

$$\tilde{\mathbf{p}} = \kappa\mathbf{p}', \tag{2.64}$$
$$\kappa = \sqrt{\theta'\sin\Theta}, \tag{2.65}$$

and return to the coordinate representation. So, we have

$$I_{ns}(\chi,\theta') = \frac{\theta'}{2}\int_0^{\pi} d\Theta\sin^2\Theta\int d\tilde{\mathbf{r}}\tilde{\psi}_{n,0,0}(\kappa\chi\tilde{\mathbf{r}})\times$$
$$\times\left(\frac{1}{\tilde{r}}-\frac{1}{\sqrt{\tilde{r}^2+\tilde{p}^2}}\right)\tilde{\psi}_{n,0,0}(\kappa\chi\tilde{\mathbf{r}}) = \frac{\theta'}{2}\int_0^{\pi} d\Theta\sin^2\Theta\int_0^{\infty} d\tilde{r}\times$$
$$\times\tilde{r}^2\tilde{R}_{n,0}(\kappa\chi\tilde{r})\left(\frac{1}{\tilde{r}}-\frac{1}{\sqrt{\tilde{r}^2+p^2_{\tilde{r}}}}\right)\tilde{R}_{n,0}(\kappa\chi\tilde{r}), \tag{2.66}$$

with $\tilde{R}_{n,0}(\kappa\chi\tilde{r})$ being radial wave function of the hydrogen atom

$$\tilde{R}_{n,0}(\kappa\chi\tilde{r}) = \sqrt{\frac{4}{n^5}}e^{-\frac{\kappa\chi\tilde{r}}{n}}L^1_{n-1}\left(\frac{2\kappa\chi\tilde{r}}{n}\right), \tag{2.67}$$

and

$$p_{\tilde{r}} = -i\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\tilde{r}. \tag{2.68}$$

Then for convenience we use notation

$$S_{ns}(\kappa\chi) = \\ = 4\int_0^\infty d\tilde{r}\tilde{r}^2 e^{-\frac{\kappa\chi\tilde{r}}{n}} L^1_{n-1}\left(\frac{2\kappa\chi\tilde{r}}{n}\right)\times \\ \times\left(\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2 + p^2_{\tilde{r}}}}\right) e^{-\frac{\kappa\chi\tilde{r}}{n}} L^1_{n-1}\left(\frac{2\kappa\chi\tilde{r}}{n}\right). \tag{2.69}$$

So, for $I_{ns}(\chi, \theta')$ we obtain

$$I_{ns}(\chi, \theta') = \frac{\theta'}{2n^5}\int_0^\pi d\Theta \sin^2\Theta S_{ns}(\kappa\chi). \tag{2.70}$$

Taking into account

$$I_{ns}(0) = \langle I_{ns}(0, \theta')\rangle_{\mathbf{a}', \mathbf{b}'}, \tag{2.71}$$

$$I_{ns}(0, \theta') = \frac{\theta'}{2n^5}\int_0^\pi d\Theta \sin^2\Theta S_{ns}(0) = \frac{\pi\theta'}{4n^5}S_{ns}(0), \tag{2.72}$$

we have

$$\Delta E^{(\theta)}_{ns} = \frac{\pi\langle\theta'\rangle\chi^2 e^2}{4a_B n^5}S_{ns}(0), \tag{2.73}$$

$$\langle\theta'\rangle = \langle\tilde{\psi}^a_{0,0,0}(\mathbf{a}')\tilde{\psi}^b_{0,0,0}(\mathbf{b}')|\sqrt{\sum_i(\theta'_i)^2}|\tilde{\psi}^a_{0,0,0}(\mathbf{a}')\tilde{\psi}^b_{0,0,0}(\mathbf{b}')\rangle = 1. \tag{2.74}$$

Note, that

$$S_{ns}(0) = S_{1s}(0)n^2. \tag{2.75}$$

On the basis of (2.74), (2.75), we find expression for the leading term in the asymptotic expansion of the corrections to the ns energy levels

$$\Delta E_{ns} = \frac{\pi\chi^2 e^2}{4a_B n^3}S_{1s}(0). \tag{2.76}$$

So, we have to calculate the integral

$$S_{1s}(0) = 4\int_0^\infty d\tilde{r}\tilde{r}^2 \left(\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2 + p_{\tilde{r}}^2}}\right). \qquad (2.77)$$

We expand 1 over the eigenfunctions of operator $\tilde{r}^2 + p_{\tilde{r}}^2$. They read

$$\phi_k = \sqrt{\frac{2k!}{\Gamma(k+\frac{3}{2})}} e^{-\frac{\tilde{r}^2}{2}} L_k^{\frac{1}{2}}(\tilde{r}^2). \qquad (2.78)$$

We have

$$1 = \sum_{k=0}^{\infty} C_k \phi_k, \qquad (2.79)$$

C_k are the expansion coefficients, which are as follows

$$C_k = \sqrt{\frac{2k!}{\Gamma(k+\frac{3}{2})}} \int_0^\infty d\tilde{r}\tilde{r}^2 e^{-\frac{\tilde{r}^2}{2}} L_k^{\frac{1}{2}}\left(\tilde{r}^2\right) = (-1)^k \sqrt{\frac{4\Gamma(k+\frac{3}{2})}{k!}}. \qquad (2.80)$$

So, for the second term in (2.77) we obtain

$$\int_0^\infty d\tilde{r}\tilde{r}^2 \frac{1}{\sqrt{\tilde{r}^2 + p_{\tilde{r}}^2}} = \sum_{k=0}^{\infty} \frac{C_k^2}{\sqrt{\lambda_k}}, \qquad (2.81)$$

where

$$\lambda_k = 2\left(2k + \frac{3}{2}\right), \qquad (2.82)$$

are the eigenvalues of operator $\tilde{r}^2 + p_{\tilde{r}}^2$.

Let us represent the first term in (2.77) as follows

$$\int_0^\infty d\tilde{r}\tilde{r} = \sum_{k=0}^{\infty} C_k I_k, \qquad (2.83)$$

$$I_k = \sqrt{\frac{2k!}{\Gamma(k+\frac{3}{2})}} \int_0^\infty d\tilde{r}\tilde{r} e^{-\frac{\tilde{r}^2}{2}} L_k^{\frac{1}{2}}\left(\tilde{r}^2\right) =$$

$$= (-1)^k \sqrt{\frac{8k!}{\pi\Gamma(k+\frac{3}{2})}}\, {}_2F_1\left(-k, \frac{1}{2}; \frac{3}{2}; 2\right), \qquad (2.84)$$

where ${}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right)$ is the hypergeometric function. Using (2.81), (2.83), we obtain

$$S_{1s}(0)=4\sum_{k=0}^{\infty}\left(C_kI_k-\frac{C_k^2}{\sqrt{\lambda_k}}\right)=$$
$$=16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\left({}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right)-\sqrt{\frac{\pi}{8k+6}}\right). \quad (2.85)$$

It is important to mention that the two sums in $S_{1s}(0)$

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}{}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right), \quad (2.86)$$
$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\sqrt{\frac{\pi}{8k+6}}, \quad (2.87)$$

are divergent. But the value of $S_{1s}(0)$ is finite. To study the sums (2.86), (2.87) separately we consider additional multiplier η^k $(\eta<1)$

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}{}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right)\eta^k, \quad (2.88)$$
$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\sqrt{\frac{\pi}{8k+6}}\eta^k. \quad (2.89)$$

In the case of $\eta=1$ we find (2.86), (2.87).

First let us calculate (2.89). It is easy to show that

$$\sqrt{\frac{\pi}{k+\frac{3}{4}}}=2\int_0^{\infty}dze^{-(k+\frac{3}{4})z^2}. \quad (2.90)$$

Also, we can write

$$\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}t^k=\frac{\sqrt{\pi}}{2(1-t)^{\frac{3}{2}}}. \quad (2.91)$$

As a result, using (2.90), (2.91), we find

$$16\sqrt{2}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!\sqrt{8k+6}}\eta^k =$$
$$= 16\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!\sqrt{\pi}}\eta^k\int_0^{\infty}dze^{-(k+\frac{3}{4})z^2} =$$
$$= 8\int_0^{\infty}dz\frac{e^{-\frac{3}{4}z^2}}{(1-\eta e^{-z^2})^{\frac{3}{2}}}. \tag{2.92}$$

To calculate (2.88) we represent the hypergeometric function as

$$_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right) = \sum_{q=0}^{k}\frac{(-1)^q C_k^q 2^q}{2q+1}, \tag{2.93}$$

here C_k^q are the binomial coefficients. We can write

$$\frac{1}{2q+1} = \int_0^1 dzz^{2q}. \tag{2.94}$$

So, taking into account (2.93), (2.94), we find

$$_2F_1\left(-k,\frac{1}{2},\frac{3}{2},2\right) = \sum_{q=0}^{k}\int_0^1 dzC_k^q(-2)^q z^{2q} = \int_0^1 dz(1-2z^2)^k. \tag{2.95}$$

Then, using (2.91) and (2.95), we rewrite (2.88) as

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}{}_2F_1\left(-k,\frac{1}{2},\frac{3}{2},2\right)\eta^k =$$
$$= 16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\eta^k\int_0^1 dz(1-2z^2)^k =$$
$$= 8\sqrt{2}\int_0^1\frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}}. \tag{2.96}$$

We split the integral (2.96) into two integrals as

$$\int_0^1 \frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}} = I_1(\eta) + I_2(\eta), \tag{2.97}$$

$$I_1(\eta) = \int_0^{\frac{1}{\sqrt{2}}} \frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}}, \tag{2.98}$$

$$I_2(\eta) = \int_{\frac{1}{\sqrt{2}}}^1 \frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}}. \tag{2.99}$$

The integral $I_2(\eta)$ has a finite value even for $\eta = 1$, we find

$$I_2(1) = \frac{\sqrt{2}}{8}. \tag{2.100}$$

Let us represent (2.98) in the form close to (2.92). We use substitution $e^{-t^2} = 1 - 2z^2$, and obtain

$$I_1(\eta) = \frac{\sqrt{2}}{2} \int_0^\infty dt \frac{te^{-t^2}}{(1-e^{-t^2})^{\frac{1}{2}}(1-\eta e^{-t^2})^{\frac{3}{2}}}. \tag{2.101}$$

Using (2.92), (2.96), (2.100), (2.101), we can write

$$\begin{aligned} 16\sqrt{\frac{2}{\pi}} \sum_{k=0}^\infty \frac{\Gamma(k+\frac{3}{2})}{k!} {}_2F_1\left(-k, \frac{1}{2}; \frac{3}{2}; 2\right) \eta^k - \\ -16\sqrt{\frac{2}{\pi}} \sum_{k=0}^\infty \frac{\Gamma(k+\frac{3}{2})}{k!} \sqrt{\frac{\pi}{8k+6}} \eta^k = \\ = 8\sqrt{2} I_2(\eta) + 8\int_0^\infty dt \frac{te^{-t^2} - e^{-\frac{3}{4}t^2}(1-e^{-t^2})^{\frac{1}{2}}}{(1-e^{-t^2})^{\frac{1}{2}}(1-\eta e^{-t^2})^{\frac{3}{2}}}. \end{aligned} \tag{2.102}$$

The integral (2.102) is finite for $\eta = 1$. So, substituting $\eta = 1$ into (2.102), and using (2.85), (2.100), we obtain

$$S_{1s}(0) = 2 + 8\int_0^\infty dt \frac{te^{-t^2} - e^{-\frac{3}{4}t^2}\sqrt{1-e^{-t^2}}}{(1-e^{-t^2})^2} = 1.72006\ldots \tag{2.103}$$

Taking into account (2.76), we find

$$\Delta E_{ns}^{(\theta)} \simeq 1.72 \frac{\pi \chi^2 e^2}{4 a_B n^3}. \tag{2.104}$$

So, on the basis of (2.51) and (2.55), we can write

$$\Delta E_{ns}^{(\theta)} \simeq 1.72 \frac{\hbar \langle \theta \rangle \pi e^2}{8 a_B^3 n^3}, \tag{2.105}$$

$$\langle \theta \rangle = \langle \psi_{0,0,0}^a(\mathbf{a}) \psi_{0,0,0}^b(\mathbf{b}) | \sqrt{\sum_i \theta_i^2} | \psi_{0,0,0}^a(\mathbf{a}) \psi_{0,0,0}^b(\mathbf{b}) \rangle = \frac{\alpha l_p^2}{\hbar}. \tag{2.106}$$

Finally, corrections to the ns energy levels of the hydrogen atom caused by noncommutativity of coordinates and noncommutativity of momenta read

$$\Delta E_{ns} = \frac{a_B^2 n^2 \langle \eta^2 \rangle}{24 M} (5n^2 + 1) + 1.72 \frac{\hbar \langle \theta \rangle \pi e^2}{8 a_B^3 n^3}. \tag{2.107}$$

Let us analyze the corrections (2.44), (2.107). There is an important difference between the influences of coordinates noncommutativity and momentum noncommutativity on the spectrum of the hydrogen atom. In the case of large quantum numbers n , corrections caused by noncommutativity of momenta $\Delta E_{n,l}^{(\eta)}$ (2.165) are proportional to n^4, and corrections caused by noncommutativity of coordinates $\Delta E_{n,l}^{(\theta)}$ (2.164) are proportional to $1/n^3$. So, we can conclude that the energy levels with large quantum numbers n are more sensitive to the momentum noncommutativity than noncommutativity of coordinates. Energy levels with small quantum numbers n are more sensitive to the coordinates noncommutativity

Note also that ns energy levels are more sensitive to the noncommutativity of coordinates (1.27) Namely corrections to the ns energy levels (2.107) contain terms with $\langle \theta \rangle$ and $\langle \eta^2 \rangle$. Corrections to other energy levels ($l > 1$) include terms proportional to $\langle \theta^2 \rangle$ and $\langle \eta^2 \rangle$.

2.5 Energy levels of hydrogen-like exotic atoms in quantum space

We examine two particles with masses m_1, m_2 with Coulomb interaction in the frame of rotationally-invariant noncommutative algebra of canonical type (1.27)-(1.29). In this case the total Hamiltonian reads

$$H = \frac{(\mathbf{P}^{(1)})^2}{2m_1} + \frac{(\mathbf{P}^{(2)})^2}{2m_2} - \frac{\kappa}{|\mathbf{X}^{(1)} - \mathbf{X}^{(2)}|} + H^a_{osc} + H^b_{osc}. \quad (2.108)$$

Hare κ is a constant.

In general case coordinates of different particles may satisfy commutation relations of noncommutative algebra with different tensors of noncommutativity $\theta^{(n)}_{ij}$, $\eta^{(n)}_{ij}$ (n labels the particles, $n = (1, 2)$). So, in this case the relations of the algebra read

$$[X^{(n)}_i, X^{(m)}_j] = i\hbar\delta_{mn}\theta^{(n)}_{ij}, \quad (2.109)$$

$$[X^{(n)}_i, P^{(m)}_j] = i\hbar\delta_{mn}\left(\delta_{ij} + \sum_k \frac{\theta^{(n)}_{ik}\eta^{(m)}_{jk}}{4}\right), \quad (2.110)$$

$$[P^{(n)}_i, P^{(m)}_j] = i\hbar\delta_{mn}\eta^{(n)}_{ij}, \quad (2.111)$$

$n, m = (1, 2)$. Note that we also suppose that commutators for coordinates and the momenta of different particles equal zero.

Let us introduce coordinates and momenta of the center-of-mass and coordinates and momenta of the relative motion as in ordinary space

$$\mathbf{X}^c = \mu_1\mathbf{X}^{(1)} + \mu_2\mathbf{X}^{(2)}, \quad (2.112)$$

$$\mathbf{P}^c = \mathbf{P}^{(1)} + \mathbf{P}^{(2)}, \quad (2.113)$$

$$\mathbf{X}^r = \mathbf{\Delta X}^{(1)} - \mathbf{\Delta X}^{(2)} = \mathbf{X}^{(1)} - \mathbf{X}^{(2)}, \quad (2.114)$$

$$\mathbf{P}^r = \frac{1}{2}(\mathbf{\Delta P}^{(1)} - \mathbf{\Delta P}^{(2)}) = \mu_2\mathbf{P}^{(1)} - \mu_1\mathbf{P}^{(2)}. \quad (2.115)$$

So, we can rewrite the Hamiltonian of the system as

$$H_s = \frac{(\mathbf{P}^c)^2}{2M} + \frac{(\mathbf{P}^r)^2}{2\mu} - \frac{\kappa}{|\mathbf{X}^r|}, \quad (2.116)$$

where $M = m_1 + m_2$, $\mu = m_1 m_2 / M$ are the total and the reduced masses respectively, $\mu_i = m_i/M$.

Coordinates and momenta of the center-of mass X_i^c, P_i^c satisfy the following relations

$$[X_i^c, X_j^c] = i\hbar \sum_{n=1}^{2} \mu_n^2 \theta_{ij}^{(n)} = i\hbar\theta_{ij}^c, \tag{2.117}$$

$$[P_i^c, P_j^c] = i\hbar \sum_{n=1}^{2} \eta_{ij}^{(n)} = i\hbar\eta_{ij}^c, \tag{2.118}$$

$$[X_i^c, P_j^c] = i\hbar(\delta_{ij} + \sum_{n=1}^{2}\sum_{k=1}^{2} \mu_n \frac{\theta_{ik}^{(n)}\eta_{jk}^{(n)}}{4}). \tag{2.119}$$

where

$$\theta_{ij}^c = \mu_1^2\theta_{ij}^{(1)} + \mu_2^2\theta_{ij}^{(2)}, \tag{2.120}$$

$$\eta_{ij}^c = \eta_{ij}^{(1)} + \eta_{ij}^{(2)}. \tag{2.121}$$

It is important to stress that

$$i\hbar(\delta_{ij} + \sum_n\sum_k \mu_n \frac{\theta_{ik}^{(n)}\eta_{jk}^{(n)}}{4}) \neq i\hbar(\delta_{ij} + \sum_k \frac{\theta_{ik}^c\eta_{jk}^c}{4}). \tag{2.122}$$

So, commutators (2.117)-(2.119) do not correspond to noncommutative algebra (1.27)-(1.29).

It is worth mentioning that in the case when parameters of noncommutativity depend on mass as

$$c_\theta^{(n)} = \frac{\tilde{\gamma}}{m_n}, \tag{2.123}$$

$$c_\eta^{(n)} = \tilde{\alpha} m_n, \tag{2.124}$$

the tensors of noncommutativity can be rewritten as

$$\theta_{ij}^{(n)} = \frac{\tilde{\gamma} l_P^2}{\hbar m_n} \sum_k \varepsilon_{ijk}\tilde{a}_k, \tag{2.125}$$

$$\eta_{ij}^{(n)} = \frac{\tilde{\alpha}\hbar m_n}{l_P^2} \sum_k \varepsilon_{ijk}\tilde{p}_k^b, \tag{2.126}$$

and the effective tensors of noncommutativity do not depend on the masses of particles in the system. They read

$$\theta^c_{ij} = \frac{\tilde{\gamma} l_P^2}{\hbar M}\sum_k \varepsilon_{ijk}\tilde{a}_k, \tag{2.127}$$

$$\eta^c_{ij} = \frac{\tilde{\alpha}\hbar M}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}^b_k. \tag{2.128}$$

Also, due to conditions (2.123), (2.124) we can write

$$[X^c_i, P^c_j] = i\hbar(\delta_{ij} + \tilde{\gamma}\tilde{\alpha}\sum_{k,l,m}\frac{\varepsilon_{ikl}\varepsilon_{jkm}\tilde{a}_l\tilde{p}^b_m}{4}) = \\ = i\hbar(\delta_{ij} + \sum_k \frac{\theta^c_{ik}\eta^c_{jk}}{4}). \tag{2.129}$$

In the case when conditions (2.123), (2.124) hold, for coordinates and momenta of the relative motion satisfy the following relations

$$[X^r_i, X^r_j] = i\hbar\theta^r_{ij}, \tag{2.130}$$

$$[P^r_i, P^r_j] = i\hbar\eta^r_{ij}, \tag{2.131}$$

$$[X^r_i, P^r_j] = i\hbar(\delta_{ij} + \frac{1}{4}\sum_k \theta^r_{ik}\eta^r_{jk}), \tag{2.132}$$

where

$$\theta^r_{ij} = \theta^{(1)}_{ij} + \theta^{(2)}_{ij}, \tag{2.133}$$

$$\eta^r_{ij} = \mu_2^2\eta^{(1)}_{ij} + \mu_1^2\eta^{(2)}_{ij}. \tag{2.134}$$

It is also important to stress that coordinates and momenta of the center-of-mass commute with coordinates and momenta of the relative motion due to conditions (2.123), (2.124)

$$[X^c_i, X^r_j] = [P^c_i, P^r_j] = 0. \tag{2.135}$$

Taking into account (2.125), (2.126), (2.133), (2.134) we can write

$$\theta^r_{ij} = \frac{c^r_\theta l_P^2}{\hbar}\varepsilon_{ijk}\tilde{a}_k = \frac{\tilde{\gamma} l_P^2}{\mu\hbar}\varepsilon_{ijk}\tilde{a}_k, \tag{2.136}$$

$$\eta^r_{ij} = \frac{c^r_\eta \hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}^b_k = \frac{\tilde{\alpha}\mu\hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}^b_k, \tag{2.137}$$

where

$$c^r_\theta = c^{(1)}_\theta + c^{(2)}_\theta, c^r_\eta = \mu_2^2 c^{(1)}_\eta + \mu_1^2 c^{(2)}_\eta. \tag{2.138}$$

We also have

$$\theta^c_{ij} = \frac{c^c_\theta l_P^2}{\hbar}\varepsilon_{ijk}\tilde{a}_k = \frac{\tilde{\gamma} l_P^2}{M\hbar}\varepsilon_{ijk}\tilde{a}_k, \tag{2.139}$$

$$\eta^c_{ij} = \frac{c^c_\eta \hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}^b_k = \frac{\tilde{\alpha} M\hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}^b_k, \tag{2.140}$$

with

$$c^c_\theta = \mu_1^2 c^{(1)}_\theta + \mu_2^2 c^{(2)}_\theta, \tag{2.141}$$

$$c^c_\eta = c^{(1)}_\eta + c^{(2)}_\eta. \tag{2.142}$$

So, from (2.136)-(2.140) we can conclude that in the case when conditions (2.123), (2.124) are satisfied the tensors of noncommutativity describing of the center-of-mass θ^c_{ij}, η^c_{ij} and relative motion θ^r_{ij}, η^r_{ij} depend on the total and reduced masses, respectively.

Note also that conditions (2.123), (2.124) are also satisfied for constants c^c_θ, c^c_η, c^r_θ, c^r_η. Namely, we can write

$$c^c_\theta M = c^r_\theta \mu = c^{(1)}_\theta m_1 = c^{(2)}_\theta m_2 = \tilde{\gamma} = \text{const}, \tag{2.143}$$

$$\frac{c^c_\eta}{M} = \frac{c^r_\eta}{\mu} = \frac{c^{(1)}_\eta}{m_1} = \frac{c^{(2)}_\eta}{m_2} = \tilde{\alpha} = \text{const}. \tag{2.144}$$

For noncommutative coordinates and noncommutative momenta of the center-of-mass and noncommutative coordinates and noncommutative momenta of the relative motion can be represented as

$$X^c_i = x^c_i - \frac{1}{2}\theta_{ij}p^c_j = x^c_i + \frac{1}{2}[\boldsymbol{\theta}^c \times \mathbf{p}^c]_i, \tag{2.145}$$

$$P^c_i = p^c_i + \frac{1}{2}\eta^c_{ij}x^c_j = p^c_i - \frac{1}{2}[\boldsymbol{\eta}^c \times \mathbf{x}^c]_i, \tag{2.146}$$

$$X^r_i = x^r_i - \frac{1}{2}\theta^r_{ij}p^r_j = x^r_i + \frac{1}{2}[\boldsymbol{\theta}^r \times \mathbf{p}^r]_i, \tag{2.147}$$

$$P^r_i = p^r_i + \frac{1}{2}\eta^r_{ij}x^r_j = p^r_i - \frac{1}{2}[\boldsymbol{\eta}^r \times \mathbf{x}^r]_i. \tag{2.148}$$

For coordinates x_i^r, x_i^c and momenta p_i^r, p_i^c we have the ordinary commutation relations

$$[x_i^c, x_j^c] = [p_i^c, p_j^c] = [x_i^r, x_j^r] = [p_i^r, p_j^r] = 0, \tag{2.149}$$
$$[x_i^c, x_j^r] = [p_i^c, p_j^r] = [x_i^r, p_j^c] = [p_i^r, x_j^c] = 0, \tag{2.150}$$
$$[x_i^c, p_j^c] = [x_i^r, p_j^r] = i\hbar\delta_{ij}. \tag{2.151}$$

So, the Hamiltonian in the representation (2.145)-(2.148) reads

$$H_s = \frac{(\mathbf{p}^c)^2}{2M} + \frac{(\mathbf{p}^r)^2}{2\mu} + \frac{(\boldsymbol{\eta}^c \cdot \mathbf{L^c})}{2M} + \frac{[\boldsymbol{\eta}^c \times \mathbf{x^c}]^{\mathbf{2}}}{8M} + \\ + \frac{(\boldsymbol{\eta}^r \cdot \mathbf{L^r})}{2\mu} + \frac{[\boldsymbol{\eta}^r \times \mathbf{x^r}]^{\mathbf{2}}}{8\mu} - \\ - \frac{\kappa}{\sqrt{(x^r)^2 - (\boldsymbol{\theta}^r \cdot \mathbf{L}^r) + \frac{1}{4}[\boldsymbol{\theta}^r \times \mathbf{p}^r]^2}}. \tag{2.152}$$

Here we consider notation

$$\mathbf{L}^c = [\mathbf{x^c} \times \mathbf{p^c}], \tag{2.153}$$
$$\mathbf{L}^r = [\mathbf{x}^r \times \mathbf{p}^r]. \tag{2.154}$$

Up to the second order in the parameters of noncommutativity the Hamiltonian of the system reads

$$H_s = \frac{(\mathbf{p}^c)^2}{2M} + \frac{(\mathbf{p}^r)^2}{2\mu} - \frac{\kappa}{x^r} + \\ + \frac{(\boldsymbol{\eta}^c \cdot \mathbf{L}^c)}{2M} + \frac{[\boldsymbol{\eta}^c \times \mathbf{x}^c]^2}{8M} + \frac{(\boldsymbol{\eta}^r \cdot \mathbf{L}^r)}{2\mu} + \frac{[\boldsymbol{\eta}^r \times \mathbf{x}^r]^2}{8\mu} - \\ - \frac{\kappa}{2(x^r)^3}(\boldsymbol{\theta}^r \cdot \mathbf{L}^r) - \frac{3\kappa}{8(x^r)^5}(\boldsymbol{\theta}^r \cdot \mathbf{L}^r)^2 + \\ + \frac{\kappa}{16}\left(\frac{1}{(x^r)^2}[\boldsymbol{\theta}^r \times \mathbf{p}^r]^2 \frac{1}{x^r} + \frac{1}{x^r}[\boldsymbol{\theta}^r \times \mathbf{p}^r]^2 \frac{1}{(x^r)^2} + \frac{\hbar^2}{(x^r)^7}[\boldsymbol{\theta}^r \times \mathbf{x}^r]^2\right). \tag{2.155}$$

After averaging over the eigenfunctions of the harmonic oscillators

$\psi^a_{0,0,0}$, $\psi^b_{0,0,0}$ we find

$$\langle H_s\rangle_{ab} = \frac{(\mathbf{p}^c)^2}{2M} + \frac{(x^c)^2\langle(\eta^c)^2\rangle}{12M} + \\ + \frac{(\mathbf{p}^r)^2}{2\mu} - \frac{\kappa}{x^r} + \frac{(x^r)^2\langle(\eta^r)^2\rangle}{12\mu} - \frac{\kappa(L^r)^2\langle(\theta^r)^2\rangle}{8(x^r)^5} + \\ + \frac{\kappa}{24}\left(\frac{1}{(x^r)^2}(p^r)^2\frac{1}{x^r} + \frac{1}{x^r}(p^r)^2\frac{1}{(x^r)^2} + \frac{\hbar^2}{(x^r)^5}\right)\langle(\theta^r)^2\rangle. \quad (2.156)$$

Up to the second order in the parameters of noncommutativity we can examine H_0

$$H_0 = \langle H_c\rangle_{ab} + \langle H_r\rangle_{ab} + H^a_{osc} + H^b_{osc}, \quad (2.157)$$

$$\langle H_c\rangle_{ab} = \frac{(\mathbf{p}^c)^2}{2M} + \frac{(x^c)^2\langle(\eta^c)^2\rangle}{12M}, \quad (2.158)$$

$$\langle H_r\rangle_{ab} = \frac{(\mathbf{p}^r)^2}{2\mu} - \frac{\kappa}{x^r} + \frac{(x^r)^2\langle(\eta^r)^2\rangle}{12\mu} - \frac{\kappa(L^r)^2\langle(\theta^r)^2\rangle}{8(x^r)^5} + \\ + \frac{\kappa}{24}\left(\frac{1}{(x^r)^2}(p^r)^2\frac{1}{x^r} + \frac{1}{x^r}(p^r)^2\frac{1}{(x^r)^2} + \frac{\hbar^2}{(x^r)^5}\right)\langle(\theta^r)^2\rangle. \quad (2.159)$$

Operators $\langle H_c\rangle_{ab}$, $\langle H_r\rangle_{ab}$ describe the motion of the center-of-mass and the relative motion.

It is important that

$$[\langle H_c\rangle_{ab}, \langle H_r\rangle_{ab}] = [\langle H_c\rangle_{ab}, H^a_{osc} + H^b_{osc}] = 0. \quad (2.160)$$

So, we can study $\langle H_c\rangle_{ab}$ independently. Operator $\langle H_c\rangle_{ab}$ corresponds to the Hamiltonian of three-dimensional harmonic oscillator of mass M and frequency

$$\omega = \frac{\sqrt{2}\langle(\eta^c)^2\rangle}{\sqrt{3}M} \quad (2.161)$$

The spectrum of the oscillator is well known

$$E_{n^c_1,n^c_2,n^c_3} = \frac{\hbar\sqrt{2}\langle(\eta^c)^2\rangle}{\sqrt{3}M}\left(n^c_1 + n^c_2 + n^c_3 + \frac{3}{2}\right), \quad (2.162)$$

here n^c_1, n^c_2, n^c_3 are quantum numbers.

According to the perturbation theory we have the following corrections to the energy levels caused by noncommutativity of coordinates and noncommutativity of momenta

$$\Delta E^{(\theta\eta)}_{n,l} = \langle\psi^{(0)}_{n,l,m}|V|\psi^{(0)}_{n,l,m}\rangle = \Delta E^{(\eta)}_{n,l} + \Delta E^{(\theta)}_{n,l}, \quad (2.163)$$

$$\Delta E^{(\eta)}_{n,l} = \langle\psi^{(0)}_{n,l,m}|V^{\eta}|\psi^{(0)}_{n,l,m}\rangle = \\ = \frac{\kappa a^3 n^2\langle(\eta^r)^2\rangle}{24\hbar^2}(5n^2+1-3l(l+1)), \quad (2.164)$$

$$\Delta E^{(\theta)}_{n,l} = \langle\psi^{(0)}_{n,l,m}|V^{\theta}|\psi^{(0)}_{n,l,m}\rangle = \\ = -\frac{\hbar^2\kappa\langle(\theta^r)^2\rangle}{a^5n^5}\left(-\frac{6n^2-2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)}+\right. \\ \frac{1}{6l(l+1)(2l+1)}+\frac{5n^2-3l(l+1)+1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)}- \\ \left.-\frac{5}{6}\frac{5n^2-3l(l+1)+1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right), \quad (2.165)$$

where

$$a = \frac{\hbar^2}{\mu\kappa}. \quad (2.166)$$

Corrections to the energy levels with $l=0$ reads

$$\Delta E^{(\theta\eta)}_{n,0} = \frac{a^3\kappa\langle(\eta^r)^2\rangle}{24\hbar^2}n^2(5n^2+1) + 1.72\frac{\hbar\langle\theta^r\rangle\pi\kappa}{8a^3n^3}. \quad (2.167)$$

Let us examine effect of noncommutativity on hydrogen-like atoms. Corrections caused by noncommutativity of momenta (2.164) are proportional to $\langle(\eta^r)^2\rangle a^3$. From (2.137) follows that

$$\langle(\eta^r)^2\rangle a^3 \sim \frac{1}{\mu}. \quad (2.168)$$

In corrections to the energy levels caused by noncommutativity of coordinates we have proportionality to $\langle\theta^r\rangle/a^3$ in the case of ns energy levels, or proportionality to $\langle(\theta^r)^2\rangle/a^5$ for energy levels with $l>1$, (2.165). From (2.136), we can write

$$\frac{\langle\theta^r\rangle}{a^3} \sim \mu^2, \tag{2.169}$$

$$\frac{\langle\theta^r\rangle}{a^5} \sim \mu^3. \tag{2.170}$$

So, the effect of coordinate noncommutativity can be better examined in the spectrum of atoms with large reduced masses, especially for energy levels with $l = 0$ and small quantum numbers n. The effect of momentum noncommutativity better appears in energy levels with large quantum numbers of atoms with small reduced masses. Also, it is worth mentioning that in the case of atoms with large reduced masses the differences in effects of momentum and coordinates noncommutativity appear better.

Let us examine the muonic hydrogen which is a system of proton and muon. We have that

$$\frac{\mu_{\mu p}}{\mu_H} \simeq \frac{m_\mu}{m_e} = 206.8 \tag{2.171}$$

where $\mu_{\mu p}$, μ_H are reduced mass of muonic hydrogen and hydrogen atoms, m_e, m_μ are the mass of electron and the mass of muon. Because of this ratio the corrections to the energy levels of muonic hydrogen in the case of $l > 1$ (2.165) are $(m_\mu/m_e)^3 = 8.8 \cdot 10^6$ times larger than that for the hydrogen atom. So, noncommutativity of coordinates can be better examined in the case of muonic hydrogen. Corrections (2.164) are 206.8 times smaller in the case of muonic hydrogen than in the case of hydrogen atom.

2.6 Upper bounds on the parameters of coordinates and momentum noncommutativity obtained based on studies of hydrogen atom and antiprotonic helium

To find upper bounds for the parameters of coordinate and momentum noncommutativity we assume that corrections to the hydrogen atom transition energies in quantum space do not exceed the accuracy of the transitions measurements. In paper [48] the authors presented experimental result for $1s - 2s$ transition frequency $f_{1s-2s} =$

2466061413187018(11)Hz with relative uncertainty of 4.5×10^{-15}. So, we can write the following inequality

$$\left|\frac{\Delta^{\theta}_{1,2}+\Delta^{\eta}_{1,2}}{E^{(0)}_2-E^{(0)}_1}\right|\leq 4.5\times10^{-15}, \tag{2.172}$$

here $E^{(0)}_n$ are well known energy levels of the hydrogen atom in the ordinary space. To estimate the order of the upper bounds for the parameters of noncommutativity, we consider

$$\left|\frac{\Delta^{\theta}_{1,2}}{E^{(0)}_2-E^{(0)}_1}\right|\leq 2.25\times10^{-15}, \tag{2.173}$$

$$\left|\frac{\Delta^{\eta}_{1,2}}{E^{(0)}_2-E^{(0)}_1}\right|\leq 2.25\times10^{-15}. \tag{2.174}$$

Using (2.107) we have

$$\Delta^{\theta}_{1,2}=-\frac{3\hbar\langle\theta\rangle\pi e^2}{16a_B^3}, \tag{2.175}$$

$$\Delta^{\eta}_{1,2}=\frac{13a_B^2\langle\eta^2\rangle}{4M}. \tag{2.176}$$

So, the upper bounds read

$$\hbar\langle\theta\rangle\leq 10^{-36}\,\mathrm{m}^2, \tag{2.177}$$

$$\hbar\sqrt{\langle\eta^2\rangle}\leq 10^{-61}\,\mathrm{kg}^2\mathrm{m}^2/\mathrm{s}^2. \tag{2.178}$$

The obtained results are in agreement with that obtained on the basis of studies of the spectrum of gravitation quantum well [49]. Also they are in agreement with the results obtained from the spectrum of hydrogen atom considered in noncommutative space of canonical type [50], and examining the Lamb shift [27]. Note the ratio $m_p/m_e=1836$, therefore $\mu\simeq m_e$. Therefore the orders of the upper bounds do not change if we take into account the effect of reduced mass of the hydrogen atom.

Let us examine exotic atom known as antiprotonic helium $\bar{p}^4He^+$. It is composed of an antiproton, an electron and a helium nucleus. In papers [51, 52] it was shown that the transition frequency of the atom can be approximately written as transitions of the hydrogen atom effective nuclear charge $Z_{eff} < 2$. The charge describes the shielding of the nuclear charge by the electron. Of course the difference of massesees of hydrogen and antiprotonic helium atoms has to be taken into consideration. So, the obtained results for effect of noncommutativity of coordinate and noncommutativity of momenta (2.164), (2.165) can be used for estimation of the upper bounds. Atom $\bar{p}^4He^+$ has a large reduced mass. So, effect of coordinate noncommutativity on the spectrum of the exotic atom is larger than on the hydrogen atom. So, the antiprotonic helium is an attractive candidate for studies of noncommutativity of coordinates

Experimental results for transition frequency $(n, l) = (36, 34) \to (34, 32)$ of antiprotonic helium reads $f = 1522107062$ MHz. The result is obtained with the total experimental error 3.5 MHz [53]. Assuming that effect of noncommutativity on the energy levels is smaller than the accuracy of measurements we have

$$|\Delta^{(\theta)} + \Delta^{(\eta)}| \leq 3.5\text{MHz}, \tag{2.179}$$

$$\Delta^{\theta} = \Delta E^{(\theta)}_{36,34} - \Delta E^{(\theta)}_{34,32}, \tag{2.180}$$

$$\Delta^{\eta} = \Delta E^{(\eta)}_{36,34} - \Delta E^{(\eta)}_{34,32}, \tag{2.181}$$

and $\Delta E^{(\theta)}_{n,l}$, $\Delta E^{(\eta)}_{n,l}$ read (2.164), (2.165). To estimate the upper bounds we write

$$|\Delta^{\theta}| \leq 1.75\text{MHz}, \tag{2.182}$$

$$|\Delta^{\eta}| \leq 1.75\text{MHz}. \tag{2.183}$$

We also consider $Z = 2$, $a = m_e a_B / m_{\bar{p}}$, where $m_{\bar{p}}$ is the mass of antiproton, a_B is the Bohr radius of the hydrogen atom in (2.164), (2.165). As a result we find

$$\hbar\langle\theta^r\rangle \leq 10^{-27}\,\text{m}^2, \tag{2.184}$$

$$\hbar\sqrt{\langle(\eta^r)^2\rangle} \leq 10^{-50}\,\text{kg}^2\text{m}^2/\text{s}^2. \tag{2.185}$$

Because of not hight precision of the measurements of the spectrum of $\bar{\text{p}}^4\text{He}^+$ the obtained upper bound do not lead to strong restriction on the values of parameters of noncommutativity. But, it is worth stressing that effect of coordinates noncommutativity on $\bar{\text{p}}^4\text{He}^+$ is three orders larger than that on hydrogen atom. So, improvement of precision of measurements of the spectrum of the exotic atom opens a possibility to find stringent upper bound for the parameter of coordinates noncomutativity.

2.7 Conclusions

Hydrogen atom spectrum has been examined noncommutative phase space with preserver rotational symmetry (1.27)-(1.29). Effect of noncommutativity of coordinates and noncommutativity of moment on the energy levels of the atom has been obtained (2.44). We conclude that the effect of momentum noncommutativity is larger in the case of energy levels with large principal quantum numbers.

Effect of coordinates noncommutativity can be better studied on the basis of energy levels of the hydrogen atom with small quantum numbers n. We have also found that corrections to the ns-energy levels (2.107) are proportional to $\langle\theta\rangle$. For energy levels with $l > 1$ (2.44) we have proportionality to $\langle\theta^2\rangle$. So, ns energy levels of hydrogen atom are more sensitive to the coordinates noncommutativity.

We have also studied effect of noncommutativity of coordinates and noncommutativity of momenta on the spectrum of hydrogen-like atoms.

We have examined a general case when different particles feel effects of space quantization with different tensors of noncommutativity. The problem of description of a system of particles in rotationally-invariant noncommutative phase space has been considered.

It has been shown that in the case when tensors of noncommutativity corresponding to different particles are determined by their masses for coordinates and momenta of the center-of-mass of a system we have noncommutative algebra with effective tensors of noncommutativity. Also, in the case when the conditions hold the effective tensors of noncommutativity no do not depend on the composition of the system and are determined by their total mass (2.127), (2.128).

It is important to stress that idea of relation of parameters of noncommutativity with mass opens a possibility to solve fundamental principles in noncommutative space of canonical type [54, 55], noncommutative phase phase of canonical type [56, 57], deformed space with minimal length [58–60]. The proposed conditions on the tensors of noncommutativity (2.123), (2.124) are similar to that

$\theta m = \gamma =$ const, $\eta/m = \alpha =$ const proposed in the noncommutative phase space of canonical type [56, 57]. They lead to solving of problem of violation of the properties of the kinetic energy, and violation of the weak equivalence principle in the space.

We have obtained corrections to the spectrum of two-particle system with Colomb interaction caused by noncommutativity of coordinates and noncommutativity of momenta. It has been obtained that the corrections caused by noncommutativity of coordinates and corrections caused by noncommutativity of momenta have different dependencies on the reduced mass μ and parameter of interaction κ. So, one can choose system with good sensitivity to the particular type of noncommutativity. We have found that the effect of momentum noncommutativity can be better examined for the ns energy levels with large quantum numbers of atoms with small reduced masses. Studies ns energy levels with small quantum numbers of atoms with large reduced masses are important for finding the effect of coordinate noncommutativity. We have also shown that antiprotonic helium is an attractive candidate for studies of the effect of coordinate noncommutativity.

Upper bounds for parameters of noncommutativity have been found on the basis of studies of hydrogen atom and antiprotonic helium The upper bounds obtained based on studies of the hydrogen atom are in agreement with those presented in the literature.

Chapter 3

System of harmonic oscillators in rotationally-invariant noncommutative phase space

To find new effects of noncommutativity of coordinates and noncommutativity of momenta in the properties of a wide class of physical systems it is important to examine many-particle systems Studies of harmonic oscillator in noncommutative space have received much attention (see, for instance, [35, 61–74]). Two coupled harmonic oscillators were studied in noncommutative space [75, 76], noncommutative phase space [77, 78]. System of free particles was examined in [79, 80] in noncommutative phase space of canonical type. Classical problems of many particles were examined in [81] in the case of space-time noncommutativity.

It is worth noting that system of harmonic oscillators has various applications. Such studies have importance in nuclei physics [82–84], in quantum chemistry and molecular spectroscopy [85–88]. Also networks of harmonic oscillators are used in quantum information [89–91].

In this chapter we study a system of interacting oscillators in uniform field in the frame of rotationally-invariant noncommutative algebra. Hamiltonian of the system in rotationally-invariant noncommutative phase space is analyzed. The spectrum of a system of harmonic oscillators is obtained up to the second order in the parameters of coordinate and momentum noncommutativity.

The chapter is organized as follows. In section Hamiltonian of a system of interacting harmonic oscillators is analyzed in noncommutative phase space. Section is devoted to studies of the energy levels of a system of interacting harmonic oscillators in uniform field in the frame of noncommutative algebra. Particular cases of a system of particles with harmonic oscillator interaction and a system of free particles are analyzed. A system of two interacting oscillators and three interacting oscillators are examined in section and section respectively. The effect of noncommutativity of coordinates and noncommutativity of momenta on the harmonic oscillator chain is studied in section . Conclusions are presented in section . Results presented in this chapter are published in [43, 92, 93].

3.1 Hamiltonian of a system of oscillators in rotationally-invariant noncommutative phase space

We consider a system of N interacting harmonic oscillators of masses m and frequencies ω in uniform field in the frame of rotationally-invariant noncommutatove algebra of canonical type (1.27)-(1.29). The system is described with the following Hamiltonian

$$H_s = \sum_n \frac{(\mathbf{P}^{(n)})^2}{2m} + \sum_n \frac{m\omega^2 (\mathbf{X}^{(n)})^2}{2} + \frac{k}{2} \sum_{\substack{m,n \\ m\neq n}} (\mathbf{X}^{(n)} - \mathbf{X}^{(m)})^2 + \\ + \kappa \sum_n X_1^{(n)}, \quad (3.1)$$

where κ, k are constants. For convenience, we choose the direction of the field to coincide with the X_1 axis direction. In the vase of $\kappa = 0$, Hamiltonian (3.1) describes nondissipative symmetric network of coupled harmonic oscillators [90].

Coordinates and momenta of harmonic oscillators satisfy relations of rotationally-invariant noncommutative algebra

$$[X_i^{(n)}, X_j^{(m)}] = i\hbar\delta_{mn}\theta_{ij}^{(n)}, \tag{3.2}$$

$$[X_i^{(n)}, P_j^{(m)}] = i\hbar\delta_{mn}\left(\delta_{ij} + \sum_k \frac{\theta_{ik}^{(n)}\eta_{jk}^{(m)}}{4}\right), \tag{3.3}$$

$$[P_i^{(n)}, P_j^{(m)}] = i\hbar\delta_{mn}\eta_{ij}^{(n)}, \tag{3.4}$$

$$\theta_{ij}^{(n)} = \frac{c_\theta^{(n)} l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, \tag{3.5}$$

$$\eta_{ij}^{(n)} = \frac{c_\eta^{(n)}\hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b. \tag{3.6}$$

Here indexes $m, n = (1...N)$ label the oscillators.

If masses of harmonic oscillators are equal m. Using (3.5), (3.6), (2.124), we can write

$$\theta_{ij}^{(n)} = \theta_{ij} = \frac{c_\theta l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, \tag{3.7}$$

$$\eta_{ij}^{(n)} = \eta_{ij} = \frac{c_\eta\hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b, \tag{3.8}$$

$$c_\theta = \frac{\tilde{\gamma}}{m}, \tag{3.9}$$

$$c_\eta = \tilde{\alpha}m. \tag{3.10}$$

Using representation of noncommutative coordinates and noncommutative momenta over coordinates and momenta satisfying the ordinary commutation relations, one has

$$H_s =$$

$$+\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m} + \frac{m\omega^2(\mathbf{x}^{(n)})^2}{2} + \kappa x_1^{(n)}\right) + \frac{k}{2}\sum_{\substack{m,n\\ m\neq n}}(\mathbf{x}^{(n)} - \mathbf{x}^{(m)})^2 +$$

$$+\sum_n\left(-\frac{(\boldsymbol{\eta}\cdot\mathbf{L}^{(n)})}{2m} - \frac{m\omega^2(\boldsymbol{\theta}\cdot\mathbf{L}^{(n)})}{2} + \frac{\kappa}{2}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]_1 +\right.$$

$$+\frac{m\omega^2}{8}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2+\frac{[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2}{8m}\Bigg)-\frac{k}{2}\sum_{\substack{m,n\\ m\neq n}}\boldsymbol{\theta}\cdot$$

$$\cdot[(\mathbf{x}^{(n)}-\mathbf{x}^{(m)})\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]+\sum_{\substack{m,n\\ m\neq n}}\frac{k}{8}[\boldsymbol{\theta}\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]^2, \tag{3.11}$$

The total Hamiltonian reads

$$H=H_s+H^a_{osc}+H^b_{osc}=H_0+\Delta H. \tag{3.12}$$

We have

$$\langle[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2\rangle_{ab}=\frac{2}{3}\langle\eta^2\rangle(\mathbf{x}^{(n)})^2, \tag{3.13}$$

$$\langle[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2\rangle_{ab}=\frac{2}{3}\langle\theta^2\rangle(\mathbf{p}^{(n)})^2, \tag{3.14}$$

$$\langle[\boldsymbol{\theta}\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]^2\rangle_{ab}=\frac{2}{3}\langle\theta^2\rangle(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})^2. \tag{3.15}$$

So, for ΔH we can write

$$\Delta H=\sum_n\Bigg(-\frac{(\boldsymbol{\eta}\cdot\mathbf{L}^{(n)})}{2m}-\frac{m\omega^2(\boldsymbol{\theta}\cdot\mathbf{L}^{(n)})}{2}+\frac{\kappa}{2}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]_1+$$

$$+\frac{m\omega^2}{8}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2+\frac{[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2}{8m}\Bigg)-$$

$$-\frac{k}{2}\sum_{\substack{m,n\\ m\neq n}}\boldsymbol{\theta}\cdot[(\mathbf{x}^{(n)}-\mathbf{x}^{(m)})\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]+$$

$$+\sum_{\substack{m,n\\ m\neq n}}\frac{k}{8}[\boldsymbol{\theta}\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]^2-\sum_n\Bigg(\frac{\langle\eta^2\rangle(\mathbf{x}^{(n)})^2}{12m}+$$

$$+\frac{\langle\theta^2\rangle m\omega^2(\mathbf{p}^{(n)})^2}{12}\Bigg)-\frac{k}{12}\sum_{\substack{m,n\\ m\neq n}}\langle\theta^2\rangle(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})^2. \tag{3.16}$$

So, up to the second order in ΔH (or up to the second order in the parameters of noncommutativity) the Hamiltonian of a system of interacting harmonic oscillators in uniform field reads

$$H_0 = \sum_n \left(\frac{(\mathbf{p}^{(n)})^2}{2m} + \frac{m\omega^2 (\mathbf{x}^{(n)})^2}{2} + \kappa x_1^{(n)} \right) + \frac{k}{2} \sum_{\substack{m,n \\ m \neq n}} (\mathbf{x}^{(n)} - \mathbf{x}^{(m)})^2 + \\ + \sum_n \left(\frac{\langle \eta^2 \rangle (\mathbf{x}^{(n)})^2}{12m} + \frac{\langle \theta^2 \rangle m\omega^2 (\mathbf{p}^{(n)})^2}{12} \right) + \\ + \frac{k}{12} \sum_{\substack{m,n \\ m \neq n}} \langle \theta^2 \rangle (\mathbf{p}^{(n)} - \mathbf{p}^{(m)})^2 + H^a_{osc} + H^b_{osc}. \tag{3.17}$$

3.2 Effect of noncommutativity on spectrum of interacting oscillators

For convenience, let us introduce

$$m_{eff} = m \left(1 + \frac{m^2\omega^2 \langle \theta^2 \rangle}{6} \right)^{-1}, \tag{3.18}$$

$$\omega_{eff} = \left(\omega^2 + \frac{\langle \eta^2 \rangle}{6m^2} \right)^{\frac{1}{2}} \left(1 + \frac{m^2\omega^2 \langle \theta^2 \rangle}{6} \right)^{\frac{1}{2}}. \tag{3.19}$$

So, Hamiltonian (4.198) can be rewritten as

$$H_0 = \sum_n \left(\frac{(\mathbf{p}^{(n)})^2}{2m_{eff}} + \frac{m_{eff}\omega_{eff}^2 (\tilde{\mathbf{x}}^{(n)})^2}{2} \right) - \frac{N\kappa^2}{2m_{eff}\omega^2_{eff}} + \\ + \frac{k}{2} \sum_{\substack{m,n \\ m \neq n}} (\tilde{\mathbf{x}}^{(n)} - \tilde{\mathbf{x}}^{(m)})^2 + \frac{k}{12} \sum_{\substack{m,n \\ m \neq n}} \langle \theta^2 \rangle (\mathbf{p}^{(n)} - \mathbf{p}^{(m)})^2 + \\ + H^a_{osc} + H^b_{osc}. \tag{3.20}$$

Here $\tilde{\mathbf{x}}^{(n)}$ is defined as

$$\tilde{\mathbf{x}}^{(n)} = \left(x_1^{(n)} + \frac{\kappa}{m_{eff}\omega^2_{eff}}, x_2^{(n)}, x_3^{(n)} \right). \tag{3.21}$$

For operators $\tilde{\mathbf{x}}^{(n)}$, $\mathbf{p}^{(n)}$ we have the ordinary commutation relations

$$[\tilde{x}_i^{(n)}, \tilde{x}_j^{(m)}] = 0, \tag{3.22}$$

$$[\tilde{x}_i^{(n)}, p_j^{(m)}] = i\hbar\delta_{nm}\delta_{ij}, \tag{3.23}$$

$$[p_i^{(n)}, p_j^{(m)}] = 0. \tag{3.24}$$

It is also important to mention that

$$[H_0, H^a_{osc}] = [H_0, H^b_{osc}] = 0. \tag{3.25}$$

So, the energy levels of H_0 are

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \sum_{a=1}^{N} \hbar\omega_a \left(n_1^{(a)} + n_2^{(a)} + n_3^{(a)} + \frac{3}{2}\right) - \\ -\frac{N\kappa^2}{2m_{eff}\omega_{eff}^2} + 3\hbar\omega_{osc}. \tag{3.26}$$

Here $n_i^{(a)}$ are quantum numbers ($n_i^{(a)} = 0, 1, 2...$) and

$$\omega_1 = \omega_{eff}, \tag{3.27}$$

$$\omega_2 = \omega_3 = ... = \omega_N = \\ = \left(\omega_{eff}^2 + \frac{2kN}{m_{eff}} + \frac{kN\langle\theta^2\rangle m_{eff}\omega_{eff}^2}{3} + \frac{2k^2\langle\theta^2\rangle N^2}{3}\right)^{\frac{1}{2}}. \tag{3.28}$$

The spectrum of the center-of-mass of the system of the harmonic oscillators is represented by the first term in (3.26). The spectrum of the relative motion is described by the terms with $a = 2..N$. To show this let us introduce coordinates and moments of the center of mass

$$\mathbf{x}^c = \frac{\sum_n \mathbf{x}^{(n)}}{N}, \tag{3.29}$$

$$\mathbf{p}^c = \sum_n \mathbf{p}^{(n)}, \tag{3.30}$$

coordinates and momenta of he relative motion

$$\Delta\mathbf{x}^{(n)} = \mathbf{x}^{(n)} - \mathbf{x}^c, \tag{3.31}$$

$$\Delta\mathbf{p}^{(n)} = \frac{\mathbf{p}^{(n)} - \mathbf{p}^c}{N}. \tag{3.32}$$

Taking into account (3.80), we have

$$H_0 = H^c + H_{rel} + H^a_{osc} + H^b_{osc}, \quad (3.33)$$

$$H^c = \frac{(\mathbf{p}^c)^2}{2Nm_{eff}} + \frac{Nm_{eff}\omega^2_{eff}(\tilde{\mathbf{x}}^c)^2}{2} - \frac{N\kappa^2}{2m_{eff}\omega^2_{eff}}, \quad (3.34)$$

$$H_{rel} = \sum_n \left(\frac{(\Delta\mathbf{p}^{(n)})^2}{2m_{eff}} + \frac{m_{eff}\omega^2_{eff}(\Delta\mathbf{x}^{(n)})^2}{2} \right) +$$
$$+\frac{k}{2}\sum_{\substack{m,n\\ m\neq n}}(\Delta\mathbf{x}^{(n)} - \Delta\mathbf{x}^{(m)})^2 + \frac{k}{12}\sum_{\substack{m,n\\ m\neq n}}\langle\theta^2\rangle(\Delta\mathbf{p}^{(n)} - \Delta\mathbf{p}^{(m)})^2, \quad (3.35)$$

$$[H^c, H_{rel}] = [H^c, H^a_{osc} + H^b_{osc}] = [H_{rel}, H^a_{osc} + H^b_{osc}] = 0. \quad (3.36)$$

Here $\tilde{\mathbf{x}}^c$ reads

$$\tilde{\mathbf{x}}^c = \left(x^c_1 + \kappa/(m_{eff}\omega^2_{eff}), x^c_2, x^c_3\right). \quad (3.37)$$

Let us analyze the obtained result. From (3.26) follows that frequencies in the spectra of the center-of-mass and relative motion of the system of interacting oscillators are affected by the noncommutativity of coordinates and noncommutativity of momenta. The uniform field causes to the shift of the spectrum of the system on a constant.

Considering limit $\langle\theta^2\rangle \to 0$, $\langle\eta^2\rangle \to 0$ form $E_{\{n_1\},\{n_2\},\{n_3\}}$ one obtains well known expression

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \hbar\omega\left(n^{(1)}_1 + n^{(1)}_2 + n^{(1)}_3 + \frac{3}{2}\right) +$$
$$+\sum_{a=2}^{N}\hbar\left(\omega^2 + \frac{2Nk}{m}\right)^{\frac{1}{2}}\left(n^{(a)}_1 + n^{(a)}_2 + n^{(a)}_3 + \frac{3}{2}\right) - \frac{N\kappa^2}{2m\omega^2}. \quad (3.38)$$

On the basis of (3.26) we can write the spectrum of a system of N particles of mass m with harmonic oscillator interaction. Considering

$\omega = 0$, we have

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \frac{\hbar\langle\eta^2\rangle}{6m^2}\left(n_1^{(1)} + n_2^{(1)} + n_3^{(1)} + \frac{3}{2}\right) + \\ +\hbar\left(\frac{2kN}{m} + \frac{\langle\eta^2\rangle}{6m^2} + \frac{2k^2\langle\theta^2\rangle N^2}{3}\right)^{\frac{1}{2}} \\ \sum_{a=2}^{N}\left(n_1^{(a)} + n_2^{(a)} + n_3^{(a)} + \frac{3}{2}\right) - \frac{3N\kappa^2 m}{\langle\eta^2\rangle} + 3\hbar\omega_{osc}. \qquad (3.39)$$

The spectrum of the center-of-mass of the system is described by (3.39). It is important to note that this spectrum is discreet, that is caused by momentum noncommutativity. The spectrum of the center-of-mass of the system if the spectrum of harmonic oscillator with a frequency determined by the parameter of momentum noncommutativity $\hbar\langle\eta^2\rangle/6m^2$. The spectrum of the relative motion of the system is affected by noncommutativity of coordinates and noncommutativity of momenta (see second term in (3.39)).

It is important to stress that from (3.26) and (3.39) follows that the influence of noncommutativity on the spectrum increases with increasing of the number of particles N .

In the case of $k = 0$ we obtain energy levels of a system of N free particles in uniform field in a space with noncommutativity of coordinates and noncommutativity of momenta

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \sum_{a=1}^{N}\frac{\hbar\langle\eta^2\rangle}{6m^2}\left(n_1^{(a)} + n_2^{(a)} + n_3^{(a)} + \frac{3}{2}\right) - \\ -\frac{3N\kappa^2 m}{\langle\eta^2\rangle} + 3\hbar\omega_{osc}. \qquad (3.40)$$

The expression corresponds to the spectrum of N oscillators with frequencies $\hbar\langle\eta^2\rangle/6m^2$. Noncommutativity of coordinates does not affect on the energy levels of free particle system

3.3 Energy levels of two interacting oscillators

We consider a system of two oscillators with masses m_1, m_2 and frequencies ω_1, ω_2. The Hamiltonian of the system reads

$$H_s = \frac{(\mathbf{P}^{(1)})^2}{2m_1} + \frac{(\mathbf{P}^{(2)})^2}{2m_2} + \frac{m_1\omega_1^2(\mathbf{X}^{(1)})^2}{2} + \frac{m_2\omega_2^2(\mathbf{X}^{(2)})^2}{2} + \\ + k(\mathbf{X}^{(1)} - \mathbf{X}^{(2)})^2. \quad (3.41)$$

Coordinates and momenta $\mathbf{X}^{(n)}$, $\mathbf{P}^{(n)}$ satisfy relations of noncommutative algebra (3.2)-(3.4).

It is worth noting that system of two coupled harmonic oscillators is considered as a model in molecular physics [85, 86]. It is also is used for description of states of light in the framework of two-photon quantum optics [94, 95].

In the case of two interacting oscillators we can write

$$H_0 = \frac{(\mathbf{p}^{(1)})^2}{2m_{eff}^{(1)}} + \frac{(\mathbf{p}^{(2)})^2}{2m_{eff}^{(2)}} + \\ + \frac{m_{eff}^{(1)}(\omega_{eff}^{(1)})^2(\mathbf{x}^{(1)})^2}{2} + \frac{m_{eff}^{(2)}(\omega_{eff}^{(2)})^2(\mathbf{x}^{(2)})^2}{2} + \\ + k(\mathbf{x}^{(1)} - \mathbf{x}^{(2)})^2 + \frac{k}{6}\Big(\langle(\theta^{(1)})^2\rangle(\mathbf{p}^{(1)})^2 + \langle(\theta^{(2)})^2\rangle(\mathbf{p}^{(2)})^2 - \\ - 2\langle\theta^{(1)}\theta^{(2)}\rangle(\mathbf{p}^{(1)}\cdot\mathbf{p}^{(2)})\Big) + H_{osc}^a + H_{osc}^b. \quad (3.42)$$

Here

$$m_{eff}^{(n)} = m_n\left(1 + \frac{m_n^2\omega_n^2\langle(\theta^{(n)})^2\rangle}{6}\right)^{-1}, \quad (3.43)$$

$$\omega_{eff}^{(n)} = \left(\omega_n^2 + \frac{\langle(\eta^n)^2\rangle}{6m_n^2}\right)^{\frac{1}{2}}\left(1 + \frac{m_n^2\omega_n^2\langle(\theta^{(n)})^2\rangle}{6}\right)^{\frac{1}{2}}, \quad (3.44)$$

$$\langle\theta^{(n)}\theta^{(m)}\rangle = \frac{c_\theta^{(n)}c_\theta^{(m)}l_P^4}{\hbar^2}\langle\psi_{0,0,0}^a|\tilde{a}^2|\psi_{0,0,0}^a\rangle = \frac{3c_\theta^{(n)}c_\theta^{(m)}l_P^4}{2\hbar^2}, \quad (3.45)$$

$$\langle(\eta^{(n)})^2\rangle = \frac{\hbar^2(c_\eta^{(n)})^2}{l_P^4}\langle\psi_{0,0,0}^b|(\tilde{p}^b)^2|\psi_{0,0,0}^b\rangle = \frac{3\hbar^2(c_\eta^{(n)})^2}{2l_P^4}. \quad (3.46)$$

For coordinates and momenta $x_i^{(n)}$, $p_i^{(n)}$ we have the ordinary commutation relations. Therefore, the energy levels of H_0 are

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \hbar\omega_+ \left(n_1^{(1)} + n_2^{(1)} + n_3^{(1)} + \frac{3}{2}\right) + \\ +\hbar\omega_- \left(n_1^{(2)} + n_2^{(2)} + n_3^{(2)} + \frac{3}{2}\right) + 3\hbar\omega_{osc}, \tag{3.47}$$

with

$$\omega_\pm^2 = \frac{1}{2}\sum_n \left((\omega_{eff}^{(n)})^2 + \frac{2k}{m_{eff}^{(n)}} + \frac{k m_{eff}^{(n)} (\omega_{eff}^{(n)})^2 \langle(\theta^{(n)})^2\rangle}{3} + \right. \\ \left. + \frac{2k^2}{3}\left(\langle(\theta^{(n)})^2\rangle + \langle\theta^{(1)}\theta^{(2)}\rangle\right)\right) \pm \frac{1}{2}\sqrt{D}, \tag{3.48}$$

$$D = \left(\sum_n (\omega_{eff}^{(n)})^2 + \sum_n \frac{2k}{m_{eff}^{(n)}} + \sum_n \frac{k m_{eff}^{(n)} (\omega_{eff}^{(n)})^2 \langle(\theta^{(n)})^2\rangle}{3} + \right. \\ \left. + \sum_n \frac{2k^2}{3}\left(\langle(\theta^{(n)})^2\rangle + \langle\theta^{(1)}\theta^{(2)}\rangle\right)\right)^2 - 4\prod_n \left((\omega_{eff}^{(n)})^2 + \frac{2k}{m_{eff}^{(n)}} + \right. \\ \left. + \frac{k m_{eff}^{(n)} (\omega_{eff}^{(n)})^2 \langle(\theta^{(n)})^2\rangle}{3} + \frac{2k^2}{3}\left(\langle(\theta^{(n)})^2\rangle + \langle\theta^{(1)}\theta^{(2)}\rangle\right)\right) + \\ +4\left(\frac{2k}{m_{eff}^{(2)}} + \frac{k m_{eff}^{(1)} (\omega_{eff}^{(1)})^2 \langle\theta^{(1)}\theta^{(2)}\rangle}{3} + \frac{2k^2}{3}\left(\langle(\theta^{(2)})^2\rangle + \langle\theta^{(1)}\theta^{(2)}\rangle\right)\right) \times \\ \left(\frac{2k}{m_{eff}^{(1)}} + \frac{k m_{eff}^{(2)} (\omega_{eff}^{(2)})^2 \langle\theta^{(1)}\theta^{(2)}\rangle}{3} + \frac{2k^2}{3}\left(\langle(\theta^{(1)})^2\rangle + \langle\theta^{(1)}\theta^{(2)}\rangle\right)\right). \tag{3.49}$$

If the masse of the oscillators are the same $m_1 = m_2$, we obtain

$$m_{eff}^{(n)} = m_{eff}, \tag{3.50}$$

$$\omega_{eff}^{(n)} = \omega_{eff}, \tag{3.51}$$

and

$$\omega_- = \omega_{eff}, \tag{3.52}$$

$$\omega_+ = \left(\omega_{eff}^2 + \frac{4k}{m_{eff}} + \frac{2k\langle\theta^2\rangle m_{eff}\omega_{eff}^2}{3} + \frac{8k^2\langle\theta^2\rangle}{3}\right)^{\frac{1}{2}}, \tag{3.53}$$

which is in agreement with the results presented in the previous section (3.27), (3.28) with $N = 2$.

3.4 Effect of noncommutativity on the energy levels of system of three interacting oscillators

We study three interacting oscillators with masses m_1, $m_2 = m_3 = m$, and frequencies ω_1, $\omega_2 = \omega_3 = \omega$ descried with the following Hamiltonian

$$H_s = \frac{(\mathbf{P}^{(1)})^2}{2m_1} + \frac{(\mathbf{P}^{(2)})^2}{2m} + \frac{(\mathbf{P}^{(3)})^2}{2m} + \\ + \frac{m_1\omega_1^2(\mathbf{X}^{(1)})^2}{2} + \frac{m\omega^2(\mathbf{X}^{(2)})^2}{2} + \frac{m\omega^2(\mathbf{X}^{(3)})^2}{2} + \\ + k(\mathbf{X}^{(1)} - \mathbf{X}^{(2)})^2 + k(\mathbf{X}^{(2)} - \mathbf{X}^{(3)})^2 + k(\mathbf{X}^{(3)} - \mathbf{X}^{(3)})^2. \tag{3.54}$$

If $\omega_n = 0$ the model (3.54) is used for the description of confining forces between quarks [82–84]. Up to the second order in the parameters of noncommutativity we can study Hamiltonian

$$H_0 = \\ = \sum_n \frac{(\mathbf{p}^{(n)})^2}{2m_{eff}^{(n)}} + \sum_n \frac{m_{eff}^{(n)}(\omega_{eff}^{(n)})^2(\mathbf{x}^{(n)})^2}{2} + + \frac{k}{2}\sum_{\substack{m,n\\ m\neq n}}(\mathbf{x}^{(n)} - \mathbf{x}^{(m)})^2 + \\ + \frac{k}{12}\sum_{\substack{m,n\\ m\neq n}}\Big(\langle(\theta^{(n)})^2\rangle(\mathbf{p}^{(n)})^2 + \langle(\theta^{(m)})^2\rangle(\mathbf{p}^{(m)})^2 - \\ -2\langle\theta^{(n)}\theta^{(m)}\rangle(\mathbf{p}^{(n)}\cdot\mathbf{p}^{(m)})\Big) + H_{osc}^a + H_{osc}^b, \tag{3.55}$$

with $m_{eff}^{(n)}$, $\omega_{eff}^{(n)}$, $\langle\theta^{(n)}\theta^{(m)}\rangle$ given by (3.43)-(3.45).

The energy levels of Hamiltonian (3.55) are the following

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \sum_{a=1}^{3} \hbar\tilde{\omega}_a \left(n_1^{(a)} + n_2^{(a)} + n_3^{(a)} + \frac{3}{2}\right) + 3\hbar\omega_{osc}, \quad (3.56)$$

$$\tilde{\omega}_1 = \frac{1}{\sqrt{2}} \left(\omega_{eff}^2 + (\omega_{eff}^{(1)})^2 + \frac{2k}{m_{eff}} + \frac{4k}{m_{eff}^{(1)}} + A_1 - \sqrt{D}\right)^{\frac{1}{2}}, \quad (3.57)$$

$$\tilde{\omega}_2 = \frac{1}{\sqrt{2}} \left(\omega_{eff}^2 + (\omega_{eff}^{(1)})^2 + \frac{2k}{m_{eff}} + \frac{4k}{m_{eff}^{(1)}} + A_1 + \sqrt{D}\right)^{\frac{1}{2}}, \quad (3.58)$$

$$\tilde{\omega}_3 = \left(\omega_{eff}^2 + \frac{6k}{m_{eff}}\right)^{\frac{1}{2}} \left(1 + km_{eff}\langle\theta^2\rangle\right)^{\frac{1}{2}}, \quad (3.59)$$

where

$$D = \left(\omega_{eff}^2 - (\omega_{eff}^{(1)})^2 + \frac{4k}{m_{eff}} - \frac{4k}{m_{eff}^{(1)}} + A_2\right)^2 +$$
$$+ \left(\frac{2k}{m} + A_3\right) \left(2(\omega_{eff}^{(1)})^2 - 2\omega_{eff}^2 - \frac{6k}{m} + \frac{8k}{m_{eff}^{(1)}} + \right.$$
$$\left. +8\left(\frac{2k}{m} + A_4\right)\left(\frac{2k}{m_1} + A_5\right)\left(\frac{2k}{m} + A_3\right)^{-1} + A_6\right), \quad (3.60)$$

$$A_1 = \left(\frac{km_{eff}\omega_{eff}^2}{3} + \frac{2k^2}{3}\right)\langle\theta^2\rangle +$$
$$+ \left(\frac{2km_{eff}^{(1)}(\omega_{eff}^{(1)})^2}{3} + \frac{8k^2}{3}\right)\langle(\theta^{(1)})^2\rangle + \frac{8k^2}{3}\langle\theta\theta^{(1)}\rangle, \quad (3.61)$$

$$A_2 = \left(\frac{2km_{eff}\omega_{eff}^2}{3} + \frac{10k^2}{3}\right)\langle\theta^2\rangle -$$
$$- \left(\frac{2km_{eff}^{(1)}(\omega_{eff}^{(1)})^2}{3} + \frac{8k^2}{3}\right)\langle(\theta^{(1)})^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle, \tag{3.62}$$

$$A_3 = \left(\frac{8k^2}{3} + \frac{km_{eff}\omega_{eff}^2}{3}\right)\langle\theta^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle, \tag{3.63}$$

$$A_4 = \left(\frac{km_{eff}^{(1)}(\omega_{eff}^{(1)})^2}{3} + \frac{4k^2}{3}\right)\langle\theta\theta^{(1)}\rangle + \frac{2k^2}{3}\langle,\theta^2\rangle, \tag{3.64}$$

$$A_5 = \left(\frac{km_{eff}(\omega_{eff}^2)}{3} + \frac{2k^2}{3}\right)\langle\theta\theta^{(1)}\rangle + \frac{4k^2}{3}\langle(\theta^{(1)})^2\rangle, \tag{3.65}$$

$$A_6 = -\left(km_{eff}\omega_{eff}^2 + 4k^2\right)\langle\theta^2\rangle +$$
$$+ \left(\frac{4km_{eff}^{(1)}(\omega_{eff}^{(1)})^2}{3} + \frac{16k^2}{3}\right)\langle(\theta^{(1)})^2\rangle + \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle. \tag{3.66}$$

For convenience we introduce notations

$$m_{eff} = m_{eff}^{(2)} = m_{eff}^{(3)}, \omega_{eff} = \omega_{eff}^{(2)} = \omega_{eff}^{(3)}, \tag{3.67}$$
$$\theta = \theta^{(2)} = \theta^{(3)}. \tag{3.68}$$

Considering $m_1 = m$, $\omega_1 = \omega$, on the basis of (3.56) we obtain (3.26) with $N = 3$. Namely, we can write

$$\tilde{\omega}_1 = \omega_{eff}, \tag{3.69}$$
$$\tilde{\omega}_2 = \tilde{\omega}_3 = \left(\omega_{eff}^2 + \frac{6k}{m_{eff}} + k\langle\theta^2\rangle m_{eff}\omega_{eff}^2 + 6k^2\langle\theta^2\rangle\right)^{\frac{1}{2}}. \tag{3.70}$$

If $\omega_n = 0$ in Hamiltonian (3.54) the spectrum is given by (3.56) with (3.57), (3.58), (3.59) and $m^{(1)}_{eff} = m_1$, $m_{eff} = m$,

$$\omega^{(1)}_{eff} = \frac{\sqrt{\langle(\eta^1)^2\rangle}}{\sqrt{6m_1^2}}, \tag{3.71}$$

$$\omega_{eff} = \frac{\sqrt{\langle(\eta)^2\rangle}}{\sqrt{6m^2}}. \tag{3.72}$$

It is worth mentioning that spectrum of the center-of-mass of the system is discrete. It has the form of the spectrum of harmonic oscillator with frequency $\tilde{\omega}_1$ (3.57).

If we consider algebra with commutation relations (3.2), (3.3) and commutative momenta $[P_i^{(n)}, P_j^{(m)}] = 0$), the spectrum of a system (3.54) with $\omega_n = 0$ reads (3.56) where $\tilde{\omega}_i$ are given by

$$\tilde{\omega}_1 = 0, \tag{3.73}$$

$$\tilde{\omega}_2 = \frac{1}{\sqrt{2}}\left(\frac{2k}{m} + \frac{4k}{m^{(1)}} + \frac{2k^2}{3}\langle\theta^2\rangle + \right.$$
$$\left. + \frac{8k^2}{3}\langle(\theta^{(1)})^2\rangle + \frac{8k^2}{3}\langle\theta\theta^{(1)}\rangle + \sqrt{D}\right)^{\frac{1}{2}}, \tag{3.74}$$

$$\tilde{\omega}_3 = \left(\frac{6k}{m} + 6k^2\langle\theta^2\rangle\right)^{\frac{1}{2}}. \tag{3.75}$$

Here we have

$$D = \left(\frac{4k}{m} - \frac{4k}{m^{(1)}} + \frac{10k^2}{3}\langle\theta^2\rangle - \frac{8k^2}{3}\langle(\theta^{(1)})^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\right)^2 +$$
$$+ \left(\frac{2k}{m} + \frac{8k^2}{3}\langle\theta^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\right) \times$$
$$\times \left(-\frac{6k}{m} + \frac{8k}{m^{(1)}} + 8\left(\frac{2k}{m} + \frac{4k^2}{3}\langle\theta\theta^{(1)}\rangle + \frac{2k^2}{3}\langle\theta^2\rangle\right) \times \right.$$
$$\times \left(\frac{2k}{m_1} + \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle + \frac{4k^2}{3}\langle(\theta^{(1)})^2\rangle\right) \times$$
$$\times \left(\frac{2k}{m} + \frac{8k^2}{3}\langle\theta^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\right)^{-1} -$$

$$-4k^2\langle\theta^2\rangle + \frac{16k^2}{3}\langle(\theta^{(1)})^2\rangle + \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\Bigg). \tag{3.76}$$

It is worth mentioning that noncommutativity of coordinates does not affect the spectrum of the center-of-mass of the system (3.73). Space quantization affects the frequencies of the relative motion (3.74), (3.75).

3.5 Harmonic oscillator chain in noncommutative phase space with preserved rotational symmetry

Let us study Hamiltonian as follows

$$H_s = \sum_{n=1}^{N}\frac{(\mathbf{P}^{(n)})^2}{2m} + \sum_{n=1}^{N}\frac{m\omega^2(\mathbf{X}^{(n)})^2}{2} + \\ +k\sum_{n=1}^{N}(\mathbf{X}^{(n+1)} - \mathbf{X}^{(n)})^2 \tag{3.77}$$

with periodic boundary conditions $\mathbf{X}^{(N+1)} = \mathbf{X}^{(1)}$, k is a constant. The Hamiltonian corresponds to N interacting harmonic oscillator chain, m are the masses of oscillators and ω are frequencies

The Hamiltonian H_s can be represented as

$$H_s = \sum_{n=1}^{N}\Bigg(\frac{(\mathbf{p}^{(n)})^2}{2m} + \frac{m\omega^2(\mathbf{x}^{(n)})^2}{2} + \\ +k(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)})^2 - \frac{(\boldsymbol{\eta}\cdot[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2m} - \\ -\frac{m\omega^2(\boldsymbol{\theta}\cdot[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2} - \\ -k(\boldsymbol{\theta}\cdot[(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)})\times(\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})]) + \\ +\frac{[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2}{8m} + \frac{m\omega^2}{8}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2 + \\ +\frac{k}{4}[\boldsymbol{\theta}\times(\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})]^2\Bigg). \tag{3.78}$$

Also, for the harmonic oscillator chain we can write

$$\Delta H = \sum_{n=1}^{N} \left(\frac{[\boldsymbol{\eta} \times \mathbf{x}^{(n)}]^2}{8m} + \frac{m\omega^2}{8}[\boldsymbol{\theta} \times \mathbf{p}^{(n)}]^2 - \right.$$
$$- \frac{m\omega^2(\boldsymbol{\theta} \cdot [\mathbf{x}^{(n)} \times \mathbf{p}^{(n)}])}{2} - \frac{(\boldsymbol{\eta} \cdot [\mathbf{x}^{(n)} \times \mathbf{p}^{(n)}])}{2m} -$$
$$-k\boldsymbol{\theta} \cdot [(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}) \times (\mathbf{p}^{(n+1)} - \mathbf{p}^{(n+1)})] +$$
$$+ \frac{k}{4}[\boldsymbol{\theta} \times (\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})]^2 - \frac{\langle \eta^2 \rangle (\mathbf{x}^{(n)})^2}{12m} -$$
$$\left. - \frac{\langle \theta^2 \rangle m\omega^2 (\mathbf{p}^{(n)})^2}{12} - \frac{k}{6}\langle \theta^2 \rangle (\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})^2 \right). \tag{3.79}$$

So, up to the second order in the parameters of noncommutativity one can study Hamiltonian H_0 as follows

$$H_0 = \sum_{n=1}^{N} \left(\frac{(\mathbf{p}^{(n)})^2}{2m_{eff}} + \frac{m_{eff}\omega_{eff}^2 (\mathbf{x}^{(n)})^2}{2} + \right.$$
$$+ k(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)})^2 +$$
$$\left. + \frac{k}{6}\langle \theta^2 \rangle (\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})^2 + H_{osc}^a + H_{osc}^b \right), \tag{3.80}$$

where

$$m_{eff} = m \left(1 + \frac{m^2\omega^2\langle \theta^2 \rangle}{6} \right)^{-1}, \tag{3.81}$$

$$\omega_{eff} = \left(\omega^2 + \frac{\langle \eta^2 \rangle}{6m^2} \right)^{\frac{1}{2}} \left(1 + \frac{m^2\omega^2\langle \theta^2 \rangle}{6} \right)^{\frac{1}{2}}. \tag{3.82}$$

Note that $[H_{osc}^a + H_{osc}^b, H_0] = 0$. Coordinates and momenta $\mathbf{x}^{(n)}$, $\mathbf{p}^{(n)}$ satisfy the ordinary commutation relations. It is convenient to rewrite the Hamiltonian as follows

$$H_0 = \frac{\hbar\omega_{eff}}{2}\sum_n\left(1+\frac{4km_{eff}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)\tilde{\mathbf{p}}^{(n)}(\tilde{\mathbf{p}}^{(n)})^\dagger + \\ +\frac{\hbar\omega_{eff}^2}{2}\sum_n\left(1+\frac{8k}{m_{eff}\omega_{eff}^2}\sin^2\frac{\pi n}{N}\right)\tilde{\mathbf{x}}^{(n)}(\tilde{\mathbf{x}}^{(n)})^\dagger, \tag{3.83}$$

where

$$\mathbf{x}^{(n)} = \sqrt{\frac{\hbar}{Nm_{eff}\omega_{eff}}}\sum_{l=1}^{N}\exp\left(\frac{2\pi inl}{N}\right)\tilde{\mathbf{x}}^{(l)}, \tag{3.84}$$

$$\mathbf{p}^{(n)} = \sqrt{\frac{\hbar m_{eff}\omega_{eff}}{N}}\sum_{l=1}^{N}\exp\left(-\frac{2\pi inl}{N}\right)\tilde{\mathbf{p}}^{(l)}, \tag{3.85}$$

(see, for example, [91]). Introducing

$$a_j^{(n)} = \frac{1}{\sqrt{2w_n}}\left(w_n\tilde{x}_j^{(n)} + i\tilde{p}_j^{(n)}\right), \tag{3.86}$$

$$w_n = \left(1+\frac{8k}{m_{eff}\omega_{eff}^2}\sin^2\frac{\pi n}{N}\right)^{\frac{1}{2}}\times \\ \times\left(1+\frac{4km_{eff}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)^{-\frac{1}{2}}, \tag{3.87}$$

we obtain

$$H_0 = \hbar\omega_{eff}\sum_{n=1}^{N}\sum_{j=1}^{3}\left(1+\frac{4km_{eff}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)^{\frac{1}{2}}\times \\ \times\left(1+\frac{8k}{m_{eff}\omega_{eff}^2}\sin^2\frac{\pi n}{N}\right)^{\frac{1}{2}}\left((a_j^{(n)})^\dagger a_j^{(n)}+\frac{1}{2}\right). \tag{3.88}$$

So, the energy levels of H_0 are given by

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \hbar \sum_{a=1}^{N} \left(\omega_{eff}^2 + \frac{8k}{m_{eff}} \sin^2 \frac{\pi a}{N} \right)^{\frac{1}{2}} \times \\ \times \left(1 + \frac{4k m_{eff} \langle \theta^2 \rangle}{3} \sin^2 \frac{\pi a}{N} \right)^{\frac{1}{2}} \left(n_1^{(a)} + n_2^{(a)} + \right. \\ \left. + n_3^{(a)} + \frac{3}{2} \right) = \sum_{a=1}^{N} \hbar\omega_a \left(n_1^{(a)} + n_2^{(a)} + n_3^{(a)} + \frac{3}{2} \right). \tag{3.89}$$

Here $n_i^{(a)}$ are quantum numbers ($n_i^{(a)} = 0, 1, 2...$). Using (3.81), (3.82), we have the following expressions for the frequencies

$$\omega_a^2 = \left(\omega^2 + \frac{\langle \eta^2 \rangle}{6m^2} \right) \left(1 + \frac{m^2 \omega^2 \langle \theta^2 \rangle}{6} + \right. \\ \left. + \frac{4k^2 m \langle \theta^2 \rangle}{3} \sin^2 \frac{\pi a}{N} \right) + \frac{8k}{m} \sin^2 \frac{\pi a}{N} + \\ + \frac{32k^2 \langle \theta^2 \rangle}{3} \sin^4 \frac{\pi a}{N}. \tag{3.90}$$

Let us also study a particular case of $\omega = 0$. So, up to the second order in the parameters of noncommutativity for a system of particles with harmonic oscillator interaction we have

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \\ = \sum_{a=1}^{N} \hbar\omega_a \left(n_1^{(a)} + n_2^{(a)} + n_3^{(a)} + \frac{3}{2} \right), \tag{3.91}$$

where

$$\omega_a^2 = \frac{8k}{m} \sin^2 \frac{\pi a}{N} + \frac{\langle \eta^2 \rangle}{6m^2} + \frac{32k^2 \langle \theta^2 \rangle}{3} \sin^4 \frac{\pi a}{N}. \tag{3.92}$$

If momenta commutes $\eta_{ij} = 0$ the spectrum of a chain of particles with harmonic oscillator interaction in a space with noncommutativity of coordinates has the form (3.91) with frequencies

$$\omega_a^2 = \frac{8k}{m} \sin^2 \frac{\pi a}{N} + \frac{32k^2 \langle \theta^2 \rangle}{3} \sin^4 \frac{\pi a}{N}. \tag{3.93}$$

From (3.91), (3.92) we have that the spectrum of the center-of-mass of the system is the spectrum of harmonic oscillator with frequency

$$\omega_N^2 = \frac{\langle \eta^2 \rangle}{6m^2}. \tag{3.94}$$

Note that in the limit $\langle \theta^2 \rangle \to 0$, $\langle \eta^2 \rangle \to 0$ on the basis of (3.90) we have

$$\omega_a^2 = \omega^2 + \frac{8k}{m} \sin^2 \frac{\pi a}{N}. \tag{3.95}$$

that is well known result in ordinary space.

3.6 Conclusions

We have examined energy levels of a system of N harmonic oscillators with harmonic oscillator interaction in uniform field in rotationally-invariant noncommutative phase space of canonical type.

Up to the second order in the parameters of noncommutativity we have obtained influence of noncommutativity of coordinates and noncommutativity of momenta on the energy levels of the system. We have concluded that space quantization affects on the frequencies of the system (3.26). Uniform field shifts of the spectrum of the system on a constant (3.26). Particular case of a system of two interacting oscillators and a system of three interacting oscillators have been examined. We have found energy levels of the systems in rotationally-invariant noncommutative phase space (3.47), (3.56).

On the basis of the obtained results a system of particles with harmonic oscillator interaction and a system of free particles in uniform field have been examined. We have concluded that up to the second orders in the parameters of noncommutativity, the noncommutativity of coordinates does not affect the spectrum of free particle system in uniform field. The spectrum of free particles in uniform field has the form of the spectrum of a system of N harmonic oscillators with frequencies determined by parameters of momentum noncommutativity a s $\hbar\langle \eta^2 \rangle / 6m^2$ (3.40). We have also shown that a spectrum of the center-of-mass of a system of particles with harmonic oscillator interaction in uniform field corresponds to the spectrum of harmonic oscillator (see first term in (3.39)) and is affected only by noncommutativity of momenta. We have also found that the spectrum of the relative motion of the system of interacting harmonic oscillators corresponds to the spectrum of harmonic oscillators with frequencies that depends on the parameters of noncommutativity (see second term in (3.39)). We have also showed that effect of coordinates noncommutativity on the spectra of systems with harmonic oscillator interaction increases with increasing of the number of particles (3.26), (3.39).

Also, the harmonic oscillator chain has been studied. We have obtained that noncommutativity of coordinates and noncommutativity of momenta does not change the form of the spectrum of the system (3.89). The frequencies of the system are affected by the space quantization as (3.90).

Chapter 4

Time reversal symmetry in noncommutative phase space of canonical type

4.1 Introduction

In ordinary space commutation relations for coordinates and momenta

$$[X_1, X_2] = 0 \tag{4.1}$$
$$[X_1, P_1] = [X_2, P_2] = i\hbar, \tag{4.2}$$
$$[P_1, P_2] = 0, \tag{4.3}$$

are invariant upon time reversal [96].

If we consider transformations of coordinates and momenta upon time reversal as in the ordinary case

$$X_i \to X_i, \tag{4.4}$$
$$P_i \to -P_i, \tag{4.5}$$

taking into account that in the quantum case the time reversal operation involves also the operation of complex conjugation [96], in the case of noncommutative algebra of canonical type

$$[X_1, X_2] = i\hbar\theta, \tag{4.6}$$
$$[X_1, P_1] = [X_2, P_2] = i\hbar(1+\gamma), \tag{4.7}$$
$$[P_1, P_2] = i\hbar\eta, \tag{4.8}$$

we find

$$[X_1, X_2] = -i\hbar\theta, \tag{4.9}$$
$$[X_1, P_1] = [X_2, P_2] = i\hbar(1+\gamma), \tag{4.10}$$
$$[P_1, P_2] = -i\hbar\eta. \tag{4.11}$$

So, algebra (4.6)-(4.8) is not invariant upon the time reversal. Because of this, the transformation of coordinates and moment X_i, P_i after time reversal depends on representation. Noncommutative coordinates and momenta satisfying (4.6)-(4.8) can be represented by coordinates and momenta that satisfy the ordinary commutation relations as

$$X_1 = \varepsilon\left(x_1 - \theta_1' p_2\right), \tag{4.12}$$
$$X_2 = \varepsilon\left(x_2 + \theta_2' p_1\right), \tag{4.13}$$
$$P_1 = \varepsilon\left(p_1 + \eta_1' x_2\right), \tag{4.14}$$
$$P_2 = \varepsilon\left(p_2 - \eta_2' x_1\right). \tag{4.15}$$

Here ε, θ_1', θ_2', η_2', η_2' are constants.

After time reversal, if we consider transformations for coordinates and momenta as in the ordinary space $x_i \to x_i$, $p_i \to -p_i$, we obtain

$$X_1 \to X_1' = \varepsilon\left(x_1 + \theta_1' p_2\right), \tag{4.16}$$
$$X_2 \to X_2' = \varepsilon\left(x_2 - \theta_2' p_1\right), \tag{4.17}$$
$$P_1 \to -P_1' = \varepsilon\left(-p_1 + \eta_1' x_2\right), \tag{4.18}$$
$$P_2 \to -P_2' = \varepsilon\left(-p_2 - \eta_2' x_1\right). \tag{4.19}$$

The results (4.16)-(4.19) depend on the parameters ε, θ_1', θ_2', η_2', η_2'. So, the transformation of the noncommutative coordinates depends on the representation.

One can choose parameters ε, θ_1', θ_2', η_2', η_2' in different ways. On the basis of (4.12)-(4.15) we can write

$$[X_1, X_2] = i\hbar\varepsilon^2(\theta_1' + \theta_2'), \tag{4.20}$$
$$[X_1, P_1] = i\hbar\varepsilon^2(1 + \theta_1'\eta_1') \tag{4.21}$$
$$[X_2, P_2] = i\hbar\varepsilon^2(1 + \theta_2'\eta_2'), \tag{4.22}$$
$$[P_1, P_2] = i\hbar\varepsilon^2(\eta_1' + \eta_2'). \tag{4.23}$$

Comparing (4.20)-(4.23) and (4.6)-(4.8) we obtain

$$\varepsilon^2 = 1, \quad \theta'_1\eta'_1 = \theta'_2\eta'_2 = \gamma, \tag{4.24}$$

$$\theta'_1 + \theta'_2 = \theta, \tag{4.25}$$

$$\eta'_1 + \eta'_2 = \eta. \tag{4.26}$$

Based on the equations we find

$$\theta'_1 = \frac{1}{2}\left(\theta \pm \sqrt{\theta^2 - 4\frac{\theta\gamma}{\eta}}\right), \tag{4.27}$$

$$\theta'_2 = \frac{1}{2}\left(\theta \mp \sqrt{\theta^2 - 4\frac{\theta\gamma}{\eta}}\right), \tag{4.28}$$

$$\eta'_1 = \frac{1}{2}\left(\eta \mp \sqrt{\eta^2 - 4\frac{\eta\gamma}{\theta}}\right), \tag{4.29}$$

$$\eta'_2 = \frac{1}{2}\left(\eta \pm \sqrt{\eta^2 - 4\frac{\eta\gamma}{\theta}}\right), \tag{4.30}$$

and $\gamma \leq \theta\eta/4$. So, we have two different representations for noncommutative coordinates and noncommutative momenta. These representations determine two different transformations after time reversal (4.16)-(4.19).

Well-known is the symmetric representation

$$\varepsilon = 1, \tag{4.31}$$

$$\theta'_1 = \theta'_2 = \frac{\theta}{2}, \tag{4.32}$$

$$\eta'_1 = \eta'_2 = \frac{\eta}{2} \tag{4.33}$$

In this case

$$\gamma = \frac{\theta\eta}{4}, \tag{4.34}$$

see [21] If $\gamma = 0$, one has the ordinary commutation relation for coordinates and momenta. The commutator for coordinates and momenta is equal to $i\hbar$. Taking into account (4.20)-(4.23), (4.6)-(4.8),

$\gamma = 0$ we have

$$\varepsilon^2 = \frac{1}{1+\theta'_1\eta'_1}, \tag{4.35}$$

$$\theta'_1\eta'_1 = \theta'_2\eta'_2, \tag{4.36}$$

$$\varepsilon^2(\theta'_1 + \theta'_2) = \theta, \tag{4.37}$$

$$\varepsilon^2(\eta'_1 + \eta'_2) = \eta, \tag{4.38}$$

One has one free parameter. Namely, five parameters ε, θ'_1, θ'_2, η'_1, η'_2 are related with four equations (4.35)-(4.38). So, choosing one of the parameters one can obtain different representations for noncommutative coordinates and momenta which satisfy (4.6)-(4.8) with $\gamma = 0$. So, one can write different transformations after time reversal (4.16)-(4.19).

If we choose $\theta'_2 = 0$ we find $\varepsilon = 1$, $\eta'_1 = 0$, $\eta'_2 = \eta$, $\theta'_1 = \theta$. So, the representation is the following

$$X_1 = x_1 - \theta p_2, \tag{4.39}$$

$$X_2 = x_2, \tag{4.40}$$

$$P_1 = p_1, \tag{4.41}$$

$$P_2 = p_2 - \eta x_1. \tag{4.42}$$

So, upon time reversal the coordinate X_2, and momentum P_1 transform as in the ordinary space $X_2 \to X_2$, $P_1 \to -P_1$. For coordinates and momenta X_1, P_2 we obtain

$$X_1 \to X'_1 = x_1 + \theta p_2, \tag{4.43}$$

$$P_2 \to -P'_2 = -p_2 - \eta x_1. \tag{4.44}$$

If we choose

$$\varepsilon = (1+\theta'\eta')^{-\frac{1}{2}}, \tag{4.45}$$

$$\theta'_1 = \theta'_2 = \frac{1 \pm \sqrt{1-\theta\eta}}{\eta}, \tag{4.46}$$

$$\eta'_1 = \eta'_2 = \frac{1 \pm \sqrt{1-\theta\eta}}{\theta} \tag{4.47}$$

we can write two symmetric representations (4.12)-(4.15) [21, 57]. These representations also lead to different transformations under the time reversal.

Obvious example for studies of time-reversal symmetry is the circular motion. Considering Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - \frac{k}{X}, \tag{4.48}$$

(here $X = \sqrt{X_1^2 + X_2^2}$) and taking into account that coordinates and momenta X_i, P_i satisfy relations of noncommutative algebra of canonical type, we find

$$\dot{X}_1 = \frac{P_1}{m}(1+\gamma) + \frac{k\theta X_2}{X^3}, \tag{4.49}$$

$$\dot{X}_2 = \frac{P_2}{m}(1+\gamma) - \frac{k\theta X_1}{X^3}, \tag{4.50}$$

$$\dot{P}_1 = \frac{\eta P_2}{m} - \frac{kX_1}{X^3}(1+\gamma), \tag{4.51}$$

$$\dot{P}_2 = -\frac{\eta P_1}{m} - \frac{kX_2}{X^3}(1+\gamma). \tag{4.52}$$

Solutions of the equations that correspond to the circular motion read

$$X_1(t) = R_0 \cos(\omega t), \tag{4.53}$$

$$X_2(t) = R_0 \sin(\omega t), \tag{4.54}$$

$$P_1(t) = -P_0 \sin(\omega t), \tag{4.55}$$

$$P_2(t) = P_0 \cos(\omega t). \tag{4.56}$$

Here R_0 is the radii of the circle. The momentum reads

$$P_0 = \frac{m\omega R_0^3 + km\theta}{R_0^2(1+\gamma)}, \tag{4.57}$$

and frequency is defined as

$$\omega = \frac{1}{2}\left(\sqrt{\frac{4k}{mR_0^3}\left((1+\gamma)^2 - \theta\eta\right) + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} - \frac{\eta}{m} - \frac{k\theta}{R_0^3}\right). \tag{4.58}$$

For the period of motion, we have

$$T = 4\pi\left(\sqrt{\frac{4k}{mR_0^3}\left((1+\gamma)^2 - \theta\eta\right) + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} - \frac{\eta}{m} - \frac{k\theta}{R_0^3}\right)^{-1} \tag{4.59}$$

If we study the motion in the opposite direction with the same radii R_0, we find

$$X_1(t) = R_0 \cos(\omega t), \tag{4.60}$$
$$X_2(t) = -R_0 \sin(\omega t), \tag{4.61}$$
$$P_1(t) = P_0' \sin(\omega t), \tag{4.62}$$
$$P_2(t) = P_0' \cos(\omega t). \tag{4.63}$$

Here we use notions P_0' to distinguish momentum in the case of motion in opposite direction. Using (4.60)-(4.63), (4.49)-(4.220) we find

$$\omega' = \frac{1}{2}\left(\sqrt{\frac{4k}{mR_0^3}\left((1+\gamma)^2 - \theta\eta\right) + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} + \frac{\eta}{m} + \frac{k\theta}{R_0^3}\right) \tag{4.64}$$

$$T' = 4\pi\left(\sqrt{\frac{4k}{mR_0^3}\left((1+\gamma)^2 - \theta\eta\right) + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} + \frac{\eta}{m} + \frac{k\theta}{R_0^3}\right)^{-1} \tag{4.65}$$

and the momentum reads

$$P_0' = -\frac{m\omega' R_0^3 - km\theta}{R_0^2\,(1+\gamma)}. \tag{4.66}$$

It is important to stress that the expressions (4.58), (4.59), (4.64), (4.65) are different. We have

$$\Delta\omega = \omega' - \omega = \frac{\eta}{m} + \frac{k\theta}{R_0^3}. \tag{4.67}$$

Expressions for ω', T' contain terms with parameters of noncommutativity with opposite signs in comparison to (4.58), (4.59). It is also important to stress that $P_0' \neq -P_0$. All these conclusions are caused by the time-reversal symmetry breaking in noncommutative phase space of canonical type.

In the present chapter, we propose algebra with noncommutativity of coordinates and noncommutativity of momenta which does not lead to violation of the rotational and time-reversal symmetries and is equivalent to noncommutative algebra of canonical type. In the frame of the algebra, the motion of a system of free particles is studied, also the spectrum of a particles in uniform field is find. Also, the motion in the gravitational filed is analyzed. We obtain a stringent upper bound for the momentum scale on the basis of studies of the perihelion shift of the Mercury planet.

The chapter is organized as follows. In section 4.2 algebra which is rotationally invariant and does not lead to time-reversal symmetry breaking is constructed. In section 4.3 motion of a free particle system is studied in noncommutative phase space with preserved rotational and time-reversal symmetries. Section 4.4 is devoted to studies of the energy of a particle in uniform field in noncommutative phase space. In 4.5 motion of a particle in a uniform gravitational field is analyzed and the weak equivalence principle is studied. In section 4.6 equivalence principle is examined in the case of motion in non-uniform gravitational field. Section 4.7 is devoted to calculations of the perihelion shift of the Mercury planet in rotationally-invariant and time-reversal invariant noncommutative phase space. In section 4.8 the upper bound for the parameters of coordinate noncommutativity and parameters of momentum noncommutativity are obtained. Conclusions are presented in section 4.9.

Results presented in this chapter are published in [97–99].

4.2 Noncommutative phase space with preserved time reversal and rotational symmetries

To preserve the time reversal symmetry and rotational symmetry in noncommutative phase space we consider the idea of construction of the tensors of noncommutativity with the help of additional coordinates and additional momenta. To construct algebra which is rotationally-invariant, tensors of noncommutativity θ_{ij}, η_{ij} have to transform under the time reversal as follows

$$\theta_{ij} \to -\theta_{ij}, \tag{4.68}$$

$$\eta_{ij} \to -\eta_{ij}. \tag{4.69}$$

So, from the view of simplicity we construct tensors of noncommutativity as follows

$$\theta_{ij} = \frac{c_\theta}{\hbar}\sum_k \varepsilon_{ijk} p_k^a, \tag{4.70}$$

$$\eta_{ij} = \frac{c_\eta}{\hbar}\sum_k \varepsilon_{ijk} p_k^b. \tag{4.71}$$

Here c_θ, c_η are constants, and p_i^a, p_i^b are additional momenta that correspond to harmonic oscillators (1.23), (1.24) So, rotationally-invariant and time-reversal invariant algebra reads

$$[X_i, X_j] = ic_\theta \sum_k \varepsilon_{ijk} p_k^a, \tag{4.72}$$

$$[X_i, P_j] = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar^2}(\mathbf{p}^a \cdot \mathbf{p}^b)\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar^2} p_j^a p_i^b\right), \tag{4.73}$$

$$[P_i, P_j] = ic_\eta \sum_k \varepsilon_{ijk} p_k^b. \tag{4.74}$$

Additional coordinates and additional momenta satisfy the ordinary commutation relations.

It is important to stress that independently of representation coordinates and momenta upon time reversal transforms as $X_i \to X_i$, $P_i \to -P_i$. Coordinates and momenta which satisfy relations of noncommutative algebra (4.72)-(4.74) can be represented as

$$X_i = x_i + \frac{c_\theta}{2\hbar}[\mathbf{p}^a \times \mathbf{p}]_i, \tag{4.75}$$

$$P_i = p_i - \frac{c_\eta}{2\hbar}[\mathbf{x} \times \mathbf{p}^b]_i, \tag{4.76}$$

where operators x_i, p_i satisfy the ordinary relations

$$[x_i, x_j] = [p_i, p_j] = 0, \tag{4.77}$$

$$[x_i, p_j] = i\hbar\delta_{ij}. \tag{4.78}$$

Upon time reversal we have

$$x_i \to x_i, \tag{4.79}$$

$$p_i \to -p_i, \tag{4.80}$$

$$p_i^a \to -p_i^a, \tag{4.81}$$

$$p_i^b \to -p_i^b. \tag{4.82}$$

So, form (4.75), (4.76) we obtain that upon time reversal noncommutative coordinates and noncommutative momenta transform as

$$X_i \to X_i, \tag{4.83}$$
$$P_i \to -P_i. \tag{4.84}$$

Also, it is important that algebra (4.72)-(4.74) is rotationally invariant. After transformations

$$X'_i = U(\varphi)X_iU^+(\varphi), \tag{4.85}$$
$$P'_i = U(\varphi)P_iU^+(\varphi), \tag{4.86}$$
$$p^{a\prime}_i = U(\varphi)p^a_iU^+(\varphi), \tag{4.87}$$
$$p^{b\prime}_i = U(\varphi)p^b_iU^+(\varphi), \tag{4.88}$$

we have

$$[X'_i, X'_j] = ic_\theta \sum_k \varepsilon_{ijk} p^{a\prime}_k, \tag{4.89}$$
$$[X'_i, P'_j] = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar}(\mathbf{p}^{a\prime}\cdot\mathbf{p}^{b\prime})\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar}p^{a\prime}_j p^{b\prime}_i\right), \tag{4.90}$$
$$[P'_i, P'_j] = ic_\eta \sum_k \varepsilon_{ijk} p^{b\prime}_k, \tag{4.91}$$

where $U(\varphi) = \exp(i\varphi(\mathbf{n}\cdot\mathbf{L^t})/\hbar)$, with $\mathbf{L^t} = [\mathbf{x}\times\mathbf{p}]+[\mathbf{a}\times\mathbf{p}^a]+[\mathbf{b}\times\mathbf{p}^b]$.
The algebra is consistent. This follows from the explicit representation (4.75), (4.76).

4.3 Effect of noncommutativity of momentum on the motion of a system of free particles in time reversal and rotationally invariant noncommutative space

Let us consider a system of N particles in time-reversal and rotationally invariant noncommutative phase space. The Hamiltonian reads

$$H = \sum_n \frac{(\mathbf{P}^{(n)})^2}{2m_n} + H^a_{osc} + H^b_{osc}. \tag{4.92}$$

Here index n labels the particles. Using representation for noncommutative momenta with coordinates and momenta satisfying the ordinary commutation relation we can write

$$H=\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_n}-\frac{(\boldsymbol{\eta}^{(n)}\cdot\mathbf{L}^{(n)})}{2m_n}+\frac{[\boldsymbol{\eta}^{(n)}\times\mathbf{x}^{(n)}]^2}{8m_n}\right)+$$

$$+\hbar\omega_{osc}\left(\frac{(\tilde{p}^a)^2}{2}+\frac{\tilde{a}^2}{2}\right)+\hbar\omega_{osc}\left(\frac{(\tilde{p}^b)^2}{2}+\frac{\tilde{b}^2}{2}\right),\qquad(4.93)$$

where $\mathbf{L}^{(n)}$ reads

$$\mathbf{L}^{(n)}=[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}].\qquad(4.94)$$

In the case of system of free particles we have the following expressions for H_0 and ΔH

$$H_0=\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_n}+\frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m_n}\right)+\qquad(4.95)$$

$$+\hbar\omega_{osc}\left(\frac{(\tilde{p}^a)^2}{2}+\frac{\tilde{a}^2}{2}\right)+\hbar\omega_{osc}\left(\frac{(\tilde{p}^b)^2}{2}+\frac{\tilde{b}^2}{2}\right),\qquad(4.96)$$

$$\Delta H=$$

$$=\sum_n\left(-\frac{(\boldsymbol{\eta}^{(n)}\cdot\mathbf{L}^{(n)})}{2m_n}+\frac{[\boldsymbol{\eta}^{(n)}\times\mathbf{x}^{(n)}]^2}{8m_n}-\frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m}\right).\qquad(4.97)$$

So, up to the second order in the parameter of momentum noncommutativity we can study Hamiltonian H_0.

It is important that the following commutation relation is satisfied

$$\left[\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_n}+\frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m_n}\right),H^a_{osc}+H^b_{osc}\right]=0.\qquad(4.98)$$

It is important also to note that coordinates $x_i^{(n)}$ and momenta $p_i^{(n)}$ satisfy the ordinary commutation relations and therefore in the classical limit they satisfy the ordinary Poisson brackets. We have

$$\{x_i^{(n)},x_j^{(m)}\}=0,\qquad(4.99)$$

$$\{x_i^{(n)},p_j^{(m)}\}=\delta_{ij}\delta_{nm},\qquad(4.100)$$

$$\{p_i^{(n)},p_j^{(m)}\}=0.\qquad(4.101)$$

So, the Hamiltonian that describes a system of free particles reads

$$H_s = \sum_n \left(\frac{(\mathbf{p}^{(n)})^2}{2m_n} + \frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m_n} \right). \quad (4.102)$$

It corresponds to a Hamiltonian of a system of harmonic oscillators with frequencies determined by the parameters of momentum noncommutativity $\langle(\eta^{(n)})^2\rangle$ in the following way

$$\omega_n = \sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}. \quad (4.103)$$

On the basis of expression (4.102) we can write the following equations

$$x_i^{(n)}(t) = x_{0i}^{(n)} \cos\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right) + \\ +\upsilon_{0i}^{(n)}\sqrt{\frac{6m_n^2}{\langle(\eta^{(n)})^2\rangle}} \sin\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right), \quad (4.104)$$

where $x_{0i}^{(n)}$, $\upsilon_{0i}^{(n)}$ are the initial coordinates and initial velocity. It is important to stress that the trajectory of free particle (4.147) depends on mass. This is because of noncommutativity of momenta. As a result, even in the case when all particles have the same velocities $\upsilon_{0i}^{(n)} = \upsilon_{0i}$ the free particles fly away. For the trajectory of the center-of-mass and the relative motion we have the following expressions

$$\tilde{x}_i(t) = \sum_n \mu_n x_{0i}^{(n)} \cos\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right) + \\ + \sum_n \mu_n \upsilon_{0i}^{(n)}\sqrt{\frac{6m_n^2}{\langle(\eta^{(n)})^2\rangle}} \sin\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right), \quad (4.105)$$

$$\Delta x_i^{(n)}(t) =$$
$$= x_{0i}^{(n)} \cos\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right) + \upsilon_{0i}^{(n)}\sqrt{\frac{6m_n^2}{\langle(\eta^{(n)})^2\rangle}} \sin\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right) -$$
$$-\sum_l \mu_l x_{0i}^{(l)} \cos\left(\sqrt{\frac{\langle(\eta^{(l)})^2\rangle}{6m_l^2}}t\right) +$$
$$+\sum_l \mu_l \upsilon_{0i}^{(l)}\sqrt{\frac{6m_l^2}{\langle(\eta^{(l)})^2\rangle}} \sin\left(\sqrt{\frac{\langle(\eta^{(l)})^2\rangle}{6m_l^2}}t\right), \tag{4.106}$$

where $\mu_n = m_n/\sum_l m_l$. It is important to stress that if the tensor of momentum noncommutativity is defined as

$$\eta_{ij}^{(n)} = \frac{\tilde{\alpha} m_n \hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b, \tag{4.107}$$

(here constant $\tilde{\alpha}$ does not depend on mass) we can write

$$\frac{\langle(\eta^{(n)})^2\rangle}{m_n^2} = \frac{3\hbar^2\tilde{\alpha}^2}{2l_P^4} = B. \tag{4.108}$$

Here we use notation B for a constant which is the same for particles with different masses. Taking into account (4.192), we have the following expression for the trajectory

$$x_i^{(n)}(t) = x_{0i}^{(n)} \cos\left(\sqrt{\frac{B}{6}}t\right) + \upsilon_{0i}^{(n)}\sqrt{\frac{6}{B}} \sin\left(\sqrt{\frac{B}{6}}t\right). \tag{4.109}$$

If the initial velocities are the same

$$\upsilon_{0i}^{(n)} = \upsilon_{0i}, \tag{4.110}$$

the trajectory of the center-of-mass reads

$$\tilde{x}_i(t) = \tilde{x}_{0i} \cos\left(\sqrt{\frac{B}{6}}t\right) + \upsilon_{0i}\sqrt{\frac{6}{B}} \sin\left(\sqrt{\frac{B}{6}}t\right). \tag{4.111}$$

Here

$$\tilde{x}_{0i} = \sum_n \mu_n x_{0i}^{(n)}, \quad (4.112)$$

and the relative coordinates of particles do not depend on time

$$\Delta x_i^{(n)}(t) = x_{0i}^{(n)} - \tilde{x}_{0i}. \quad (4.113)$$

So, dependence of parameter of momentum noncommutativity on mass is important for solving the problem of flying away from a system of free particles.

4.4 Exact results for energy and wavefunctions of a particle in uniform field in noncommutative phase space

We examine a particle with mass m in uniform field. The Hamiltonian reads

$$H_p = \frac{P^2}{2m} - \alpha X_3, \quad (4.114)$$

α is a constant. Without loss of generality, we study the case when the field is pointed in the X_3 direction (in (4.114). The total Hamiltonian reads

$$H = \frac{P^2}{2m} - \alpha X_3 + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega_{osc}^2 a^2}{2}. \quad (4.115)$$

Using representation for noncommutative coordinates and noncommutative momenta we can write

$$H = \frac{p^2}{2m} - \alpha x_3 - \frac{1}{2}[\boldsymbol{\theta} \times \mathbf{p}]_3 + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega_{osc}^2 a^2}{2} =$$
$$= \frac{p^2}{2m} - \alpha x_3 - \frac{\alpha c_\theta}{2\hbar}(p_1^a p_2 - p_2^a p_1) + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega_{osc}^2 a^2}{2}. \quad (4.116)$$

Let us rewrite Hamiltonian (4.116) as follows

$$H=\left(1-\frac{\alpha^2c_\theta^2mm_{osc}}{4\hbar^2}\right)\frac{p_1^2}{2m}+$$
$$+\left(1-\frac{\alpha^2c_\theta^2mm_{osc}}{4\hbar^2}\right)\frac{p_2^2}{2m}+\frac{p_3^2}{2m}-\alpha x_3+$$
$$+\frac{1}{2m_{osc}}\left(p_1^a-\frac{\alpha c_\theta m_{osc}}{2\hbar}p_2\right)^2+\frac{1}{2m_{osc}}\left(p_2^a+\frac{\alpha c_\theta m_{osc}}{2\hbar}p_1\right)^2+$$
$$+\frac{(p_3^a)^2}{2m_{osc}}++\frac{m_{osc}\omega_{osc}^2a_1^2}{2}+\frac{m_{osc}\omega_{osc}^2a_2^2}{2}+\frac{m_{osc}\omega_{osc}^2a_3^2}{2}. \quad (4.117)$$

It is important to note that operators

$$\tilde{H}_p=\left(1-\frac{\alpha^2c_\theta^2m}{4\hbar\omega_{osc}l_P^2}\right)\frac{p_1^2}{2m}+$$
$$+\left(1-\frac{\alpha^2c_\theta^2m}{4\hbar\omega_{osc}l_P^2}\right)\frac{p_2^2}{2m}+\frac{p_3^2}{2m}-\alpha x_3, \quad (4.118)$$
$$\tilde{H}_{osc}=\frac{1}{2m_{osc}}\left(p_1^a-\frac{\alpha c_\theta}{2\omega_{osc}l_P^2}p_2\right)^2+$$
$$+\frac{1}{2m_{osc}}\left(p_2^a+\frac{\alpha c_\theta}{2\omega_{osc}l_P^2}p_1\right)^2+$$
$$+\frac{(p_3^a)^2}{2m_{osc}}+\frac{m_{osc}\omega_{osc}^2a_1^2}{2}+\frac{m_{osc}\omega_{osc}^2a_2^2}{2}+\frac{m_{osc}\omega_{osc}^2a_3^2}{2}, \quad (4.119)$$

commute

$$[\tilde{H}_p,\tilde{H}_{osc}]=0. \quad (4.120)$$

Hamiltonian of the particle $\tilde{H}_p$ can be rewritten as

$$\tilde{H}_p=\tilde{H}_1+\tilde{H}_2+\tilde{H}_3, \quad (4.121)$$

where

$$\tilde{H}_1 = \frac{p_1^2}{2m_{eff}}, \tag{4.122}$$

$$\tilde{H}_2 = \frac{p_2^2}{2m_{eff}}, \tag{4.123}$$

$$\tilde{H}_3 = \frac{p_3^2}{2m} - \alpha x_3, \tag{4.124}$$

$$[\tilde{H}_1, \tilde{H}_2] = [\tilde{H}_2, \tilde{H}_3] = [\tilde{H}_1, \tilde{H}_3] = 0, \tag{4.125}$$

with effective mass

$$m_{eff} = m\left(1 - \frac{\alpha^2 c_\theta^2 m m_{osc}}{4\hbar^2}\right)^{-1} = m\left(1 - \frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right)^{-1}. \tag{4.126}$$

It is important to mention that x_3, p_3 in $\tilde{H}_3$ satisfy the ordinary commutation relations. So, Hamiltonian $\tilde{H}_3$ is the Hamiltonian of a particle in uniform filed in the ordinary space. Let us introduce

$$\tilde{p}_1^a = p_1^a - \frac{\alpha c_\theta}{2\omega_{osc} l_P^2} p_2, \tag{4.127}$$

$$\tilde{p}_2^a = p_2^a + \frac{\alpha c_\theta}{2\omega_{osc} l_P^2} p_1, \tag{4.128}$$

$$\tilde{p}_3^a = p_3^a. \tag{4.129}$$

So, we can write (4.119) as follows

$$\tilde{H}_{osc} = \frac{(\tilde{p}^a)^2}{2m_{osc}} + \frac{m_{osc}\omega_{osc}^2 a^2}{2}. \tag{4.130}$$

For operators a_i and $\tilde{p}_i^a$ we have the ordinary commutation relations

$$[a_i, a_j] = [\tilde{p}_i^a, \tilde{p}_j^a] = 0, \tag{4.131}$$

$$[a_i, \tilde{p}_j^a] = i\hbar\delta_{ij}. \tag{4.132}$$

For operators $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_{osc}$ we have (4.120), (4.125). So, exact expression for the spectrum of a particle in uniform filed reads

$$E = \frac{\hbar^2 k_1^2}{2m}\left(1 - \frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right) + \\ + \frac{\hbar^2 k_2^2}{2m}\left(1 - \frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right) + E_3 + \frac{3}{2}\hbar\omega_{osc}. \tag{4.133}$$

It is important to mentioning that we have free motion of a particle in the directions perpendicular to the field. Values k_1, k_2 are components of the wave vector that correspond to this free motion. Notation E_3 is used for denoting continious eigenvalues of Hamiltonian $\tilde{H}_3$. in (4.133) the last term corresponds to the ground state of the harmonic oscillator.

The eigenfunctions of the total Hamiltonian (4.117) can be written as

$$\psi(\mathbf{x}, \mathbf{a}) = \tilde{\psi}_1(x_1)\tilde{\psi}_2(x_2)\tilde{\psi}_3(x_3)\tilde{\psi}(\mathbf{a}). \tag{4.134}$$

Here $\tilde{\psi}_i(x_i)$ are eigenfunctions of $\tilde{H}_i$ that are defined as (4.122)-(4.124). Eigenfunction of a particle in the uniform field in the space with commutative coordinates and commutative momenta $\psi^{(3)}(x_3)$ reads

$$\psi^{(3)}(x_3) = \left(\frac{4m^2}{\pi^3\alpha\hbar^4}\right)^{\frac{1}{6}} \Phi\left(\left(\frac{2m\alpha}{\hbar^2}\right)^{\frac{1}{3}}\left(-x_3 - \frac{E_3}{\alpha}\right)\right), \tag{4.135}$$

where Φ is the Airy function

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{t^3}{3} + tx\right) dt. \tag{4.136}$$

Functions $\tilde{\psi}(\mathbf{a})$ denote eigenfunctions of Hamiltonian

$$\begin{aligned} H'_{osc} = \frac{1}{2m_{osc}}\left(p_1^a - \frac{\alpha c_\theta \hbar k_2}{2\omega_{osc} l_P^2}\right)^2 + \\ + \frac{1}{2m_{osc}}\left(p_2^a + \frac{\alpha c_\theta \hbar k_1}{2\omega_{osc} l_P^2}\right)^2 + \\ + \frac{(p_3^a)^2}{2m_{osc}} + \frac{m_{osc}\omega_{osc}^2 a_1^2}{2} + \frac{m_{osc}\omega_{osc}^2 a_2^2}{2} + \frac{m_{osc}\omega_{osc}^2 a_3^2}{2}. \end{aligned} \tag{4.137}$$

Note that expression for Hamiltonian (4.137) is obtained replacing p_1 by $\hbar k_1$ and p_2 by $\hbar k_2$, in (4.119). The eigenfunction of harmonic oscillator (4.137) corresponding to the ground state as follows

$$\tilde{\psi}(\mathbf{a}) = \frac{1}{\pi^{\frac{3}{4}} l_P^{\frac{3}{2}}} e^{-\frac{a^2}{2l_P^2} - i\beta(k_1 a_2 - k_2 a_1)}. \tag{4.138}$$

Here we introduce the notation

$$\beta = \frac{\alpha c_\theta}{2\omega_{osc} l_P^2}. \tag{4.139}$$

So, for the total Hamiltonian (4.117) we have the following eigenfunctions

$$\psi(\mathbf{x}, \mathbf{a}) =$$
$$= C e^{ik_1 x_1} e^{ik_2 x_2} \Phi \left(\left(\frac{2m\alpha}{\hbar^2} \right)^{\frac{1}{3}} \left(-x_3 - \frac{E_3}{\alpha} \right) \right) e^{-\frac{a^2}{2l_P^2} - i\beta(k_1 a_2 - k_2 a_1)} \tag{4,140}$$

where C is the normalization constant.

It is important to stress that noncommutativity affects the motion of a particle in the directions perpendicular to the direction of the field Namely, it affects on the mass of the particle in uniform field.

4.5 Motion of a particle in the uniform gravitational field in noncommutative phase space with preserved time reversal and rotational symmetries

Let us consider the motion of a particle of mass m in rotationally-invariant and time-reversal invariant noncommutative phase space (4.72)-(4.74). The Hamiltonian of a particle in a uniform field is as follows

$$H_p = \frac{\mathbf{P}^2}{2m} + mgX_1. \tag{4.141}$$

In the Hamiltonian we considered X_1 axis to be directed along the field direction. The total Hamiltonian in terms of commuting coordinates and commuting momenta rads

$$H = \frac{\mathbf{p}^2}{2m} + mgx_1 - \frac{(\boldsymbol{\eta} \cdot \mathbf{L})}{2m} + \frac{mg}{2}[\boldsymbol{\theta} \times \mathbf{p}]_1 + \\ + \frac{[\boldsymbol{\eta} \times \mathbf{x}]^2}{8m} + H_{osc}^a + H_{osc}^b. \tag{4.142}$$

This Hamiltonian can be represented as $H = H_0 + \Delta H$ $H_0 = \langle H_p \rangle_{ab} + H^a_{osc} + H^b_{osc}$, $\Delta H = H - H_0 = H_p - \langle H_p \rangle_{ab}$

$$H_0 = \frac{\mathbf{p}^2}{2m} + mgx_1 + \frac{\langle \eta^2 \rangle \mathbf{x}^2}{12m} + H^a_{osc} + H^b_{osc}, \tag{4.143}$$

$$\Delta H = -\frac{(\boldsymbol{\eta} \cdot \mathbf{L})}{2m} + \frac{mg}{2}[\boldsymbol{\theta} \times \mathbf{p}]_1 + \frac{[\boldsymbol{\eta} \times \mathbf{x}]^2}{8m} - \frac{\langle \eta^2 \rangle \mathbf{x}^2}{12m}. \tag{4.144}$$

Up to the second order in ΔH one can study Hamiltonian H_0. In this approximation one can write the equations of motion of the particle

$$\dot{x}_i = \frac{p_i}{m}, \tag{4.145}$$

$$\dot{p}_i = -mg\delta_{i,1} - \frac{\langle \eta^2 \rangle x_i}{6m}. \tag{4.146}$$

Solution of the equations is as follows

$$x_i(t) = \left(x_{0i} + 6g\frac{m^2}{\langle \eta^2 \rangle}\delta_{1,i} \right) \cos\left(\sqrt{\frac{\langle \eta^2 \rangle}{6m^2}}t \right) +$$
$$+ \upsilon_{0i}\sqrt{\frac{6m^2}{\langle \eta^2 \rangle}} \sin\left(\sqrt{\frac{\langle \eta^2 \rangle}{6m^2}}t \right) - 6g\frac{m^2}{\langle \eta^2 \rangle}\delta_{1,i}, \tag{4.147}$$

where we considered notations x_{0i}, υ_{0i} for initial coordinates and velocities of the particle. Note that only momentum noncommutativity affects on the motion of a particle in gravitational field Considering limit $\langle \eta^2 \rangle \to 0$ we find the well-known result in the ordinary space

$$x_i(t) = \delta_{1,i}\frac{gt^2}{2} + x_{0i}. \tag{4.148}$$

Analyzing (4.147) we have that the weak equivalence principle is violated because of noncommutativity. According to the principle the velocity and position of a point mass in a gravitational field are independent of mass.

If we consider the parameter of momentum noncommutativity to be dependent on mass as

$$\frac{\langle \eta^2 \rangle}{m^2} = \frac{3\hbar^2 \tilde{\alpha}^2}{2l_P^4} = B = \text{const}, \tag{4.149}$$

where B does not depend on mass, one obtains the following trajectory

$$x_i(t) = \left(x_{0i} + \frac{6g}{B}\delta_{1,i}\right)\cos\left(\sqrt{\frac{B}{6}}t\right) + \\ +\upsilon_{0i}\sqrt{\frac{6}{B}}\sin\left(\sqrt{\frac{B}{6}}t\right) - \frac{6g}{B}\delta_{1,i}. \quad (4.150)$$

So, due to condition (2.124) the trajectory of a particle in the gravitational field does not depend on mass, and the weak equivalence principle is preserved.

Let us study more general case. For a composite system in the gravitational field we have the following Hamiltonian

$$H_s = \frac{(\mathbf{P}^c)^2}{2M} + MgX_1^{(c)} + H_{rel}, \quad (4.151)$$

where $\mathbf{X^{(c)}}$, $\mathbf{P}^c$ are coordinates and momenta of the center-of-mass of the composite system. Hamiltonian H_{rel} represents the relative motion. In the case when conditions (2.123), (2.124) are satisfied we can represent the Hamiltonian as follows

$$H_0 = \frac{(\mathbf{p}^c)^2}{2M} + Mgx_1^c + \frac{\langle(\eta^c)^2\rangle(\mathbf{x}^c)^2}{12M} + \langle H_{rel}\rangle_{ab} + \\ +H_{osc}^{(a)} + H_{osc}^{(b)}. \quad (4.152)$$

Taking into account that

$$[H_0, \langle H_{rel}\rangle_{ab}] = 0, \quad (4.153)$$

we can write

$$x_i^c(t) = \left(x_{0i}^c + 6g\frac{M^2}{\langle(\eta^c)^2\rangle}\delta_{1,i}\right)\cos\left(\sqrt{\frac{\langle(\eta^c)^2\rangle}{6M^2}}t\right) + \\ +\upsilon_{0i}^c\sqrt{\frac{6M^2}{\langle(\eta^c)^2\rangle}}\sin\left(\sqrt{\frac{\langle(\eta^c)^2\rangle}{6M^2}}t\right) - 6g\frac{M^2}{\langle(\eta^c)^2\rangle}\delta_{1,i}, \quad (4.154)$$

Due to condition (2.124) the trajectory can be rewritten as

$$\frac{\langle(\eta^c)^2\rangle}{M^2} = \frac{3\hbar^2\tilde{\alpha}^2}{2l_P^4} = B = \text{const}, \tag{4.155}$$

$$x_i^c(t) = \left(x_{0i}^c + \frac{6g}{B}\delta_{1,i}\right)\cos\left(\sqrt{\frac{B}{6}}t\right) + \\ +\upsilon_{0i}\sqrt{\frac{6}{B}}\sin\left(\sqrt{\frac{B}{6}}t\right) - \frac{6g}{B}\delta_{1,i}, \tag{4.156}$$

So, the weak equivalence principle is satisfied.

Using (4.147) for the trajectory of the center-of-mass of a system of N non-interacting particles in uniform gravitational field we have

$$x_i^c(t) = \sum_a \mu_a x_i^{(a)}(t) = -\sum_a 6g\mu_a \frac{m_a^2}{\langle(\eta^{(a)})^2\rangle}\delta_{1,i} + \\ +\sum_a \mu_a\left(x_{0i}^{(a)} + 6g\frac{m_a^2}{\langle(\eta^{(a)})^2\rangle}\delta_{1,i}\right)\cos\left(\sqrt{\frac{\langle(\eta^{(a)})^2\rangle}{6m_a^2}}t\right) + \\ +\sum_a \mu_a\upsilon_{0i}^{(a)}\sqrt{\frac{6m_a^2}{\langle(\eta^a)^2\rangle}}\sin\left(\sqrt{\frac{\langle(\eta^a)^2\rangle}{6m_a^2}}t\right), \tag{4.157}$$

Here m_a is the mass of particle a, $x_{0i}^{(a)}$, $\upsilon_{0i}^{(a)}$ are initial coordinates and initial velocities. Note, that due to condition (2.124), taking into account

$$x_{0i}^{(c)} = \sum_a \mu_a x_{0i}^{(a)}, \tag{4.158}$$

$$\upsilon_{0i}^{(c)} = \sum_a \mu_a \upsilon_{0i}^{(a)}, \tag{4.159}$$

one find that expression (4.157) reduces to (4.154).

4.6 Motion in the non-uniform gravitational field in rotationally- and time-reversal invariant noncommutative phase space

For a particle in non-uniform gravitational field we have the following Hamiltonian

$$H_p = \frac{P^2}{2m} - \frac{G\tilde{M}m}{X}, \tag{4.160}$$

where m is the mass of the particle,

$$X = |\mathbf{X}| = \sqrt{\sum_i X_i^2}. \tag{4.161}$$

Similarly as in the previous sections, up to the second order in the parameters of noncommutativity, we can consider Hamiltonian as follows

$$H_0 = \frac{p^2}{2m} - \frac{G\tilde{M}m}{x} + \frac{\langle\eta^2\rangle x^2}{12m} - \frac{G\tilde{M}mL^2\langle\theta^2\rangle}{8x^5} + \\ + \frac{G\tilde{M}m\langle\theta^2\rangle}{24}\left(\frac{2}{x^3}p^2 + \frac{6i\hbar}{x^5}(\mathbf{x}\cdot\mathbf{p}) - \frac{\hbar^2}{x^5}\right) + \\ + H^a_{osc} + H^b_{osc}. \tag{4.162}$$

So, in this approximation of a particle in non-uniform gravitational field read we have the following equations of motion

$$\dot{\mathbf{x}} = \frac{\mathbf{p}}{m} - \frac{G\tilde{M}m\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\mathbf{p} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\mathbf{p})\right), \tag{4.163}$$

$$\dot{\mathbf{p}} = -\frac{G\tilde{M}m\mathbf{x}}{x^3} - \frac{\langle\eta^2\rangle\mathbf{x}}{6m} - \frac{G\tilde{M}m\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\mathbf{p})\mathbf{p} - \right. \\ \left. -\frac{2\mathbf{x}}{x^5}p^2 + \frac{5\mathbf{x}}{2x^7}L^2 + \frac{5\hbar^2\mathbf{x}}{6x^7} - \frac{5i\hbar}{x^7}\mathbf{x}(\mathbf{x}\cdot\mathbf{p})\right). \tag{4.164}$$

In the limit $\hbar \to 0$ we can write

$$\dot{\mathbf{x}} = \boldsymbol{\upsilon} - \frac{G\tilde{M}m^2\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\boldsymbol{\upsilon} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right), \qquad (4.165)$$

$$\dot{\boldsymbol{\upsilon}} = -\frac{G\tilde{M}\mathbf{x}}{x^3} - \frac{\langle\eta^2\rangle\mathbf{x}}{6m^2} -$$
$$-\frac{G\tilde{M}m^2\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\boldsymbol{\upsilon} - \frac{2\mathbf{x}}{x^5}\upsilon^2 + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{\upsilon}]^2\right). \qquad (4.166)$$

Here we use notation

$$\boldsymbol{\upsilon} = \frac{\mathbf{p}}{m} \qquad (4.167)$$

Note that the obtained results depend on $m^2\langle\theta^2\rangle$ and $\langle\eta^2\rangle/m^2$. So, if we consider conditions (2.123), (2.124) we can write

$$\dot{\mathbf{x}} = \boldsymbol{\upsilon} - \frac{G\tilde{M}A}{12}\left(\frac{1}{x^3}\boldsymbol{\upsilon} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right), \qquad (4.168)$$

$$\dot{\boldsymbol{\upsilon}} = -\frac{G\tilde{M}\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} -$$
$$-\frac{G\tilde{M}A}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\boldsymbol{\upsilon} - \frac{2\mathbf{x}}{x^5}\upsilon^2 + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{\upsilon}]^2\right). \qquad (4.169)$$

Here we take into account (4.149), (2.123) and

$$\langle\theta^2\rangle m^2 = \frac{3\alpha^2 l_P^4 m^2}{2\hbar^2} = A = \text{const.} \qquad (4.170)$$

Constant A does not depend on mass.

Results for the equations of motion (4.168), (4.169) depend on constants A, B. The constants are the same for different particles. So, conditions (2.123), (2.124) open a possibility to recover the weak equivalence principle.

Let us also consider a quantum case. If relations (2.123), (2.124) are satisfied the equations (4.163), (4.164) can be rewritten as

$$\dot{\mathbf{x}} = \boldsymbol{\upsilon} - \frac{G\tilde{M}B}{12}\left(\frac{1}{x^3}\boldsymbol{\upsilon} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right), \quad (4.171)$$

$$\dot{\boldsymbol{\upsilon}} = -\frac{G\tilde{M}\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} - \frac{G\tilde{M}A}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\boldsymbol{\upsilon} - \frac{2\mathbf{x}}{x^5}\upsilon^2 + \right.$$
$$\left. + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{\upsilon}]^2 + \frac{5\hbar^2\mathbf{x}}{6m^2x^7} - \frac{5i\hbar}{mx^7}\mathbf{x}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right). \quad (4.172)$$

Note, that these equations depend on $\hbar/m$, as it has to be. This is due to commutation relation

$$[\mathbf{x}, \boldsymbol{\upsilon}] = i\hbar\frac{\hat{I}}{m}. \quad (4.173)$$

(see [100] for the details).

So, if relations (2.123), (2.124) hold, the motion of a particle in gravitational field is independent of its mass, and the weak equivalence principle is preserved.

The same conclusion can be made in the case of motion of a composite system. We have

$$H_s = \frac{(P^c)^2}{2M} - \frac{G\tilde{M}M}{(X^c)^2} + H_{rel}, \quad (4.174)$$

$$H_0 = \frac{(p^c)^2}{2M} - \frac{G\tilde{M}M}{x^c} + \frac{\langle(\eta^c)^2\rangle(x^c)^2}{12M} -$$
$$-\frac{G\tilde{M}M(L^c)^2\langle\theta^2\rangle}{8(x^c)^5} + \frac{G\tilde{M}M\langle(\theta^c)^2\rangle}{24}\left(\frac{2}{(x^c)^3}(p^c)^2 + \right.$$
$$\left. + \frac{6i\hbar}{(x^c)^5}(\mathbf{x}^c\cdot\mathbf{p}^c) - \frac{\hbar^2}{(x^c)^5}\right) + \langle H_{rel}\rangle_{ab} + H^a_{osc} + H^b_{osc}. \quad (4.175)$$

In the case when conditions (2.123), (2.124) are satisfied we can write

$$\dot{\mathbf{x}}^c = \boldsymbol{\upsilon}^c - \frac{G\tilde{M}B}{12}\left(\frac{1}{(x^c)^3}\boldsymbol{\upsilon}^c - \frac{3\mathbf{x}^c}{(x^c)^5}(\mathbf{x}^{\mathbf{c}}\cdot\boldsymbol{\upsilon}^c)\right), \quad (4.176)$$

$$\dot{\boldsymbol{\upsilon}}^c = -\frac{G\tilde{M}\mathbf{x}^c}{(x^c)^3} - \frac{B\mathbf{x}^c}{6} - \frac{G\tilde{M}A}{4}\left(\frac{1}{(x^c)^5}(\mathbf{x}^c\cdot\boldsymbol{\upsilon}^c)\boldsymbol{\upsilon}^c - \right.$$
$$\left. - \frac{2\mathbf{x}^c}{(x^c)^5}(\upsilon^c)^2 + \frac{5\mathbf{x}^c}{2(x^c)^7}[\mathbf{x}^c\times\boldsymbol{\upsilon}^c]^2\right). \quad (4.177)$$

It is important to stress that if relations (2.123), (2.124) are not preserved, the equations of motion of a composite system depend on its mass and parameters $\langle(\theta^c)^2\rangle$, $\langle(\eta^c)^2\rangle$. The parameters are defined as

$$\theta^c_{ij} = \sum_n \mu_n^2 \theta^{(n)}_{ij}, \tag{4.178}$$

$$\eta^c_{ij} = \sum_n \eta^{(n)}_{ij}, \tag{4.179}$$

and depend on the composition. So, this in addition causes violation of the weak equivalence principle in quantum space.

4.7 Studies of the Mercury motion in noncommutative phase space

Let us first consider a particle of mass m in the gravitational filed $-k/X$ in noncommutative phase space with preserved rotational and time-reversal symmetries (4.72)-(4.74). So, the total Hamiltonian reads

$$H = H_p + H^a_{osc} + H^b_{osc}, \tag{4.180}$$

$$H_p = \frac{P^2}{2m} - \frac{mk}{X}. \tag{4.181}$$

here X_i, P_i satisfy relations (4.72)-(4.74), terms H^a_{osc}, H^b_{osc} are Hamiltonians of harmonic oscillators. Up to the second order in the parameters of noncommutativity we can consider Hamiltonian as follows

$$\langle H_p\rangle_{ab} = \frac{p^2}{2m} - \frac{mk}{x} + \frac{\langle\eta^2\rangle x^2}{12m} - \frac{\langle\theta^2\rangle mkL^2}{8x^5} + \frac{\langle\theta^2\rangle mkp^2}{12x^3}. \tag{4.182}$$

Noncommutativity of coordinates and noncommutativity of momenta cause the precession of the orbit of the particle. To find the precession rate of the orbit we consider

$$\mathbf{u} = \frac{\mathbf{p}}{m} - \frac{mk[\mathbf{L}\times\mathbf{x}]}{xL^2}, \tag{4.183}$$

and calculate

$$\mathbf{\Omega} = \frac{[\mathbf{u}\times\dot{\mathbf{u}}]}{u^2}. \tag{4.184}$$

We obtain

$$\left\{\mathbf{u}, \frac{p^2}{2m} - \frac{mk}{x}\right\} = 0, \tag{4.185}$$

$$\dot{\mathbf{u}} = \left\{\mathbf{u}, \frac{\langle\eta^2\rangle x^2}{12m} - \frac{\langle\theta^2\rangle mkL^2}{8x^5} + \frac{\langle\theta^2\rangle mkp^2}{12x^3}\right\} =$$
$$= -\frac{\langle\eta^2\rangle\mathbf{x}}{6m^2} - \frac{k\langle\theta^2\rangle}{4}\left(\frac{(\mathbf{x}\cdot\mathbf{p})\mathbf{p}}{x^5} - \frac{2p^2\mathbf{x}}{x^5} + \frac{5L^2\mathbf{x}}{2x^7}\right) +$$
$$+\frac{m^2k^2\langle\theta^2\rangle[\mathbf{L}\times\mathbf{p}]}{12L^2x^4} - \frac{m^2k^2\langle\theta^2\rangle(\mathbf{x}\cdot\mathbf{p})[\mathbf{L}\times\mathbf{x}]}{12L^2x^6}. \tag{4.186}$$

It is known that in the ordinary space

$$u^2 = \frac{m^2k^2e^2}{L^2}, \tag{4.187}$$

here e is the eccentricity of the orbit. So, we find

$$\Omega = \langle\theta^2\rangle\left(\frac{5L^4}{8km^3x^7e^2} - \frac{p^2L^2}{2m^3x^5ke^2} +\right.$$
$$\left.+\frac{p^2}{4me^2x^4} - \frac{7L^2}{24mx^6e^2} - \frac{mk}{12x^5e^2}\right)L +$$
$$+\langle\eta^2\rangle\left(\frac{L^2}{6m^5k^2e^2} - \frac{x}{6m^3ke^2}\right)L. \tag{4.188}$$

For the perihelion shift per revolution we can write

$$\Delta\phi_p = \int_0^T \Omega dt =$$
$$= \int_0^{2\pi}\frac{\Omega}{\dot{\phi}}d\phi = \langle\theta^2\rangle\frac{\pi km^2(4+e^2)}{8a^3(1-e^2)^3} - \langle\eta^2\rangle\frac{\pi a^3\sqrt{1-e^2}}{2m^2k}, \tag{4.189}$$

with a being the semi-major axis, ϕ being the polar angle. To find (4.189) we take into account that in the ordinary space

$$L = mx^2\dot{\phi}, x = \frac{a(1-e^2)}{1+e\cos\phi}, \tag{4.190}$$

$$\frac{p^2}{2m} - \frac{mk}{x} = -\frac{mk}{2a}. \tag{4.191}$$

It is important to stress that the perihelion shift depends on the mass of the particle m. If relations (2.123), (2.124) hold, we obtain

$$\langle\theta^2\rangle m^2 = \frac{3\tilde{\gamma}^2}{2l_P^2} = A, \quad \frac{\langle\eta^2\rangle}{m^2} = \frac{3\tilde{\alpha}^2}{2l_P^2} = B, \tag{4.192}$$

where A, B are constants that do not depend on the masses of Particles.

Taking into account (4.192), (4.189) we find

$$\Delta\phi_p = A\frac{\pi k(4+e^2)}{8a^3(1-e^2)^3} - B\frac{\pi a^3\sqrt{1-e^2}}{2k}. \tag{4.193}$$

It is worth mentioning that the proposed conditions (2.123), (2.124) are important for solving the problem of violation of the weak equivalence principle in quantum space.

For a composite system with mass M in gravitational field we have

$$H_s = H_{cm} + H_{rel}, \tag{4.194}$$

$$H_{cm} = \frac{(P^c)^2}{2M} - \frac{Mk}{X^c}, \tag{4.195}$$

X_i^c, P_i^c are coordinates and momenta of the center-of-mass, H_{rel} describes the relative motion. If relations (2.123), (2.124) are satisfied commutators for coordinates and momenta corresponds to noncommutative algebra (4.72), (4.74). The coordinates and momenta of the center-of-mass can be represented as

$$X_i^c = x_i^c - \frac{\theta_{ij}^c p_j^c}{2}, \tag{4.196}$$

$$P_i^c = p_i^c + \frac{\eta_{ij}^c x_j^c}{2}. \tag{4.197}$$

So, up to the second order in the parameters of noncommutativity we can study Hamiltonian as follows

$$H_0 = \langle H_s\rangle_{ab} + H^a_{osc} + H^b_{osc} = \\ = \frac{(p^c)^2}{2M} - \frac{Mk}{x^c} + \frac{\langle(\eta^c)^2\rangle(x^c)^2}{12M} - \\ -\frac{\langle(\theta^c)^2\rangle Mk(L^c)^2}{8(x^c)^5} + \\ +\frac{\langle(\theta^c)^2\rangle Mk}{24}\left(\frac{1}{(x^c)^2}(p^c)^2\frac{1}{x^c} + \frac{1}{x^c}(p^c)^2\frac{1}{(x^c)^2} + \frac{\hbar^2}{(x^c)^5}\right) + \\ +\langle H_{rel}\rangle_{ab} + H^a_{osc} + H^b_{osc}. \quad (4.198)$$

Here

$$\mathbf{L}^c = [\mathbf{x}^c \times \mathbf{p}^c]. \quad (4.199)$$

Using definitions

$$\Delta\mathbf{X}^{(n)} = \mathbf{X}^{(n)} - \mathbf{X}^c, \quad (4.200)$$

$$\Delta\mathbf{P}^{(n)} = \mathbf{P}^{(n)} - \mu_n\mathbf{P}^c, \quad (4.201)$$

and taking into account (2.123), (2.124), we have

$$\Delta X_i^{(n)} = \Delta x_i^{(n)} - \frac{\theta_{ij}^{(n)}\Delta p_j^{(n)}}{2}, \quad (4.202)$$

$$\Delta P_i^{(n)} = \Delta p_i^{(n)} + \frac{\eta_{ij}^{(n)}\Delta x_j^{(n)}}{2}. \quad (4.203)$$

Here coordinates and momenta

$$\Delta\mathbf{x}^{(n)} = \mathbf{x}^{(n)} - \mathbf{x}^c, \quad (4.204)$$

$$\Delta\mathbf{p}^{(n)} = \mathbf{p}^{(n)} - \mu_n\mathbf{p}^c, \quad (4.205)$$

satisfy the ordinary commutation relations. It is important that $\langle H_{rel}\rangle_{ab}$ commutes with H_0. So, one can consider the following Hamiltonian

$$\langle H_{cm}\rangle_{ab} = \frac{(p^c)^2}{2M} - \frac{Mk}{x^c} + \frac{\langle(\eta^c)^2\rangle(x^c)^2}{12M} - \\ -\frac{\langle(\theta^c)^2\rangle Mk(L^c)^2}{8(x^c)^5} + \frac{\langle(\theta^c)^2\rangle Mk(p^c)^2}{12(x^c)^3}. \quad (4.206)$$

Using (4.189), for the perihelion shift of orbit of macroscopic body we can write

$$\Delta\phi_{nc} = \langle(\theta^c)^2\rangle \frac{\pi k M^2(4+e^2)}{8a^3(1-e^2)^3} - \langle(\eta^c)^2\rangle \frac{\pi a^3\sqrt{1-e^2}}{2M^2 k}, \qquad (4.207)$$

here

$$\langle(\theta^c)^2\rangle = \frac{3\tilde{\gamma}^2}{2l_P^2 M^2} = \frac{A}{M^2}, \qquad (4.208)$$

$$\langle(\eta^c)^2\rangle = \frac{3\tilde{\alpha}^2 M^2}{2l_P^2} = BM^2. \qquad (4.209)$$

4.8 Upper bounds on the parameters of noncommutativity

We apply the obtained result for the perihelion shift for the Mercury planet. We compare the perihelion shift caused by space quantization (4.207) with

$$\Delta\phi_{obs} - \Delta\phi_{GR} = 2\pi(-0.00049 \pm 0.00017)\cdot 10^{-8}\text{radians/revolution} \qquad (4.210)$$

(here $\Delta\phi_{GR}$ is perihelion precession rate from General Relativity predictions, $\Delta\phi_{obs}$ is the result of observations). We assume that $|\Delta\phi_{nc}|$ is less than $|\Delta\phi_{obs} - \Delta\phi_{GR}|$ at 3σ and write the following inequality

$$|\Delta\phi_{nc}| \leq 2\pi\cdot 10^{-11}\text{radians/revolution}, \qquad (4.211)$$

Parameter θ^c_{ij} or parameter η^c_{ij} could be equal to zero. Therefore it is sufficiently to consider the following inequalities

$$\left|\langle(\theta^c)^2\rangle \frac{\pi G M_\odot M^2(4+e^2)}{8a^3(1-e^2)^3}\right| \leq 2\pi\cdot 10^{-11}\text{radians/revolution}, \qquad (4.212)$$

$$\left|\langle(\eta^c)^2\rangle \frac{\pi a^3\sqrt{1-e^2}}{2GM_\odot M^2}\right| \leq 2\pi\cdot 10^{-11}\text{radians/revolution}, \qquad (4.213)$$

where M is the mass of Mercury, a, e are parameters of its orbit. So, we find

$$\hbar\sqrt{\langle(\theta^c)^2\rangle} < 2.3\cdot 10^{-57}\text{m}^2, \qquad (4.214)$$

$$\hbar\sqrt{\langle(\eta^c)^2\rangle} < 1.8\cdot 10^{-22}\text{kg}^2\text{m}^2/\text{s}^2. \qquad (4.215)$$

Taking into account (4.192), (4.208), (4.209), we have

$$\langle(\theta^c)^2\rangle M^2 = \langle(\theta^{(n)})^2\rangle m_n^2, \qquad (4.216)$$

$$\frac{\langle(\eta^c)^2\rangle}{M^2} = \frac{\langle(\theta^{(n)})^2\rangle}{m_n^2}, \qquad (4.217)$$

here parameters $\langle(\theta^{(n)})^2\rangle$, $\langle(\eta^{(n)})^2\rangle$ correspond to particle of mass m_n.

Based on relations (4.214), (4.215), (4.216), (4.217) one can find upper bounds on the parameters of noncommutativity of different particles. In the case of electron we have

$$\hbar\sqrt{\langle(\theta^{(e)})^2\rangle} < 8.3 \cdot 10^{-4}\text{m}^2, \qquad (4.218)$$

$$\hbar\sqrt{\langle(\eta^{(e)})^2\rangle} < 5.1 \cdot 10^{-76}\text{kg}^2\text{m}^2/\text{s}^2. \qquad (4.219)$$

We do not obtain strong upper bound for the parameter of coordinate noncommutativity. This is because the influence of noncommutativity of coordinates on the motion of macroscopic bodies is less than on the motion of particles. So, for strong upper bounds on the parameters of coordinate noncommutativity data of hight accuracy are needed.

The result (4.219) is quite strong. This result is at least ten orders less than that obtained based on studies of the hydrogen and exotic atoms [45, 46, 50]. Using (4.219), we can also estimate the minimal momentum

$$p_{min} = \sqrt[4]{\frac{3\hbar^2\langle(\eta^{(e)})^2\rangle}{2}} < 2.5 \cdot 10^{-38}\text{kg} \cdot \text{m/s}. \qquad (4.220)$$

In the case of nucleons we have

$$\frac{\langle(\eta^c)^2\rangle}{M^2} = \frac{\langle(\theta^{(nuc)})^2\rangle}{m_{nuc}^2}, \qquad (4.221)$$

$$\hbar\sqrt{\langle(\eta^{(nuc)})^2\rangle} < 9.3 \cdot 10^{-73}\text{kg}^2\text{m}^2/\text{s}^2, \qquad (4.222)$$

where m_{nuc} is the mass of nucleon The obtained result (4.222) is 6 orders less than that estimated on the basis of studies of neutrons in gravitational quantum well [101].

4.9 Conclusions

Noncommutative phase space of canonical type with preserved rotational and time reversal symmetries has been considered (4.72)-(4.74). Corresponding noncommutative algebra (4.72)-(4.74) is constructed with the help of generalization of the parameters of noncommutativity to tensors, constructed with the help of additional momenta. The momenta are governed by harmonic oscillators.

We have considered a particle in uniform field in the frame of the noncommutative algebra. Energy and wave functions of the particle have been found exactly (4.133), (4.140) We have obtained that noncommutativity affects the mass of the particle in the directions perpendicular to the field. Motion of a particle in the field direction is the same as in the ordinary space.

Effect of space quantization on a particle in Coulomb potential has been studied. We have obtained expression for the perihelion shift of orbit of the particle up to the second order in the parameters of noncommutativity. The result has been generalized to the case of motion of macroscopic bodies Upper bounds (4.214), (4.215) have been estimated based on expression for the perihelion shift of the Mercury planet in quantum space (4.207) and data for precession of Mercury's perihelion from ranging to the MESSENGER spacecraft. The obtained upper bounds for the parameters of momentum noncommutativity (4.219), (4.222), for the minimal momentum (4.220) are strong. For parameter of momentum noncommutativity of electron we have obtained upper bound (4.219) that is at least 10 orders less than that obtained based on studies of the hydrogen atom [46, 102].

Chapter 5

Conclusions

In the monograph noncommutative algebra with tensors of noncommutativity constructed with the help of additional coordinates and additional momenta has been considered (1.27)-(1.29). The algebra is rotationally-invariant and equivalent to noncommutative algebra of canonical type in the sense that tensors of noncommutativity commute with operators of coordinates and operators of momenta. In the frame of rotationally-invariant noncommutative algebra different physical systems have been studied. Among them are free particle, systems of harmonic oscillators, hydrogen and hydrogen-like atoms. Based on the obtained results for the energy levels of hydrogen atom and antiprotonic helium the upper bound for the parameters of noncommutativity have been found (see section 2.6). Also the eigenvalues of squared length operator defined in coordinate space, momentum space have been found. Based on the obtained results the minimal length in rotationally-invariant noncommutative phase space has been found (1.112), (1.115), (1.118).

Also, the time-reversal symmetry has been studied in noncommutative phase space. We have shown that noncommutative algebra of canonical type leads to violation of the symmetry. As a result the period of a circular motion in the space depends on its direction, the transformation of noncommutative coordinates and momenta upon time reversal depends on the representation. We have shown that by constructing tensors of coordinates and momentum noncommutativity with the help of additional momenta that are governed by harmonic oscillator one can build algebra that is rotationally invariant, equivalent to noncommutative algebra of canonical type and does not lead to violation of the time-reversal symmetry.

The weak equivalence principle has been studied in rotationally- and time reversal invariant noncommutative phase space. Based on studies of the motion in the gravitational field we have found expressions for the tensors of noncommutativity that gives a possibility to preserve the weak equivalence principle in noncommutative phase space. In addition on the basis of studies of the perihelion shift of the Mercury planet, we have found the upper bound for the parameter of momentum noncommutativity corresponding to electron (4.219) which is at least 10 orders less than that presented in literature.

References

[1] *Gross, D. J.* String theory beyond the planck scale / D. J. Gross, P. F. Mende // *Nucl. Phys. B.* — 1988. — Vol. 303, no. 3. — Pp. 407 – 454.

[2] *Maggiore, M.* A generalized uncertainty principle in quantum gravity / M. Maggiore // *Phys. Lett. B.* — 1993. — Vol. 304, no. 1-2. — Pp. 65 – 69.

[3] *Witten, E.* Reflections on the fate of spacetime / E. Witten // *Physics Today.* — 1996. — Vol. 49, no. 4. — Pp. 24–30.

[4] *Seiberg, N.* String theory and noncommutative geometry / N. Seiberg, E. Witten // *J. High Energy Phys.* — 1999. — Vol. 1999, no. 09. — 92 pp.

[5] *Doplicher, S.* Spacetime quantization induced by classical gravity / S. Doplicher, K. Fredenhagen, J. E. Roberts // *Phys. Lett. B.* — 1994. — Vol. 331, no. 1-2. — Pp. 39–44.

[6] *Amati, D.* Can spacetime be probed below the string size? / D. Amati, M. Ciafaloni, G. Veneziano // *Phys. Lett. B.* — 1989. — Vol. 216, no. 1. — Pp. 41 – 47.

[7] On the space–time uncertainty relations of Liouville strings and D-branes / G. Amelino-Camelia, N. E. Mavromatos, J. Ellis, D. V. Nanopoulos // *Mod. Phys. Lett. A.* — 1997. — Vol. 12, no. 27. — Pp. 2029–2035.

[8] *Snyder, H. S.* Quantized space-time / H. S. Snyder // *Phys. Rev.* — 1947. — Vol. 71, no. 1. — Pp. 38–41.

[9] *Dunne, G.* Topological (Chern-Simons) quantum mechanics / G. Dunne, R. Jackiw, L. Trugenberger // *Phys. Rev. D.* — Vol. 41, no. 2. — Pp. 661–666.

[10] *Dunne, G.* "Peierls substitution" and Chern-Simons quantum mechanics / G. Dunne, R. Jackiw // *Nuclear Physics B (Proc. Suppl.).* — 1991. — Vol. 33, no. C. — Pp. 114–118.

[11] *Jackiw, R.* Observations on noncommuting coordinates and on fields depending on them / R. Jackiw // *Ann. Henri Poincarre.* — 2003. — Vol. 4, no. 2. — Pp. 913–919.

[12] *Duval, C.* The exotic Galilei group and the"Peierls substitution" / C. Duval, P. A. Horváthy // *Phys. Lett. B.* — 2000. — Vol. 479, no. 1-3. — Pp. 284–290.

[13] *Banerjee, R.* A novel approach to noncommutativity in planar quantum mechanics / R. Banerjee // *Mod. Phys. Lett. A.* — 2002. — Vol. 17, no. 11. — P. 631–645.

[14] *Gamboa, J.* Noncommutative quantum mechanics / J. Gamboa, M. Loewe, J. C. Rojas // *Phys. Rev. D.* — 2001. — Vol. 64, no. 6.– Art. 067901. – 3 p.

[15] Noncommutative quantum mechanics: The two-dimensional central field / J. Gamboa, F. Méndez, M. Loewe, J. C. Rojas // *Int. J. of Mod. Phys. A.* — 2002. — Vol. 17, no. 19. — Pp. 2555–2565.

[16] *Nair, V. P.* Quantum mechanics on the noncommutative plane and sphere / V. P. Nair, A. P. Polychronakos // *Phys. Lett. B.* — 2001. — Vol. 505, no. 1-4. — Pp. 267–274.

[17] *Bolonek, K.* On uncertainty relations in noncommutative quantum mechanics / K. Bolonek, P. Kosiński // *Phys. Lett. B.* — 2002. — Vol. 547, no. 1-2. — Pp. 51–54.

[18] *Duval, C.* Exotic Galilean symmetry in the non-commutative plane and the Hall effect / C. Duval, P. A. Horvathy // *J. Phys. A: Math. Gen.* — 2001. — Vol. 34, no. 47. — Pp. 10097–10107.

[19] Quantum theories on noncommutative spaces with nontrivial topology: Aharonov–Bohm and Casimir effects / M. Chaichian,

A. , Demichev, P. Presnajder et al. // *Nuc. Phys. B.* — 2001. — Vol. 611, no. 1-3. — P. 383–402.

[20] *Sinha, D.* Non-commutative quantum mechanics in three dimensions and rotational symmetry / D. Sinha, B. Chakraborty, F. G. Scholtz // *Journal of Physics A: Mathematical and Theoretical.* — 2012. — Vol. 45, no. 10 – Art. 105308.– 23 p.

[21] Scaling of variables and the relation between noncommutative parameters in noncommutative quantum mechanics / O. Bertolami, J. G. Rosa, C. M. L. de Aragao et al. // *Mod. Phys. Lett. A.* — 2006. — Vol. 21, no. 10. — Pp. 795–802.

[22] *Moreno, E. F.* Spherically symmetric monopoles in noncommutative space / E. F. Moreno // *Phys. Rev. D.* — 2005. — Vol. 72, no. 4. — P. 045001.

[23] *Galikova, V.* Coulomb problem in non-commutative quantum mechanics / V. Galikova, P. Presnajder // *J. Math. Phys.* — 2013. — Vol. 54. — 20 pp.

[24] *Kupriyanov, V. G.* A hydrogen atom on curved noncommutative space / V. G. Kupriyanov // *J. Phys. A: Math. Theor.* — 2013. — Vol. 46, no. 24. — 7 pp.

[25] *Amorim, R.* Dynamical symmetries in noncommutative theories / R. Amorim // *Phys. Rev. D.* — 2008. — Vol. 78, no. 10.– Art. 105003.– 7 p. — 7 pp.

[26] *Bander, M.* Coherent states and N dimensional coordinate noncommutativity / M. Bander // *J. High Energy Phys.* — 2006. — Vol. 03, no. 2006. — 14 pp.

[27] *Chaichian, M.* Hydrogen atom spectrum and the Lamb shift in noncommutative QED / M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu // *Phys. Rev. Lett.* — 2001. — Vol. 86, no. 13. — Pp. 2716–2719.

[28] *Ho, P.-M.* Noncommutative quantum mechanics from noncommutative quantum field theory / P.-M. Ho, H.-C. Kao // *Phys. Rev. Lett.* — 2002. — Vol. 88, no. 15. — 4 pp.

[29] *Chaichian, M.* Non-commutativity of space-time and the hydrogen atom spectrum / M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu // *Eur. Phys. J. C.* — 2004. — Vol. 36, no. 2. — Pp. 251–252.

[30] *Chair, N.* The noncommutative quadratic Stark effect for the H-atom / N. Chair, M. A. Dalabeeh // *J. Phys. A.* — 2005. — Vol. 38, no. 7. — Pp. 1553–1558.

[31] *Stern, A.* Noncommutative point sources / A. Stern // *Phys. Rev. Lett.* — 2008. — Vol. 100, no. 6. — 4 pp.

[32] *Zaim, S.* Second-order corrections to the non-commutative Klein-Gordon equation with a Coulomb potential / S. Zaim, L. Khodja, Y. Delenda // *Int. J. Mod. Phys. A.* — 2011. — Vol. 26, no. 23. — Pp. 4133–4144.

[33] Dirac equation in noncommutative space for hydrogen atom / T. C. Adorno, M. C. Baldiotti, M. Chaichian et al. // *Phys. Lett. B.* — 2009. — Vol. 682, no. 2. — Pp. 235–239.

[34] *Khodja, L.* New treatment of the noncommutative Dirac equation with a Coulomb potential / L. Khodja, S. Zaim // *Int. J. Mod. Phys. A.* — 2012. — Vol. 27, no. 19. — 13 pp.

[35] *Djemai, A. E. F.* On quantum mechanics on noncommutative quantum phase space / A. E. F. Djemai, H. Smail // *Commun. Theor. Phys.* — 2004. — Vol. 41, no. 6. — Pp. 837–844.

[36] *Li, K.* Hydrogen atom spectrum in noncommutative phase space / K. Li, N. Chamoun // *Chin. Phys. Lett.* — 2006. — Vol. 23, no. 5. — Pp. 1122–1123.

[37] *Alavi, S. A.* Lamb shift and Stark effect in simultaneous space-space and momentum-momentum noncommutative quantum mechanics and θ-deformed $su(2)$ algebra / S. A. Alavi // *Mod. Phys. Lett. A.* — 2007. — Vol. 22, no. 5. — Pp. 377–383.

[38] *Balachandran, A. P.* On time-space noncommutativity for transition processes and noncommutative symmetries / A. P. Balachandran, A. Pinzul // *Mod. Phys. Lett. A.* — 2005. — Vol. 20, no. 27. — Pp. 2023–2034.

[39] *Stern, A.* Particlelike solutions to classical noncommutative gauge theory / A. Stern // *Phys. Rev. D.* — 2008. — Vol. 78, no. 6. — 11 pp.

[40] *Moumni, M.* A new limit for the noncommutative space–time parameter / M. Moumni, A. BenSlama, S. Zaim // *J. Geom. Phys.* — 2011. — Vol. 61, no. 1. — Pp. 151–156.

[41] *Moumni, M.* Spectrum of hydrogen atom in space-time noncommutativity / M. Moumni, A. BenSlama, S. Zaim // *The African Review of Physics.* — 2012. — Vol. 7, no. 0010. — Pp. 83–94.

[42] *Zaim, S.* Noncommutative of space-time and the relativistic hydrogen atom / S. Zaim, Y. Delenda // *J. Phys.: Conf. Ser.* — 2013. — Vol. 435. — 9 pp.

[43] *Gnatenko, Kh. P.* Hydrogen atom in rotationally invariant noncommutative space / Kh. P. Gnatenko, V. M. Tkachuk // *Phys. Lett. A.* — 2014. — Vol. 378, no. 47. — Pp. 3509–3515.

[44] *Gnatenko, Kh.. P.* Perturbation of the ns levels of the hydrogen atom in rotationally invariant noncommutative space / Kh.. P. Gnatenko, Yu.. S. Krynytskyi, V. Tkachuk // *Mod. Phys. Lett. A.* — 2015. — Vol. 30, no. 8. – Art. 1550033. – 12 p.

[45] *Gnatenko, Kh. P.* Composite system in rotationally invariant noncommutative phase space / Kh. P. Gnatenko, V. M. Tkachuk // *Int. J. Mod. Phys. A.* — 2018. — Vol. 33, no. 7. — P. 1850037.

[46] *Gnatenko, Kh. P.* Noncommutative phase space with rotational symmetry and hydrogen atom / Kh. P. Gnatenko, V. M. Tkachuk // *Int. J. Mod. Phys. A.* — 2017. — Vol. 32, no. 26. — P. 1750161.

[47] *Qiang, W.-C.* An alternative approach to calculating the mean values r^k for hydrogen-like atoms / W.-C. Qiang, S.-H. Dong // *Phys. Scripta.* — 2004. — Vol. 70, no. 5. — Pp. 276–279.

[48] Precision measurement of the hydrogen $1s$-$2s$ frequency via a 920-km fiber link / A. Matveev, C. G. Parthey, K. Predehl et al. // *Phys. Rev. Lett.* — 2013. — Vol. 110, no. 23. — 5 pp.

[49] Noncommutative gravitational quantum well / O. Bertolami, J. G. Rosa, C. M. L. Aragão et al. // *Phys. Rev. D.* — 2005. — Vol. 72, no. 2. – Art. – 025010. – 9 p.

[50] *Bertolami, O.* Phase-space noncommutativity and the Dirac equation / O. Bertolami, R. Queiroz // *Phys. Lett. A.* — 2011. — Vol. 375, no. 2011. — Pp. 4116–4119.

[51] *Polchinski, J.* Precision Spectroscopy of Antiprotonic Helium Atoms and Ions – Weighing the Antiproton. In: Karshenboim S.G. (eds) Precision Physics of Simple Atoms and Molecules. Lecture Notes in Physics vol 745. Springer, Berlin, Heidelberg / J. Polchinski. — Springer, Berlin, Heidelberg. — Pp. 187–201.

[52] Spectroscopy of antiprotonic helium atoms and its contribution to the fundamental physical constants / A. Matveev, C. G. Parthey, K. Predehl et al. // *Proc. Jpn. Acad. Ser. B Phys. Biol. Sci.* — 2010. — Vol. 86, no. 1. — Pp. 1–10.

[53] Two-photon laser spectroscopy of antiprotonic helium and the antiproton-to-electron mass ratio / M. Hori, A. Soter, D. Barn et al. // *Nature.* — 2011. — Vol. 475. — Pp. 484–488.

[54] *Gnatenko, Kh. P.* Composite system in noncommutative space and the equivalence principle / Kh. P. Gnatenko // *Phys. Lett. A.* — 2013. — Vol. 377, no. 43. — Pp. 3061–3066.

[55] *Gnatenko, Kh. P.* Effect of coordinate noncommutativity on the mass of a particle in a uniform field and the equivalence principle / Kh. P. Gnatenko, V. M. Tkachuk // *Mod. Phys. Lett. A.* — 2016. — Vol. 31, no. 5. — P. 1650026.

[56] *Gnatenko, Kh. P.* Weak equivalence principle in noncommutative phase space and the parameters of noncommutativity / Kh. P. Gnatenko, V. M. Tkachuk // *Phys. Lett. A.* — 2017. — Vol. 381, no. 31. — P. 2463–2469.

[57] *Gnatenko, Kh. P.* Kinematic variables in noncommutative phase space and parameters of noncommutativity / Kh. P. Gnatenko // *Mod. Phys. Lett. A.* — 2017. — Vol. 32, no. 31. — P. 1750166.

[58] *Quesne, C.* Composite system in deformed space with minimal length / C. Quesne, V. M. Tkachuk // *Phys. Rev. A.* — 2010. — Vol. 81, no. 1. — P. 012106.

[59] *Tkachuk, V. M.* Deformed Heisenberg algebra with minimal length and the equivalence principle / V. M. Tkachuk // *Phys. Rev. A.* — 2012. — Vol. 86, no. 6. — P. 062112.

[60] *Tkachuk, V. M.* Galilean and Lorentz transformations in a space with generalized uncertainty principle / V. M. Tkachuk // *Found. Phys.* — 2016. — Vol. 46, no. 12. — P. 1666–1679.

[61] *Hatzinikitas, A.* The noncommutative harmonic oscillator in more than one dimension / A. Hatzinikitas, I. Smyrnakis // *J. Math. Phys.* — 2002. — Vol. 43, no. 1. — Pp. 113–125.

[62] *Kijanka, A.* Noncommutative isotropic harmonic oscillator / A. Kijanka, P. Kosiński // *Phys. Rev. D.* — 2004. — Vol. 70, no. 12. — P. 127702.

[63] *Jing, J.* Non-commutative harmonic oscillator in magnetic field and continuous limit / J. Jing, J.-F. Chen // *Eur. Phys. J. C.* — 2009. — Vol. 60, no. 4. — Pp. 669–674.

[64] *Smailagic, A.* Feynman path integral on the non-commutative plane / A. Smailagic, E. Spallucci // *J. Phys. A: Math. Gen.* — 2003. — Vol. 36, no. 33. — Pp. L467–L471.

[65] *Smailagic, A.* Noncommutative 3D harmonic oscillator / A. Smailagic, E. Spallucci // *J. Phys. A: Math. Gen.* — 2002. — Vol. 35. — Pp. L363–L368.

[66] *Muthukumar, B.* Noncommutative oscillators and the commutative limit / B. Muthukumar, P. Mitra // *Phys. Rev. D.* — 2002. — Vol. 66, no. 2. — P. 027701.

[67] Anisotropic harmonic oscillator, non-commutative landau problem and exotic newton–hooke symmetry / P. D. Alvarez, J. Gomis, K. Kamimura, M. S. Plyushchay // *Phys. Lett. B.* — 2008. — Vol. 659, no. 5. — Pp. 906 – 912.

[68] *Dadic, I.* Harmonic oscillator on noncommutative spaces / I. Dadic, L. Jonke, S. Meljanac // *Acta Phys.Slov.* — 2005. — Vol. 55. — Pp. 149–164.

[69] *Giri, P. R.* The non-commutative oscillator, symmetry and the Landau problem / P. R. Giri, P. Roy // *Eur. Phys. J. C.* — 2008. — Vol. 57, no. 4. — Pp. 835–839.

[70] *Geloun, J. B.* Harmonic oscillator in a background magnetic field in noncommutative quantum phase-space / J. B. Geloun, S. Gangopadhyay, F. G. Scholtz // *EPL (Europhysics Letters).* — 2009. — Vol. 86, no. 5. — P. 51001.

[71] Harmonic oscillator on noncommutative spaces / E. M. C. Abreu, M. V. Marcial, A. C. R. Mendes, W. Oliveira // *JHEP.* — 2013. — Vol. 2013:138.

[72] *Saha, A.* Noncommutative quantum mechanics of a harmonic oscillator under linearized gravitational waves / A. Saha, S. Gangopadhyay, S. Saha // *Phys. Rev. D.* — 2011. — Vol. 83, no. 2. — P. 025004.

[73] *Nath, D.* Noncommutative anisotropic oscillator in a homogeneous magnetic field / D. Nath, P. Roy // *Ann. Phys.* — 2017. — Vol. 377. — Pp. 115 – 124.

[74] *Gnatenko, Kh. P.* Effect of noncommutativity on the spectrum of free particle and harmonic oscillator in rotationally invariant noncommutative phase space / Kh. P. Gnatenko, O. V. Shyiko // *Mod. Phys. Lett. A.* — 2018. — Vol. 33, no. 16. — P. 1850091.

[75] *Jellal, A.* Two coupled harmonic oscillators on noncommutative plane / A. Jellal, E. H. E. Kinani, M. Schreiber // *Int. J. Mod. Phys. A.* — 2005. — Vol. 20, no. 7. — Pp. 1515—-1529.

[76] *Jabbari, I.* Partition function of the harmonic oscillator on a noncommutative plane / I. Jabbari, A. Jahan, Z. Riazi // *Turk. J. Phys.* — 2009. — Vol. 33. — Pp. 149–154.

[77] *Lin, B.-S.* Deformation quantization for coupled harmonic oscillators on a general noncommutative space / B.-S. Lin, S.-C. Jing, T.-H. Heng // *Mod. Phys. Lett. A.* — 2008. — Vol. 23, no. 06. — Pp. 445–456.

[78] *Gnatenko, Kh. P.* Two-particle system with harmonic oscillator interaction in noncommutative phase space / Kh. P. Gnatenko,

V. M. Tkachuk // *J. Phys. Stud.* — 2017. — Vol. 21, no. 3. — P. 3001.

[79] *Santos, J. F.* Probing phase-space noncommutativity through quantum mechanics and thermodynamics of free particles and quantum rotors / J. F. Santos, A. E. Bernardini, C. Bastos // *Physica A: Statistical Mechanics and its Applications.* — 2015. — Vol. 438. — Pp. 340 – 354.

[80] *Gnatenko, Kh. P.* Features of free particles system motion in noncommutative phase space and conservation of the total momentum / Kh. P. Gnatenko, H. P. Laba, V. M. Tkachuk // *Mod. Phys. Lett. A.* — 2018. — Vol. 33, no. 23. — P. 1850131.

[81] *Daszkiewicz, M. C.* Classical mechanics of many particles defined on canonically deformed nonrelativistic space-time / M. C. Daszkiewicz, J. Walczyk // *Mod. Phys. Lett. A.* — 2011. — Vol. 26, no. 11. — Pp. 819–832.

[82] *Isgur, N.* p-wave baryons in the quark model / N. Isgur, G. Karl // *Phys. Rev. D.* — Vol. 18, no. 11. — Pp. 4187–4205.

[83] *Glozman, L.* The spectrum of the nucleons and the strange hyperons and chiral dynamics / L. Glozman, D. Riska // *Phys. Rep.* — 1996. — Vol. 268, no. 4. — Pp. 263 – 303.

[84] *Capstick, S.* Quark models of baryon masses and decays / S. Capstick, W. Roberts // *Prog. Part. Nucl. Phys.* — 2000. — Vol. 45. — Pp. S241 – S331.

[85] *Ikeda, S.* Incoherent elastic-neutron-scattering study of the vibrational dynamics and spin-related symmetry of protons in the khco$_3$ crystal / S. Ikeda, F. Fillaux // *Phys. Rev. B.* — 1999. — Vol. 59, no. 6. — Pp. 4134–4145.

[86] *Fillaux, F.* Quantum entanglement and nonlocal proton transfer dynamics in dimers of formic acid and analogues / F. Fillaux // *Chem. Phys. Lett.* — 2005. — Vol. 408, no. 4. — Pp. 302 – 306.

[87] *Hong-Yi, F.* Unitary transformation for four harmonically coupled identical oscillators / F. Hong-Yi // *Phys. Rev. A.* — 1990. — Vol. 42, no. 7. — Pp. 4377–4380.

[88] *Michelot, F.* Solution for an arbitrary number of coupled identical oscillators / F. Michelot // *Phys. Rev. A.* — 1992. — Vol. 45, no. 7. — Pp. 4271–4276.

[89] *Ponte, M. A.* Decoherence in a system of strongly coupled quantum oscillators. i. symmetric network / M. A. Ponte, M. C. Oliveira, M. H. Y. Moussa // *Phys. Rev. A.* — 2004. — Vol. 70, no. 2. Art. 022324. 16 p.

[90] *Ponte, M. A.* Networks of dissipative quantum harmonic oscillators: A general treatment / M. A. Ponte, S. S. Mizrahi, M. H. Y. Moussa // *Phys. Rev. A.* — 2007. — Vol. 76, no. 3. Art. 032101. 10 p.

[91] *Plenio, M. B.* Dynamics and manipulation of entanglement in coupled harmonic systems with many degrees of freedom / M. B. Plenio, J. Hartley, J. Eisert // *New Journal of Physics.* — 2004. — Vol. 6. Art. 36. 39 p.

[92] *Gnatenko, Kh. P.* System of interacting harmonic oscillators in rotationally invariant noncommutative phase space / Kh. P. Gnatenko // *Phys. Lett. A.* — 2018. — Vol. 382, no. 46. — Pp. 3317 – 3324.

[93] *Gnatenko, Kh. P.* Harmonic oscillator chain in noncommutative phase space with rotational symmetry / Kh. P. Gnatenko // *Ukr. J. Phys.* — 2019. — Vol. 64, no. 2. — Pp. 131–136.

[94] *Caves, C. M.* New formalism for two-photon quantum optics. I. Quadrature phases and squeezed states / C. M. Caves, B. L. Schumaker // *Phys. Rev. A.* — 1985. — Vol. 31, no. 5. — Pp. 3068–3092.

[95] *Schumaker, B. L.* New formalism for two-photon quantum optics. II. Mathematical foundation and compact notation / B. L. Schumaker, C. M. Caves // *Phys. Rev. A.* — 1985. — Vol. 31, no. 5. — Pp. 3093–3111.

[96] *Domingos, J. M.* Time reversal in classical and quantum mechanics / J. M. Domingos // *Int. J Theor. Phys.* — 1979. — Vol. 18, no. 3. — Pp. 212 – 230.

[97] *Gnatenko, Kh. P.* Time-reversal and rotational symmetries in noncommutative phase space / Kh. P. Gnatenko, M. I. Samar, V. M. Tkachuk // *Phys. Rev. A.* — 2019. — Vol. 99, no. 1. — P. 012114.

[98] *Gnatenko, Kh. P.* Rotationally invariant noncommutative phase space of canonical type with recovered weak equivalence principle / Kh. P. Gnatenko // *EPL (Europhysics Letters).* — 2018. — Vol. 123, no. 5. — P. 50002.

[99] *Gnatenko, Kh. P.* Upper bound on the momentum scale in noncommutative phase space of canonical type / Kh. P. Gnatenko, V. M. Tkachuk // *EPL (Europhysics Letters).* — 2019. — Vol. 127, no. 2. Art. 20008. 7 p.

[100] *Greenberger, D.* The role of equivalence in quantum mechanics / D. Greenberger // *Ann. Phys.* — 1968. — Vol. 47, no. 1. — Pp. 116 – 126.

[101] Noncommutative gravitational quantum well / O. Bertolami, J. G. Rosa, C. M. L. de Aragão et al. // *Phys. Rev. D.* — 2005. — Vol. 72, no. 2. Art. 025010. 9 p.

[102] *Bertolami, O.* Phase-space noncommutativity and the Dirac equation / O. Bertolami, R. Queiroz // *Phys. Lett. A.* — 2011. — Vol. 375, no. 46. — Pp. 4116–4119.